Quantum medium algebras

Joint work with Jakub Löwit

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SAGA seminar
University Paris-Saclay
May 2026

Austrian Science Fund

European Research Council
Established by the European Commission

Institute of Science and Technology Austria

Motivations from mirror symmetry and Langlands duality

$$
\begin{tikzcd}
\mathrm{Spec}(\mathcal{B}^\mu) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\rho^\mu(\mathbb{E}_c^\vee)) \arrow[l, twoheadrightarrow] \arrow[dd] \arrow[r, phantom, "\cong"] & \mathcal{H}_c^\mu(W_0^+) \arrow[r, twoheadrightarrow] \arrow[d] & \mathrm{Spec}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(\mathcal{M}^\mu) \arrow[d] \arrow[r, phantom, "\llcorner"] & & \overline{\overline{W_\mu^+}} \arrow[d] \arrow[r, phantom, "\lrcorner"] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, twoheadrightarrow] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}
$$

Filtered medium algebras $\mathcal{Z}^\mu$

- G complex connected reductive group, $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$, $U(\mathfrak{g})$ universal enveloping algebra
- $X_+^*(\mathrm{G}) \ni \mu$-highest weight representation $\rho^\mu : \mathrm{G} \to \mathrm{GL}(V^\mu)$
- $\mathcal{R}^\mu := (U(\mathfrak{g}) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}}$ *Kostant algebra*
  filtered algebra over $Z := Z(U(\mathfrak{g})) = U(\mathfrak{g})^{\mathrm{G}}$
- $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ diagonal coproduct
  $\Delta_\mu := (1 \otimes \rho^\mu) \circ \Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes \mathrm{End}(V^\mu)$
- $\mathcal{Z}^\mu := \langle z \otimes 1, \Delta_\mu(z) \rangle_{z \in Z} \subset \mathcal{R}^\mu$ *filtered medium algebra*
- (Feigin–Frenkel 1992) $\rightsquigarrow$ filtered big algebra $\mathcal{G}^\mu \subset \mathcal{R}^\mu$ maximal commutative
- $Z \subset \mathcal{Z}^\mu \subset \mathcal{G}^\mu \subset \mathcal{R}^\mu$ and $gr\mathcal{Z}^\mu \cong \mathcal{M}^\mu$, $gr\mathcal{G}^\mu = \mathcal{B}^\mu$

Equivariant topology of $\mathcal{Z}^\mu$

- $\mathrm{Gr} := \mathrm{G}^\vee((z))/\mathrm{G}^\vee[[z]] = \bigsqcup_{\mu \in X_+^*} \mathrm{G}^\vee[[z]]z^\mu$ *affine Grassmannian*
- $\mathrm{Gr}^\mu := \overline{\mathrm{G}^\vee[[z]]z^\mu} \subset \mathrm{Gr}$ *affine Schubert variety*
  "space of type μ Hecke transformations"
- $\mathrm{G}^\vee$ acts on Gr^μ and $\mathbb{C}^\times$ acts on Gr^μ by loop rotation $z \mapsto az$
- Rees construction: for $R = \bigcup_i F_i R$, $R_\hbar := \bigoplus_i \hbar^i F_i R$ over $\mathbb{C}[\hbar]$
- (Nakajima 2023) $\Longrightarrow$ (Braverman–Finkelberg 2008) $\Longrightarrow$

$$\mathcal{Z}_\hbar^\mu \cong H^{2*}_{\mathrm{G}^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu)$$

$$\Downarrow$$

$$\mathcal{Z}^\mu \cong H^{2*}_{\mathrm{G}^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu)/(\hbar - 1)$$

Representation theory of $\mathcal{Z}^\mu$

- Category $\mathcal{O}$ for $U(\mathfrak{g})$ is generated by Verma modules $M(\lambda)$, $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra
- $R := S(\mathfrak{h}) = U(\mathfrak{h})$ and $c : \mathfrak{h} \to R$ the canonical injection $U_R(\mathfrak{a}) := R \otimes_{\mathbb{C}} U(\mathfrak{a})$ for a Lie algebra $\mathfrak{a}$
- $M_R(c) := U_R(\mathfrak{g}) \otimes_{U_R(\mathfrak{b})} R_c$ *universal Verma module*

Theorem (Kostant 1975; Bernstein–Gelfand 1980; Muić–Savin 2008; Higson 2010; Hausel 2025)

1. $\mathcal{R}^\mu = (U(\mathfrak{g}) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \cong \mathrm{End}_{U_R(\mathfrak{g})}(M_R(c) \otimes V^\mu)^{W^\bullet}$
2. $\mathcal{Z}^\mu = Z(\mathcal{R}^\mu)$
3. $\mathcal{R}^\mu$ *is a spherical Hecke algebra:* $\mathrm{Rep}_{\mathrm{fd}}(\mathcal{R}^\mu) \cong \mathrm{HC}^\mu(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ *category of Harish–Chandra bimodules containing the* μ*-type*
4. $\lim_{\leftarrow \mu} \mathcal{Z}^\mu = Z(\mathcal{C}(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}))$, *center of category of* $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$*-modules*

Examples of infinitesimal characters in tensor product

$$M_{-1} \otimes V^{5\omega_1} = P_{-6} \oplus P_{-4} \oplus P_{-2}$$

$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

$$M_2 \otimes V^{5\omega_1} = \begin{array}{l} P_{-3} \oplus M_{-1} \oplus M_3 \\ \oplus M_5 \oplus M_7 \end{array}$$

$$M_3 \otimes V^{5\omega_1} = \begin{array}{l} P_{-2} \oplus M_2 \oplus M_4 \\ \oplus M_6 \oplus M_8 \end{array}$$

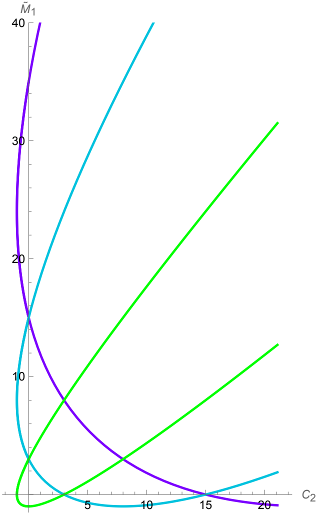


Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

Equivariant K-theory of Gr^μ

- G complex reductive, X complex projective variety, $\mathrm{G} \circlearrowright X$
- (Löwit 2025) defines *fixed point scheme*
  $\mathrm{Fix}_{\mathrm{G}}(X) : R \mapsto \{(g, x) \in \mathrm{G}(R) \times X(R) \mid gx = x\}$.
  develops $K_{\mathrm{G}}(X; \mathbb{C}) \cong \mathbb{C}[\mathrm{Fix}_{\mathrm{G}}]^{\mathrm{G}}$
  for a nice class of varieties including $\mathrm{G}^\vee \times \mathbb{C}^\times \circlearrowright \mathrm{Gr}^\mu$
- (Löwit 2026) proves (Bezrukavnikov 2023) conjecture:

  $$K_{T^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu; \mathbb{C}) = \langle b_{i,\ell} \rangle_{\ell > 0}^{1 \le i \le r}$$

  where $b_{i,\ell}$ are the *Bezrukavnikov classes*
- in the GKM description

  $$K_{T^\vee \times \mathbb{C}^\times}(\mathrm{Gr}^\mu; \mathbb{C}) \xrightarrow[\cong]{\mathrm{GKM}^\mu} \Big\{ (f_\lambda)_\lambda \in \bigoplus\nolimits_{\lambda \in wt(\mu)} \mathbb{C}[T^\vee \times \mathbb{C}^\times] : \\ (1 - z_{\beta^\vee} q^m) \mid (f_{\lambda_1} - f_{\lambda_2}) \text{ when } \lambda_1 - \lambda_2 = m\beta,\ m \in \mathbb{Z},\ \text{for } \beta \in \Phi \Big\}$$
- for $\mathrm{G} = \mathrm{GL}_n$ we have $K_{T^\vee \times \mathbb{C}}(\mathrm{Gr}^\mu; \mathbb{C}) = \langle b_{1,\ell} \rangle_{\ell \in \mathbb{Z}_{>0}}$

  $$\mathrm{GKM}^\mu(b_{1,\ell})_\lambda = \frac{\sum_a z_a^\ell - \sum_a z_a^\ell q^{\ell \lambda_a}}{1 - q^\ell} \in \mathbb{C}[z_a, q] \cong \mathbb{C}[T^\vee \times \mathbb{C}^\times]$$

Quantum groups

- restrict to $\mathrm{G} = \mathrm{GL}_n$, $\mathfrak{g} = \mathfrak{gl}_n$; $\mathbb{K} = \mathbb{C}(v)$ and $\mathbb{A} = \mathbb{C}[v, 1/v]$
- denote by $\mathrm{U}_{\mathbb{K}}(\mathfrak{gl}_n)$ the Drinfeld–Jimbo quantum group generated by

$$E_i,\ F_i \quad (1 \le i < n), \qquad K_a^{\pm 1} \quad (1 \le a \le n)$$

- write $H_i := K_i K_{i+1}^{-1}$. The basic relations are

$$K_a E_i K_a^{-1} = v^{\delta_{a,i} - \delta_{a,i+1}} E_i, \qquad K_a F_i K_a^{-1} = v^{-\delta_{a,i} + \delta_{a,i+1}} F_i,$$

$$[E_i, F_j] = \delta_{ij} \frac{H_i - H_i^{-1}}{v - v^{-1}}, \qquad K_a K_b = K_b K_a,$$

  together with quantum Serre relations.
- Hopf structure:

$$\Delta(E_i) = E_i \otimes H_i + 1 \otimes E_i, \qquad \Delta(F_i) = F_i \otimes 1 + H_i^{-1} \otimes F_i,$$

$$\Delta(K_a) = K_a \otimes K_a.$$

Integral forms of quantum groups

- define root vectors inductively by $E_{i,i+1} := E_i$ and $E_{ij} := E_{i,j-1}E_{j-1} - v^{-1}E_{j-1}E_{i,j-1}$; similarly for F_{ij}
- set $e_{ij} := (1-v^2)E_{ij}$ and $f_{ij} := (1-v^2)F_{ij}$ for all $i<j$
- (De Concini–Procesi) integral form:

$$\mathrm{V}_{\mathbb{A}}(\mathfrak{gl}_n) := \langle e_{ij}, f_{ij}, K_a^{\pm 1} \mid i<j\rangle_{\mathbb{A}}.$$

- its even part:

$$H_{ij} := K_i K_j^{-1}, \qquad \mathrm{V}_{\mathbb{A}}^{\mathrm{ev}} := \left\langle e_{ij},\ H_{ij}f_{ij}, K_a^{\pm 2} \mid i<j,\ 1\le a\le n\right\rangle_{\mathbb{A}}.$$

equivalently: fixed by $K_a \mapsto \epsilon_a K_a, \quad e_{ij}\mapsto e_{ij}, \quad f_{ij}\mapsto \epsilon_i\epsilon_j f_{ij}$.

- Lusztig's integral form:

$$\mathrm{U}_{\mathbb{A}}(\mathfrak{gl}_n) := \left\langle E_i^{(m)}, F_i^{(m)}, K_a^{\pm 1}, \begin{bmatrix} K_a;0 \\ m\end{bmatrix}\right\rangle_{\mathbb{A}},$$

$$E_i^{(m)} = \frac{E_i^m}{[m]_v!}, \qquad F_i^{(m)} = \frac{F_i^m}{[m]_v!},$$

$$\begin{bmatrix} K_a;0 \\ m\end{bmatrix} = \prod_{s=1}^{m} \frac{K_a v^{1-s} - K_a^{-1}v^{s-1}}{v^s - v^{-s}}.$$

Quantum medium algebras

- for a highest weight $\mu \in X^*_+(\mathrm{GL}_n)$ choose a Weyl lattice $V^\mu_{\mathbb{A}} \subset V^\mu$ and $\rho^\mu_{\mathbb{A}} : \mathrm{U}_{\mathbb{A}} \to \mathrm{End}_{\mathbb{A}}(V^\mu_{\mathbb{A}})$
- define the quantum Kostant algebra

$$\mathcal{R}^\mu_{\mathbb{A}} := \left(\mathrm{V}^{\mathrm{ev}}_{\mathbb{A}} \otimes \mathrm{End}_{\mathbb{A}}(V^\mu_{\mathbb{A}})\right)^{\Delta(\mathrm{U}_{\mathbb{A}})}$$

- the quantum medium algebra is

$$\mathcal{Z}^\mu_{\mathbb{A}} := \left\langle z \otimes 1,\ (1 \otimes \rho^\mu_{\mathbb{A}})\Delta(z) \;\middle|\; z \in Z(\mathrm{V}^{\mathrm{ev}}_{\mathbb{A}})\right\rangle \subset \mathcal{R}^\mu_{\mathbb{A}}.$$

  well-defined because $\Delta(\mathrm{V}^{\mathrm{ev}}_{\mathbb{A}}) \subset \mathrm{V}^{\mathrm{ev}}_{\mathbb{A}} \otimes \mathrm{U}_{\mathbb{A}}$ (Habiro–Lê 2016)
- we expect $\mathcal{Z}^\mu_{\mathbb{A}} \cong Z(\mathcal{R}^\mu_{\mathbb{A}})$

Generalised quantum Harish–Chandra map

- $S := \mathbb{C}[T^\vee \times \mathbb{C}^\times]$; $c : T \times \mathbb{C}^\times \to S^\times$ tautological character; $\lambda \in X^*$

$$\mathrm{U}_S := \mathrm{U}_\mathbb{A} \otimes_\mathbb{A} S, \quad V_S^\mu := V_\mathbb{A}^\mu \otimes_\mathbb{A} S, \quad M_S(\lambda) := \mathrm{U}_S \otimes_{\mathrm{U}_S^{\geq 0}} S_{c+\lambda}$$

$M_S(0)$ is universal quantum Verma module

- $\mathcal{R}_\mathbb{A}^\mu = (\mathrm{V}_\mathbb{A}^{\mathrm{ev}} \otimes \mathrm{End}_\mathbb{A}(V_\mathbb{A}^\mu))^{\Delta(\mathrm{U}_\mathbb{A})} \longrightarrow \mathrm{End}_{\mathrm{U}_S}(M_S(0) \otimes_S V_S^\mu)^{W^\bullet}$
- generalized Harish–Chandra map:

$$HC^\mu : \mathcal{Z}_\mathbb{A}^\mu \longrightarrow \bigoplus_{\lambda \in \mathrm{wt}(\mu)} \mathbb{C}[T^\vee \times \mathbb{C}^\times]$$

records scalar of z on the $M_S(\lambda)$-subquotient of $M_S(0) \otimes_S V_S^\mu$

Theorem (Hausel–Löwit 2026)

For $\mathrm{U}_\mathbb{A}(\mathfrak{gl}_n)$ and $\mu \in X_+^*(\mathrm{GL}_n)$,

$$\mathcal{Z}_\mathbb{A}^\mu \cong K_{\mathrm{GL}_n \times \mathbb{C}^\times}(\mathrm{Gr}^\mu; \mathbb{C})$$

via HC^μ with a ρ-shift matching GKM^μ, and $q = v^2$.

Proof by quantum Schur–Weyl duality

- $V^{\mathrm{st}}_{\mathbb{A}} := V^{(1,0,\ldots,0)}_{\mathbb{A}}$, with action $\rho_V : \mathrm{U}_{\mathbb{A}} \to \mathrm{End}_{\mathbb{A}}(V^{\mathrm{st}}_{\mathbb{A}})$
- construct the evaluated R-matrix

$$R^{\mathrm{U},V}_{12} = \Xi_V \Theta_V \in \mathrm{V}_{\mathbb{A}} \otimes \mathrm{End}(V^{\mathrm{st}}_{\mathbb{A}})$$

$$\Theta = \prod^{\rightarrow}_{1\le i<j\le n} \sum_{r\ge 0} v^{r(r-1)/2} (e_{ij})^r \otimes F^{(r)}_{ij}$$

$$\Theta_V := (1 \otimes \rho_V)(\Theta) = \prod^{\rightarrow}_{1\le i<j\le n} \left(1 + e_{ij} \otimes E^{\mathrm{mat}}_{ji}\right)$$

$$\Xi_V = \sum_{a=1}^{n} K_a \otimes E^{\mathrm{mat}}_{aa}$$

- monodromy seed in representation $V_{\mathbb{A}}$ of $\mathrm{U}_{\mathbb{A}}$ (Habiro–Lê 2016):

$$M_V := R^{\mathrm{U},V}_{21} R^{\mathrm{U},V}_{12} \in \left(\mathrm{V}^{\mathrm{ev}}_{\mathbb{A}} \otimes \mathrm{End}(V_{\mathbb{A}})\right)^{\Delta(\mathrm{U}_{\mathbb{A}})},$$

$$C_{s_\mu} := \mathrm{tr}(K_{2\rho} M_{V^\mu}) \in Z(\mathrm{V}^{\mathrm{ev}}_{\mathbb{A}}) \quad \text{(Drinfeld 1990)},$$

for any symmetric function $s = \sum_\mu a_\mu s_\mu$ we define $C_s := \sum_\mu a_\mu C_{s_\mu}$.

Affine Hecke operators

- (Orellana–Ram 2007) define $T_i, X_i \in \mathrm{V}_{\mathbb{A}} \otimes \mathrm{End}(V_{\mathbb{A}}^{\mathrm{st}\otimes k})$ by

$$T_i := \tau_{i,i+1} R^{V,V}_{i,i+1}, \quad X_1 := M_V \otimes I_V^{\otimes(k-1)}, \quad X_{i+1} := T_i X_i T_i.$$

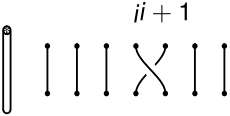
- these generate an action of the affine Hecke algebra $\mathcal{H}_k^{\mathrm{aff}}$, whose Bernstein center is $Z(\mathcal{H}_k^{\mathrm{aff}}) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_k^{\pm 1}]^{S_k}$.
- The main formula:

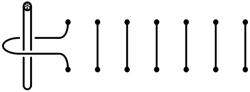
$$\frac{\Delta_{V^{\otimes k}}(C_{p_m}) - C_{p_m} \otimes I}{v^{2m} - 1} = \sum_{i=1}^{k} X_i^m.$$

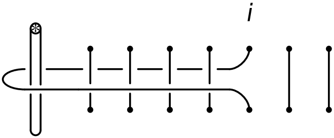
$$\frac{\Delta_{V^\mu}(C_{p_m}) - C_{p_m} \otimes I}{v^{2m} - 1} \in \mathrm{V}_{\mathbb{A}}^{\mathrm{ev}} \otimes \mathrm{End}_{\mathbb{A}}(V_{\mathbb{A}}^{\mu}).$$

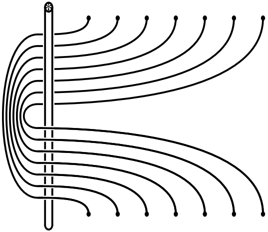
$$\mathrm{HC}^\mu\left(\frac{\Delta_{V^\mu}(C_{p_m}) - C_{p_m} \otimes 1}{v^{2m} - 1}\right) = \mathrm{GKM}^\mu(b_{1,m}),$$

completing the proof.

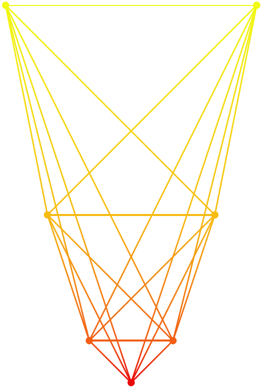
$T_i =$ 

$X_1 =$ 

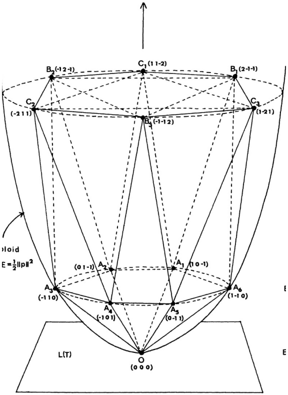
$X_i =$ 

$X_1 \cdots X_7 =$ 

Moment polytope and GKM graph



$\mathrm{Gr}_{\mathrm{SL}_2}$ (Harada–Henriquez–Holm 2006)



$\Omega SU(3)$ (Atiyah–Pressley 1983)