Minimal Output Entropy Conjecture

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Outline

Quantum channels are fundamental tools for modeling the evolution of quantum states, especially in the presence of noise or decoherence. A central representation of such channels is the Kraus decomposition, where the set of Kraus operators may often be structured by underlying symmetries. In particular, when the Kraus operators are taken to be the generators of a Lie algebra, the resulting channels—called Lie algebra channels—exhibit deep connections to representation theory. For certain representations of SU(2) and SU(N), Lieb and Solovej proved that coherent states minimize the output entropy. In this project, we investigate whether this minimal entropy property still holds for channels associated with more general Lie algebras and their representations.

Quantum Channels

Quantum Channel

A quantum channel is a mathematical model describing the evolution of quantum states under noise or interaction with an environment. Formally, a quantum channel is a completely positive and trace-preserving (CPTP) linear map between spaces of operators. Let $\mathcal{H}_d$ be a finite-dimensional Hilbert space with dimension d, and let $\mathrm{End}(\mathcal{H}_d)$ denote the space of linear maps on $\mathcal{H}_d$.

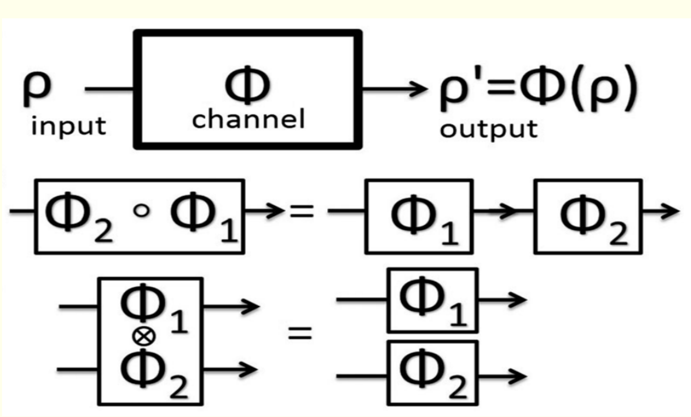

Figure 1: Quantum Channel

Definition 1. A quantum channel is a linear map

$$\mathcal{E}\colon \mathrm{End}(\mathcal{H}_{d_1}) \to \mathrm{End}(\mathcal{H}_{d_2})$$

such that:
(1) Complete positivity (CP): For every other Hilbert space $\mathcal{H}_j$, the map

$$\mathcal{E} \otimes \mathrm{id}_j\colon \mathrm{End}(\mathcal{H}_{d_1} \otimes \mathcal{H}_j) \to \mathrm{End}(\mathcal{H}_{d_2} \otimes \mathcal{H}_j)$$

is positive, i.e., it maps positive semidefinite operators to positive semidefinite operators.
(2) Trace-preserving (TP): For all density matrices $\rho \in \mathrm{End}(\mathcal{H}_d)$, $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho)$.

Von Neumann Entropy

To quantify the amount of information (or uncertainty) in a quantum state, we use the von Neumann entropy.

Definition 2. For a density matrix ρ, the von Neumann entropy is defined as $S(\rho) := -\mathrm{Tr}(\rho \log \rho)$, where ρ is a density matrix on $\mathcal{H}_d$.

This is the quantum analog of the classical Shannon entropy and is zero if and only if ρ is a pure state. However, von Neumann entropy is not the only way to define an entropy of a quantum state. In settings where one studies phase-space representations and semiclassical limits, another notion—Wehrl entropy—becomes relevant and will be introduced later.

Lie Algebra Channels and Representation Theory

Kraus Representation

Theorem 3 (Kraus's Theorem)**.** Any CPTP map $\mathcal{E}$ admits a representation of the form:

$$\mathcal{E}(\rho) = \sum_{i=1}^{k} M_i \rho M_i^\dagger,$$

where $M_i \in \mathrm{End}(\mathcal{H}_d)$ are called Kraus operators, and they satisfy the normalization condition: $\sum_{i=1}^k M_i^\dagger M_i = I$. This decomposition is not unique, but it always exists. The number k can be taken to satisfy $k \le d^2$.

Quantum Channels and Representation Theory (W. G. Ritter)

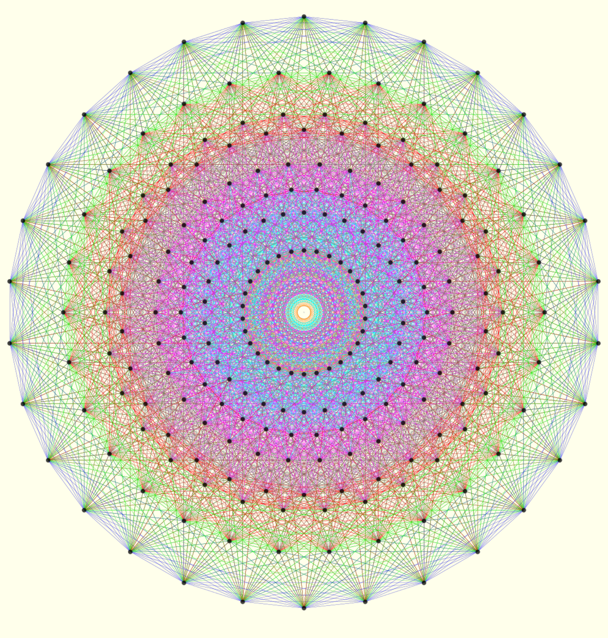

Figure 2: Lie Algebra

The Kraus operators in a quantum channel need not be arbitrary—when chosen to reflect the symmetries of a physical system, they naturally give rise to Lie algebra channels. This leads to a deep connection between quantum information theory and representation theory of Lie algebras.

Let $\mathfrak{g}$ be a real or complex Lie algebra, and let $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation on a Hilbert space V. For a basis X_i of $\mathfrak{g}$, consider the operators $\pi(X_i)$ as a set of Kraus operators. This leads to a Lie algebra channel of the form: $\mathcal{E}(\rho) = \sum_{i=1}^k \pi(X_i)\rho\pi(X_i)^\dagger$.

Ritter shows that the resulting channels often commute with the action of the Lie group G whose Lie algebra is $\mathfrak{g}$. This symmetry implies that:

- The channel $\mathcal{E}$ preserves G-invariant subspaces of V.
- Irreducible representations of G decompose the Hilbert space into sectors that evolve independently under $\mathcal{E}$.

Hence, representation theory becomes a tool to analyze the spectral and entropy properties of such channels.

Minimal Output Entropy Conjecture

Wehrl Entropy

Let $\mathcal{H}$ carry an irreducible representation of a compact Lie group G, and fix a highest weight vector $|\psi_0\rangle \in \mathcal{H}$.

Definition 4. The coherent states associated with this representation are the orbit of $|\psi_0\rangle$ under the group action: $|\Omega\rangle := g \cdot |\psi_0\rangle, \quad \text{for } g \in G.$

These states are labeled by points $\Omega \in G/H$, where $H \subset G$ is the stabilizer subgroup of $|\psi_0\rangle$. For example, when $G = \mathrm{SU}(2)$ and $\mathcal{H}$ is the spin-j representation, the space of coherent states is the Bloch sphere $S^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$.

Definition 5. Given a density matrix ρ on $\mathcal{H}$, one defines the Husimi function as: $Q_\rho(\Omega) := \langle \Omega|\rho|\Omega\rangle$, a smooth, non-negative function on G/H that integrates to one: $\int_{G/H} Q_\rho(\Omega)\, d\mu(\Omega) = 1$, where $d\mu$ is the normalized invariant measure on G/H.

Definition 6. The Wehrl entropy of ρ is then defined as the classical entropy of this distribution: $S_W(\rho) := -\int_{G/H} Q_\rho(\Omega) \log Q_\rho(\Omega)\, d\mu(\Omega)$, where $d\mu$ is the normalized invariant measure on G/H.

Minimal Output Entropy Conjecture

Conjecture 7. The minimal output entropy conjecture posits that for certain quantum channels, particularly those arising from group symmetries, the minimal output Wehrl-type entropy is achieved when the input is a coherent state.

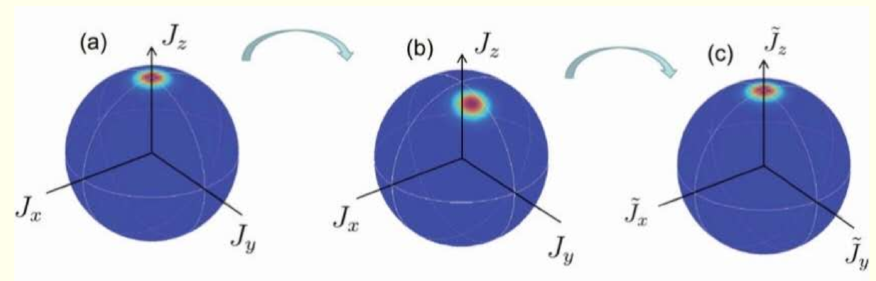

Figure 3: SU(2) Coherent State

E. H. Lieb and J. P. Solovej investigated the behavior of quantum channels arising from representations of SU(2) and symmetric representations of SU(N), and proved that coherent states minimize the output entropy for a large class of such channels. They used a general strategy for establishing the minimal output entropy conjecture: For both SU(2) representations and symmetric SU(N) representations, coherent input states produce output states whose eigenvalue sequences, arranged in decreasing order, majorize those produced by any other input state. This fundamental majorization result implies, via Karamata's theorem, that any concave function of the output is minimized by coherent states. In the semiclassical limit, this leads to a minimization of the Wehrl entropy. This confirms the conjecture that coherent states uniquely minimize the Wehrl entropy in both settings.

These results provide rigorous evidence for the minimal output entropy conjecture in the case of symmetric Lie group representations and motivate the exploration of this property in broader settings.

Relation to Big Algebras

Let $\varphi\colon \mathrm{End}(V^\mu) \to \mathrm{End}(V^\lambda)$ be a quantum channel that is G-invariant under the action of a compact Lie group G. Then the space of such channels is $\varphi \in \left(\mathrm{End}((V^\mu)^*) \otimes \mathrm{End}(V^\lambda)\right)^G \cong \left(\mathrm{End}(V^\mu) \otimes \mathrm{End}(V^\lambda)\right)^G$.

To study this space, one considers the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ and its two classical algebras:

- The symmetric algebra $S(\mathfrak{g})$,
- The universal enveloping algebra $U(\mathfrak{g})$.

For a fixed highest weight representation V^λ of $\mathfrak{g}$, Hausel then considers:

- The Kirillov algebra (classical version): $C^\lambda(\mathfrak{g}) := \left(S^*(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)\right)^G$,
- The Kostant algebra (quantum version): $R^\lambda(\mathfrak{g}) := \left(U(\mathfrak{g}) \otimes \mathrm{End}(V^\lambda)\right)^G$.

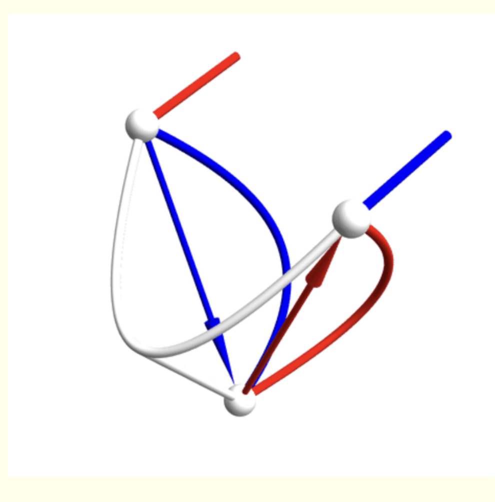

Figure 4: Triplet Crystal

The big algebra $\mathcal{B}^\lambda(\mathfrak{g})$ is a maximal commutative subalgebra of $C^\lambda(\mathfrak{g})$, while its quantum analog $\mathcal{G}^\lambda(\mathfrak{g})$ is also a maximal commutative subalgebra of $R^\lambda(\mathfrak{g})$. Thus we have $\mathcal{B}^\lambda(\mathfrak{g}) \subset C^\lambda(\mathfrak{g}) \to \left(\mathrm{End}(V^\mu) \otimes \mathrm{End}(V^\lambda)\right)^G$ and $\mathcal{G}^\lambda(\mathfrak{g}) \subset R^\lambda(\mathfrak{g}) \to \left(\mathrm{End}(V^\mu) \otimes \mathrm{End}(V^\lambda)\right)^G$, this suggests a structural bridge between quantum channels and big algebras.

Although this connection is still speculative, it raises the possibility that studying the symmetries and entropy properties of G-invariant quantum channels may yield new insight into the structure of big algebras, both classical and quantum, which still await further exploration.