Ringifying intersection cohomology

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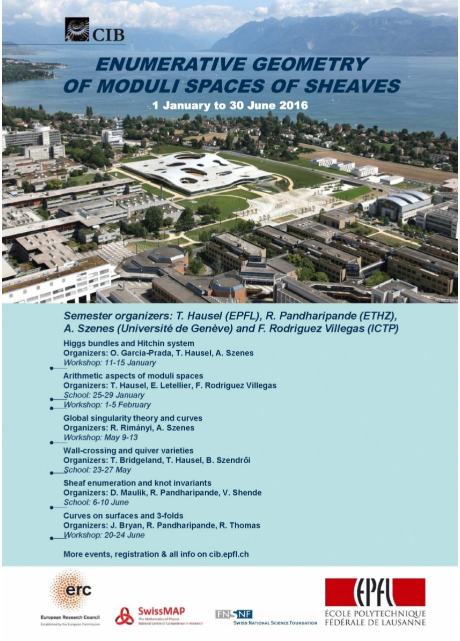
Representations, Moduli and Duality
Bernoulli center, EPFL
August 2025

FWF Austrian Science Fund

erc European Research Council
Established by the European Commission

ISTA Institute of Science and Technology Austria

Program at Bernoulli Center, EPFL, Spring 2016



- Retreat on Higgs bundles, real groups, Langlands duality and mirror symmetry - organized by Oscar García-Prada

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- Ringify! Find graded $H^*(X)$-algebra structure on $IH^*(X)$

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- *equivariantly formal*

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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

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- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^{2*}_{\mathrm{SL}_n}$ invariant polynomial

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Theorem (Hausel–Zveryk, Hausel 2022)

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Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n) \cong \mathcal{B}^{n\omega_1}(\mathfrak{sl}_2)$

$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

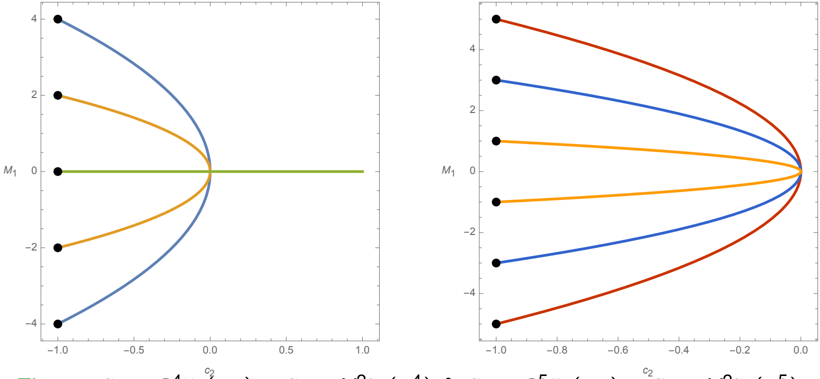

Figure: $\mathrm{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\mathrm{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^5)$.

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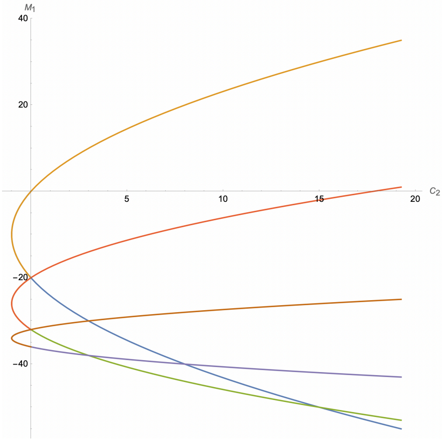

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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Examples of infinitesimal characters in tensor product

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$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

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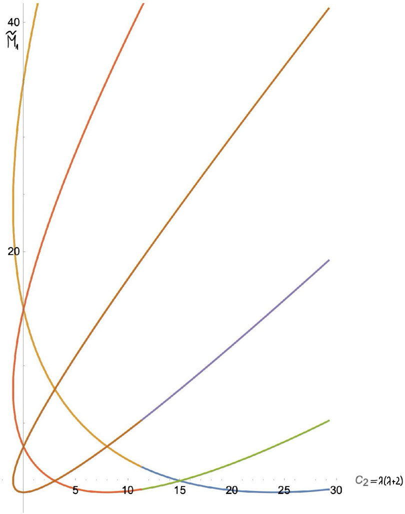

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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