Commutative avatars for baryon multiplets

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Big algebra

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Commutative avatars of representations of semisimple Lie groups

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Here we announce the construction and properties of a big commutative subalgebra of the Kirillov algebra attached to a finite dimensional irreducible representation of a complex semisimple Lie group. They are commutative finite flat algebras over the cohomology of the classifying space of the group. They are isomorphic with the equivariant intersection cohomology of affine Schubert varieties, endowing the latter with a new ring structure. Study of the finer aspects of the structure of the big algebras will also furnish the stalks of the intersection cohomology with ring structure, thus ringifying Lusztig's *q*-weight multiplicity polynomials i.e., certain affine Kazhdan–Lusztig polynomials.

representations of Lie groups | Hitchin integrable system | Higgs field | equivariant cohomology | intersection cohomology

Significance

Representations of continuous symmetry groups by matrices are fundamental to mathematical models of quantum physics and also to the Langlands program in number theory. Here, we attach a commutative matrix algebra, called big algebra, to a noncommutative irreducible matrix representation of a bounded continuous symmetry group. We show that the

1. Kirillov and Medium Algebras

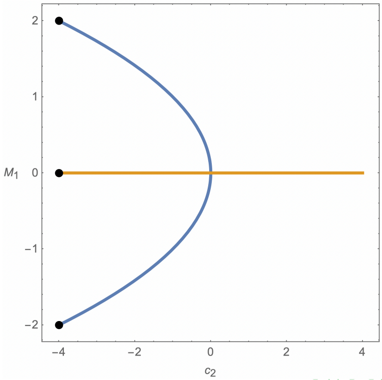
Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, which we

Big algebra

- a multiplicative $\rho : \mathrm{G} \to \mathbb{M}_{n\times n}(\mathbb{C})$ is a *matrix representation* of a compact symmetry group G
  Example: *rotation group of sphere* $\mathrm{SO}(3) \subset M_{3\times 3}(\mathbb{C})$
  or *special unitary group* $\mathrm{SU}(3) \subset M_{3\times 3}(\mathbb{C})$
- (Hausel 2024) $\rightsquigarrow$ *big algebra* of ρ:
  $\mathcal{B}^\rho \subset Maps(\mathfrak{g}, M_{n\times n}(\mathbb{C}))^{\mathrm{G}}$
  an algebra of G-equivariant maps
  from $\mathfrak{g} := Lie(\mathrm{G})$ to matrix algebra $M_{n\times n}(\mathbb{C})$ of representation
  Example: $\begin{array}{cccc} M_1 : & \mathfrak{g} & \to & M_{n\times n}(\mathbb{C}) \\ & A & \mapsto & Lie(\rho)(A) \end{array}$ then $M_1 \in \mathcal{B}^\rho$
- (Hausel 2024) $\rightsquigarrow$ wonderful properties of $\mathcal{B}^\rho$:
  1. it is a maximal commutative subalgebra
  2. free of rank n over polynomial algebra $\mathbb{C}[\mathfrak{g}]^{\mathrm{G}} = \mathbb{C}[\mathfrak{h}]^W$
  3. $\mathcal{B}^\rho \cong IH^*_{\mathrm{G}^\vee}(\mathrm{Gr}_\rho)$ equivariant intersection cohomology of affine Schubert variety for Langlands dual $\mathrm{G}^\vee$
  4. $\mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(\mathbb{C}[\mathfrak{h}]^{\mathrm{W}}) \cong \mathfrak{h}//W$ models Hitchin system on certain Lagrangians
  5. constructed using (Feigin–Frenkel 1992) center of certain vertex algebra

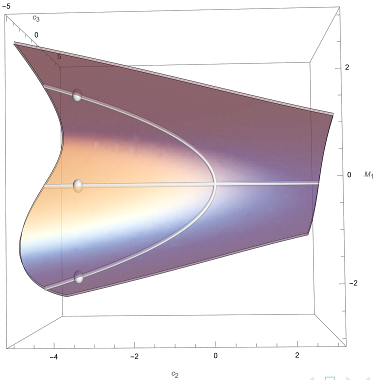
Spectrum of standard representation of $\mathrm{SO}(3)$

- $\mathrm{SO}(3)$ symmetry group of sphere
- $\rho : \mathrm{SO}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra
  $\mathcal{B}^{\rho} = \mathbb{C}[c_2, M_1]/(M_1^3 + c_2 M_1) = \mathbb{C}[c_2, M_1]/(M_1(M_1^2 + c_2))$
  by Cayley-Hamilton
- its spectrum with *principal spectrum* at $c_2 = -4$:



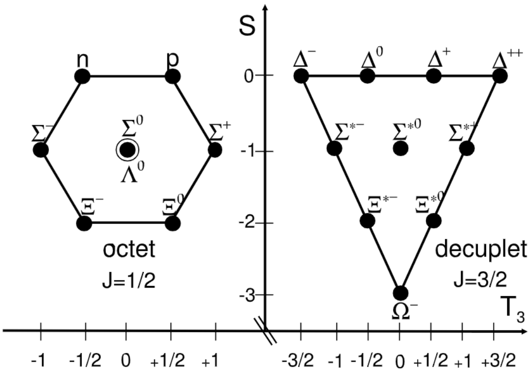
Spectrum of standard representation of $\mathrm{SU}(3)$

- $\mathrm{SU}(3)$ special unitary group
- $\rho : \mathrm{SU}(3) \to M_{3\times 3}(\mathbb{C})$ standard representation
- its big algebra is $\mathbb{C}[c_2, c_3, M_1]/(M_1^3 + c_2 M_1 + c_3)$ by Cayley-Hamilton
- its spectrum with *skeleton* at $c_3 = 0$ and *principal spectrum* at $c_2 = -4$:



Gell-Mann's spectrum

- experiments in the 1950's produced a *particle zoo* of hundreds of different hadrons
- Gell-Mann 1960s organized them into octets and decuplets



- he argued that they are composite particles made up of quarks and antiquarks, which correspond to the fundamental representations of $\mathrm{SU}(3) \rightsquigarrow$ Quantum-Chromo-Dynamics

Gell-Mann's spectrum

From: *Proc Intern Conf High Energy Phys* (CERN, 1962), p. 805

Strange Particle Physics. Strong Interactions

M. Gell-Mann

SNOW: I know that at 1.5 GeV in the CERN K^-p experiment in the fourbody reaction $\Sigma^{\pm}\pi^{\mp}\pi^-\pi^+$ one looks for $T = 2$ resonances, but when one looks at these reactions the dominant things that one sees are $T = 0$ resonances. I think the Berkeley data which has more statistical weight shows the same thing: they do not see any evidence for this.

SAKURAI: It is now plausible that the 1405 MeV Y_0^* but not the 1385 MeV Y_1^* is an *s*-wave $\bar{K}N$ bound state of the Dalitz-Tuan type. So it is worth asking what kind of dynamical mechanism is responsible for binding a $\bar{K}$ and an N in the $T = 0$ state. Along this line a student of mine, Richard Arnold at Chicago, has performed a very crude N/D calculation to show that for reasonable values of coupling constants the forces due to the exchanges of ρ and ω are sufficiently attractive to bind the $T = 0$, $\bar{K}N$ system. On this calculation the signs of the ρ and ω exchange force are fixed by the universality principle of the vector theory of strong interactions. Another interesting point is that the same mechanism predicts that the $T = 1$, KN system is strongly repulsive, in agreement with observation. I want to make another remark which is somewhat more general. It is interesting to conjecture that the isospin dependent force for any low energy scattering is dominated by the exchange of the ρ meson coupled universally to the isospin current. The ρ exchange force is then attractive whenever the isospins are antiparallel and repulsive whenever the isospin are parallel. This rule works remarkably well in five cases examined so far. In the case of *s*-wave πN scattering essentially the entire isospin dependence is due to the ρ exchange as conjectured by Cini and Fubini and by myself independently and as proved by Hamilton *et al.* In the *s*-wave $\pi\pi$ scattering case the $T = 0$ seems more attractive than the $T = 2$ state. In low energy $K\bar{K}$ scattering there is some evidence for a strong attractive interaction in the $T = 0$ state as we have just heard. In the $\bar{K}N$ case the currently accepted idea, that the 1405 MeV Y_0^* but not the 1385 MeV Y_1^* is likely to be a $\bar{K}N$ bound state resonance, shows that the $T = 0$ state is more attractive. In the $S = 1$, KN case, the $T = 1$ state is definitely more repulsive than the $T = 0$ state. In general bound states and resonances are more likely for states with lower isospins as you can see from the table of elementary particles and resonances. So there seems to be a correlation between the simplicity of quantum numbers and the possibility of bound or resonant states.

ROSENFELD: (in reply to Good): Good asked how carefully we have looked for doubly charged ($T = 2$) resonances. 1.51 GeV/c $K^- + p \to \Sigma^{\pm}\pi^{\mp}\pi^+\pi^-$ ($E^*_{CM} = 2025$ MeV, 400 events) allowed Alston *et al.*, to explore doubly charged $\Sigma\pi$ combination fairly well up to $\simeq 1600$ MeV. Actually, we saw one bump of perhaps 2 standard deviations at 1560 MeV, but nothing that looked interesting. But I repeat for negative strangeness Berkeley has not looked above 1600 MeV.

TICHO: In reply to Good, we are sensitive to $\Xi\pi$ $T = 3/2$ resonance up to Q values of 170 MeV. Within our statistics we see no evidence for resonances other than the reported Ξ^* of $T = 1/2$.

SANDWEISS: In a paper submitted to this conference, results of a π^+p study at 2 GeV are reported. In about 70 events of $\Sigma^+\pi^+K^0$, $\Sigma^0\pi^+K^+$ and $\Sigma^+\pi^0K^+$ no evidence for a $T = 2$ $\Sigma\pi$ resonance was found.

GELL-MANN: If we take the unitary symmetry model with baryon and meson octets, with first order violation giving rise to mass differences, we obtain some rules for supermultiplets. The broken symmetry picture is hard to interpret on any fundamental theoretical basis, but I hope that such a justification may be forthcoming on the basis of analytic continuation of resonant states in isotopic spin and strangeness. Instead of constructing just the inverse Regge function $E(J)$, we can consider surfaces $E(J, I, Y, \text{etc.})$. Certainly the dynamical equations are as smooth in I and Y as they are in J.

Anyway, we may look at the success of the mass rules:

$$\frac{m_N + m_\Xi}{2} = \frac{3m_\Lambda}{4} + \frac{m_\Sigma}{4}$$

and

$$m_K^2 = \frac{3m_\eta^2}{4} + \frac{m_\pi^2}{4}$$

work fine, while

$$m_M^2 = \frac{3m_\omega^2}{4} + \frac{m_\rho^2}{4}$$

does not work quite so well if $M = K^*$.

Suppose, now we try to incorporate the 3/2-3/2 nucleon resonance into the scheme. The only supermultiplet that does not lead to non-existent resonances in the $K - N$ channels is the *10* representation, which gives 4 states:

$$I = 3/2, \quad S = 0$$
$$I = 1, \quad S = -1$$
$$I = 1/2, \quad S = -2$$
$$I = 0, \quad S = -3$$

The mass rule is stronger here and yields *equal spacing* of these states. Starting with the resonance at 1238 MeV, we may conjecture that the Y_1^*, at 1385 MeV and the Ξ^* at 1535 MeV might belong to this supermultiplet. Certainly they fulfil the requirement of equal spacing. If $J = 3/2^+$ is really right for these two cases, then our speculation might have some value and we should look for the last particle, called, say, Ω^- with $S = -3$, $I = 0$. At 1685 MeV, it would be metastable and should decay by the weak interactions into $K^- + \Lambda$, $\pi^- + \Xi^0$, or $\pi^0 + \Xi^-$. Perhaps it would explain the old Eisenberg event. A beam of K^- with momentum $\gtrsim 3.5$ GeV/c could yield Ω^- by means of $K^- + p \to K^+ + K^0 + \Omega^-$.

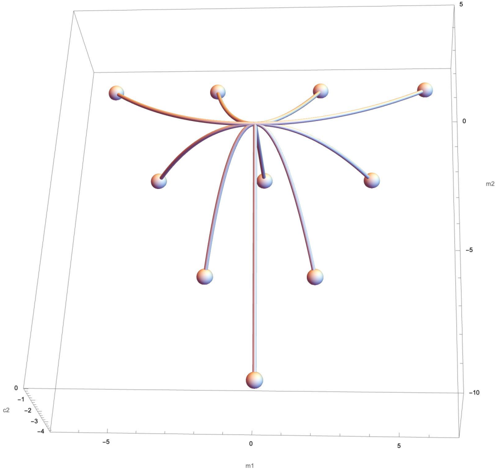
ADAIR: I would like to clarify the statements attributed to me concerning the $\Sigma - K$ parity. The conclusions of Tripp, Ferro-Luzzi, and Watson, and, implicitly, those of Capps, that the $\Sigma - K$ parity is odd, actually refer to a particular model of the K-nucleon interaction, a model which at best can be but a first approximation to the real world, and which does not exhibit certain important features of their data. In particular, this model assumes that the energy dependence of background amplitudes and resonance widths may be neglected, that charge dependent effects may be neglected, and most important, that there are no background amplitudes in the states with the J and parity of the resonance. Elementary considerations show that for the high angular momentum states of interest such approximations are quite inadequate. For example, the magnitude of background amplitudes and widths must vary with energy perhaps by a factor of two, over the resonance region. Their model results in equal resonant cross-sections for the

(Gell-Mann 1962) "If the information I've heard is really right, then our speculation might have some value, and we should look for the last particles, called, say, Omega-minus"

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:

$$\mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^4 - 6M_1^2M_2 + 4M_1^2c_2 - 18M_1c_3 + 3M_2^2 - 6M_2c_2, \\ M_1^3M_2 + M_1^3c_2 + 3M_1^2c_3 - 3M_1M_2^2 + M_1M_2c_2 + 4M_1c_2^2 - 9M_2c_3 \end{pmatrix}$$

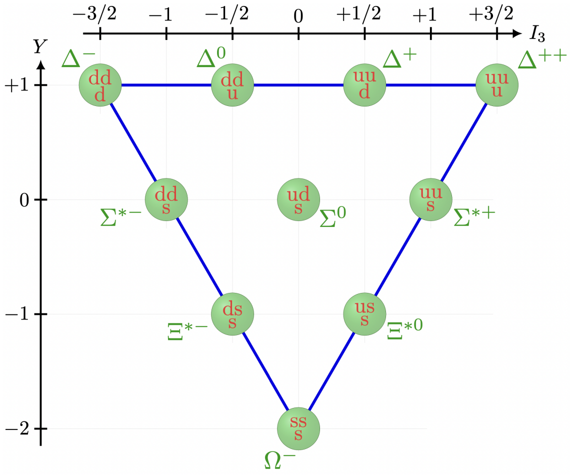


decuplet skeleton over its principal spectrum

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:

$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$$

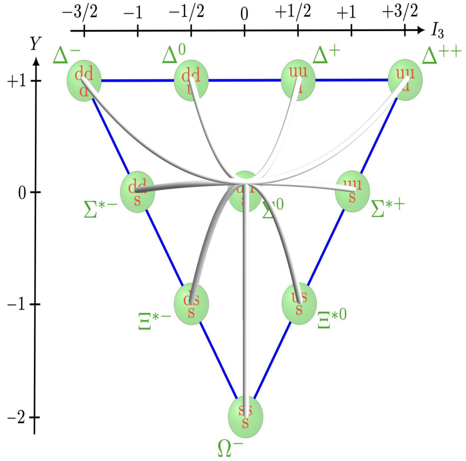


particles in baryon decuplet

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $SU(3)$:

$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18 M_1 c_3 + 3M_2^2 - 6 M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3 M_1^2 c_3 - 3 M_1 M_2^2 + M_1 M_2 c_2 + 4 M_1 c_2^2 - 9 M_2 c_3 \end{pmatrix}$$

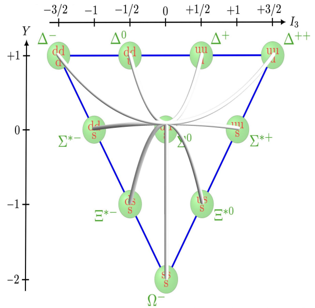


skeleton over baryon decuplet

Decuplet skeleton and baryon decuplet

- big algebra of third symmetric power of $\mathrm{SU}(3)$:

$$\mathbb{C}[c_2, c_3, M_1, M_2] / \begin{pmatrix} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{pmatrix}$$

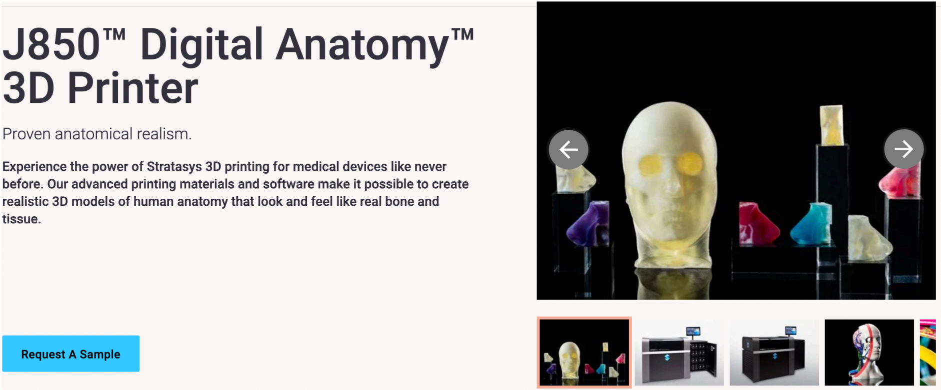


skeleton over baryon decuplet

- $\rightsquigarrow$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

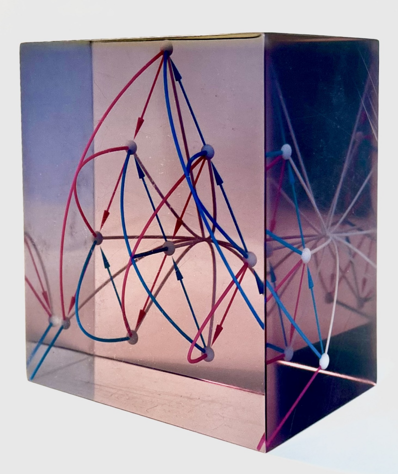
$$I_3(Y-1)(4I_3^2 - 3Y - 4) = 0$$
$$16I_3^4 - 24I_3^2 Y - 16I_3^2 + 3Y^2 + 6Y = 0$$

Digital anatomy 3D printer



3D printer for anatomy of big algebras

3D printing decuplet crystal



Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2,\ M_2 := Dc_3/2,\ N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3N_1 + c_2M_1N_1 - 9c_3M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2M_2 + c_2M_2 + 3c_3M_1, \\ M_1^4 + 4c_2M_1^2 + 3M_2^2, \\ 3M_1M_2^2 + 9c_3M_2 - c_2M_1^3 - 4c_2^2M_1 \end{pmatrix}$$

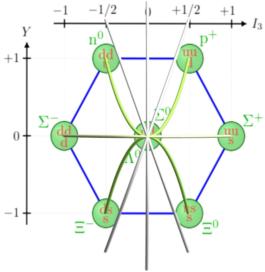


Figure: big and medium skeleton over baryon octet

- $\rightsquigarrow$ polynomial relations between I_3 and Y in baryon octet

$$Y(2I_3 - 1)(2I_3 + 1) = 0$$
$$4I_3^3 + 3I_3Y^2 - 4I_3 = 0$$
$$16I_3^4 - 16I_3^2 + 3Y^2 = 0$$