Anatomy of big algebras

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Big commmutative subalgebra of Kirillov algebra

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- for μ not weight multiplicity free we construct big (maximal) commutative subalgebra $Z(\mathcal{C}^\mu) \subset \mathcal{B}^\mu \subset \mathcal{C}^\mu$

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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative

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- $\{X_i\}$ basis for SL_n, $\{X^i\}$ dual basis wrt Killing form
- $$\begin{array}{rcl} D : \mathcal{C}^\mu & \to & \mathcal{C}^\mu = (S(\mathfrak{sl}_n^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{SL}_n} \\ A & \mapsto & \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i} \end{array}$$ Kirillov-Wei operator
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^{2*}_{\mathrm{SL}_n}$ invariant polynomial
  $t^n + c_2(a)t^{n-2} + \cdots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M-operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \ldots, M_{n-1} \rangle_{H^{2*}_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *medium algebra* $\cong Z(\mathcal{C}^\mu)$
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Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ *is commutative, cyclic*

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$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ *is commutative, cyclic, finite-free* $/H^{2*}_{\mathrm{SL}_n}$ *and Gorenstein*

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- cyclicity follows from (Feigin–Frenkel–Rybnikov 2006) in quantization of Mishchenko-Fomenko integrable systems $\rightsquigarrow \mathcal{B}^\mu$ is a quantum integrable system

Geometric properties of $\mathscr{B}^\mu$

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Corollary (Hausel 2022)

$H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu$ *as* $H^{2*}_{\mathrm{PGL}_n}$*-algebras*

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  e.g. $\mathrm{G} = \mathrm{SL}_2$, $X_\mu = \mathbb{P}(V^\mu) \rightsquigarrow H^{2*}_{\mathrm{SL}_2}(\mathbb{P}(V^\mu)) \cong \mathcal{C}^\mu \cong H^{2*}_{\mathrm{PGL}_2}(\mathrm{Gr}^\mu)$

Geometric properties of $\mathcal{B}^\mu$

- $\mathrm{Gr} := \mathrm{PGL}_n((z))/\mathrm{PGL}_n[[z]]$ affine Grassmannian of $\mathrm{G}^\vee = \mathrm{PGL}_n$
- $\mathrm{Gr}^\mu := \overline{\mathrm{PGL}_n[[z]]z^\mu} \subset \mathrm{Gr}$ affine Schubert; $\mathrm{Gr}^{\omega_k} = \mathrm{Gr}(k,n)$
- (Bezrukavnikov–Finkelberg 2008) describe the graded $H^{2*}_{\mathrm{PGL}_n}$-algebra $H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ & $IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ as module over it $\rightsquigarrow$

Corollary (Hausel 2022)

$H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu$ as $H^{2*}_{\mathrm{PGL}_n}$-algebras
$\mathrm{End}_{H^{2*}_{\mathrm{PGL}}(\mathrm{Gr}^\mu)}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \cong \mathcal{C}^\mu$
$IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{B}^\mu$ as $\mathcal{M}^\mu$-modules
$\rightsquigarrow$ graded $H^{2*}_{\mathrm{PGL}_n}$-algebra structure on $IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$

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- for non-weight multiplicity free μ topological/geometrical description of ring structure on $IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ is open

Visualizing big algebras

Visualizing big algebras

- $\mathrm{Spec}(\mathscr{B}^\mu)$ affine scheme

Visualizing big algebras

- $\mathrm{Spec}(\mathcal{B}^\mu)$ affine scheme finite flat / $\mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{g}//\mathrm{G} \cong \mathfrak{t}//W$

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- *big skeleton*: $\mathrm{Spec}(\mathcal{B}^\mu_{\mathrm{SL}_2})$

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- $3 \rightsquigarrow$ (Ginzburg, 2008) $H^{2*}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu_{e_0}$ as assoc. graded of F

Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

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- (Hausel–Rychlewicz 2022) computes $H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^n)$

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Picture of $\mathrm{Spec}(\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2))$

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$$\mathcal{B}^{n\omega_1}(\mathfrak{sl}_2) \cong \begin{cases} \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 4c_2)M_1\big) & \text{for } n \text{ even;} \\ \mathbb{C}[c_2, M_1]/\big((M_1^2 + n^2 c_2)\dots(M_1^2 + 9c_2)(M_1^2 + c_2)\big) & \text{for } n \text{ odd.} \end{cases}$$

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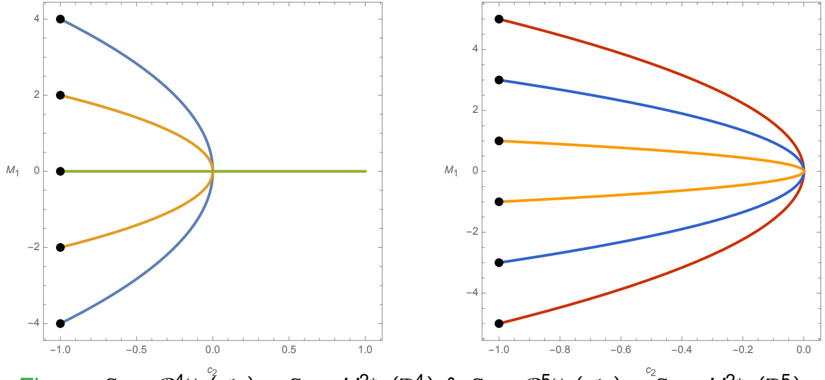

Figure: $\mathrm{Spec}\mathcal{B}^{4\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^4)$ & $\mathrm{Spec}\mathcal{B}^{5\omega_1}(\mathfrak{sl}_2) \cong \mathrm{Spec}H^{2*}_{\mathrm{SL}_2}(\mathbb{P}^5)$.

Picture of $\mathrm{Spec}(\mathscr{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

- (Panyushev 2004) $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(\mathbb{P}^2)$

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- (Panyushev 2004) $\rightsquigarrow \mathcal{B}^{\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(\mathbb{P}^2)$ or Cayley-Hamilton

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- $$\begin{array}{ccc} H^{2*}_{\mathrm{SL}_3} \cong \mathbb{C}[c_2, c_3] & \to & H^{2*}_{\mathrm{SL}_2} \cong \mathbb{C}[c_2] \\ (c_2, c_3) & \mapsto & (4c_2, 0) \end{array}$$

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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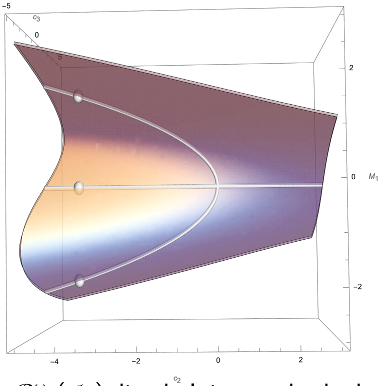

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

Picture of $\mathrm{Spec}(\mathcal{B}^{\omega_1}(\mathfrak{sl}_3))$, its skeleton and quarks

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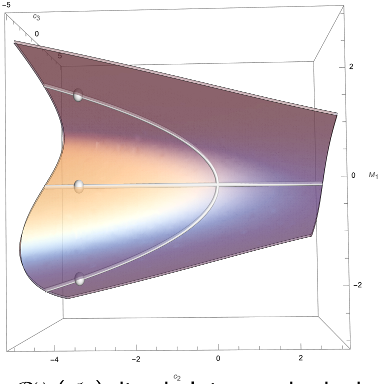

Figure: $\mathrm{Spec}\mathcal{B}^{\omega_1}(\mathfrak{sl}_3)$, its skeleton and principal spectrum

- principal spectrum can be identified with up,strange,down quarks in Gell-Mann's eightfold way

Skeleton of $\mathrm{Spec}(\mathscr{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

- (Hausel–Rychlewic 2022) $\rightsquigarrow$
  $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3) \cong H^{2*}_{\mathrm{SL}_3}(X_{3\omega_1}) \cong H^{2*}_{\mathrm{SL}_3}((\mathbb{P}^2)^3/S_3)$

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  $$\mathbb{C}[c_2, c_3, M_1, M_2]/\left(\begin{array}{l} M_1^4 - 6M_1^2 M_2 + 4M_1^2 c_2 - 18M_1 c_3 + 3M_2^2 - 6M_2 c_2, \\ M_1^3 M_2 + M_1^3 c_2 + 3M_1^2 c_3 - 3M_1 M_2^2 + M_1 M_2 c_2 + 4M_1 c_2^2 - 9M_2 c_3 \end{array} \right)$$

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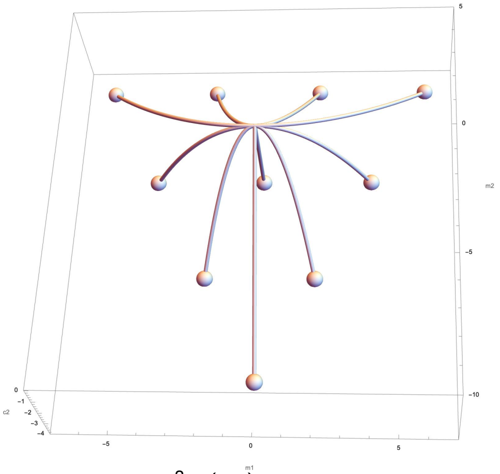

Figure: skeleton of $\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3)$ over its principal spectrum

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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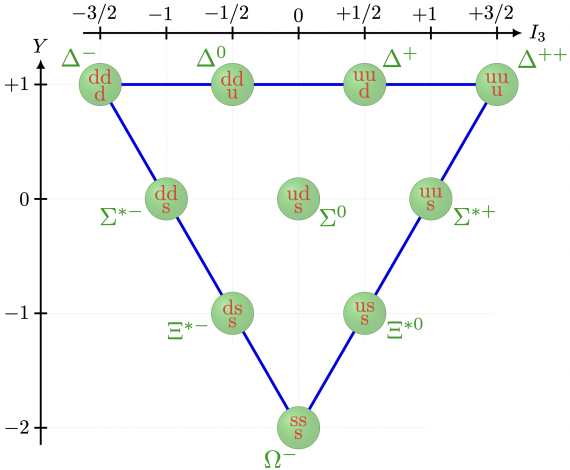

Figure: particles in baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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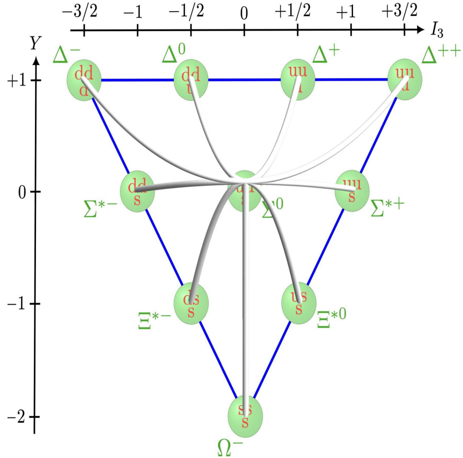

Figure: skeleton over baryon decuplet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{3\omega_1}(\mathfrak{sl}_3))$ and baryon decuplet

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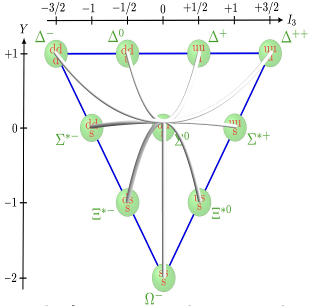

Figure: skeleton over baryon decuplet

- $\rightsquigarrow$ relations for $I_3 = \frac{(M_1)_{h_0}}{4}$ and $Y = \frac{(M_2)_{h_0}}{4}$ in baryon decuplet

$$I_3(Y-1)(4I_3^2 - 3Y - 4) = 0$$
$$16I_3^4 - 24I_3^2Y - 16I_3^2 + 3Y^2 + 6Y = 0$$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2 c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3 N_1 + c_2 M_1 N_1 - 9c_3 M_1 \end{pmatrix}$$

$$\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, M_2]/\begin{pmatrix} M_1^2 M_2 + c_2 M_2 + 3c_3 M_1, \\ M_1^4 + 4c_2 M_1^2 + 3M_2^2, \\ 3M_1 M_2^2 + 9c_3 M_2 - c_2 M_1^3 - 4c_2^2 M_1 \end{pmatrix}$$

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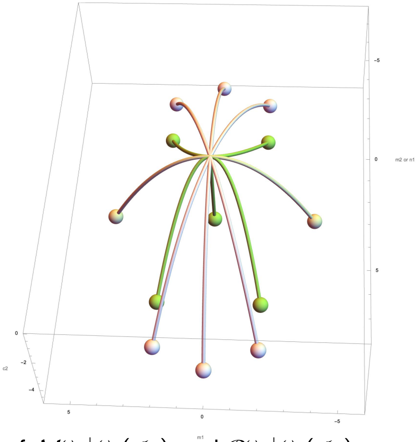

Figure: skeleton of $\mathcal{M}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ and $\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$ over principal spectra

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

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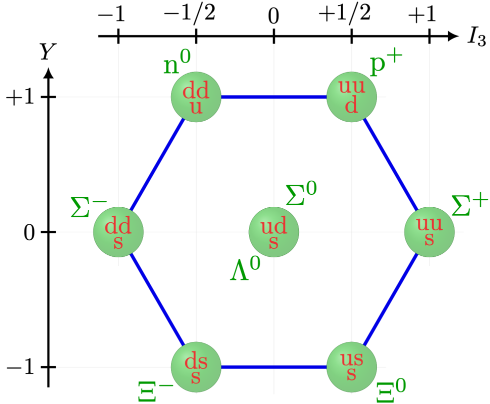

Figure: particles in baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2$, $M_2 := Dc_3/2$, $N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

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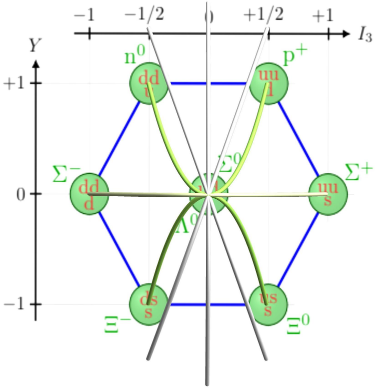

Figure: big and medium skeleton over baryon octet

Skeleton of $\mathrm{Spec}(\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3))$ and baryon octet

- $M_1 := Dc_2/2,\ M_2 := Dc_3/2,\ N_1 := D^2c_3/2$ in $\mathcal{C}^{\omega_1+\omega_2}(\mathfrak{sl}_3)$

$$\mathcal{B}^{\omega_1+\omega_2}(\mathfrak{sl}_3) \cong \mathbb{C}[c_2, c_3, M_1, N_1]/\begin{pmatrix} 3M_1^2 + N_1^2 + 12c_2, \\ M_1^3N_1 + c_2M_1N_1 - 9c_3M_1 \end{pmatrix}$$

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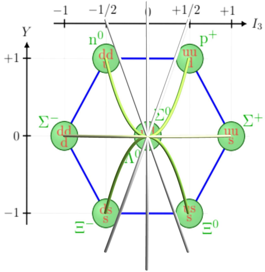

Figure: big and medium skeleton over baryon octet

- $\rightsquigarrow$ polynomial relations between I_3 and Y in baryon octet

$$Y(2I_3 - 1)(2I_3 + 1) = 0$$
$$4I_3^3 + 3I_3Y^2 - 4I_3 = 0$$
$$16I_3^4 - 16I_3^2 + 3Y^2 = 0$$

Nerves and crystals in big algebra

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- $h_0 \in \mathfrak{h}_+ \subset \mathfrak{t}_{\mathbb{R}}$ in dominant Weyl chamber

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- for $J \subset I := \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ face of dominant Weyl chamber:
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  $h_J = \mathrm{pr}_{\mathfrak{h}_+^J}(h_0)$ orthogonal projection to $\mathfrak{h}_+^J$

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- *J-nerve* is the real curve $n_J := \pi_\mu^{-1}(l_j) \subset \mathrm{Spec}(\mathcal{B}^\mu)$

Nerves and crystals in big algebra

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  for $\pi_\mu : \mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H_{\mathrm{G}}^{2*}) \cong \mathfrak{h}//W$

Nerves and crystals in big algebra

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Corollary (Hausel–Rybnikov 2024)

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The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ *induced by monodromy along the nerves* $n_{\{\alpha_i\}}$

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The I-colored oriented graph on $\mathrm{Spec}(\mathcal{B}^\mu_{h_0})(\mathbb{C})$ induced by monodromy along the nerves $n_{\{\alpha_i\}}$ determines a structure of a μ-highest weight Kashiwara crystal

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Theorem (Hausel 2024,Hausel–Zveryk 2022, Nakajima 2023)

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(Rozhkovskaya 2003)'s example of $\mathfrak{sl}_2$ Kostant algebra

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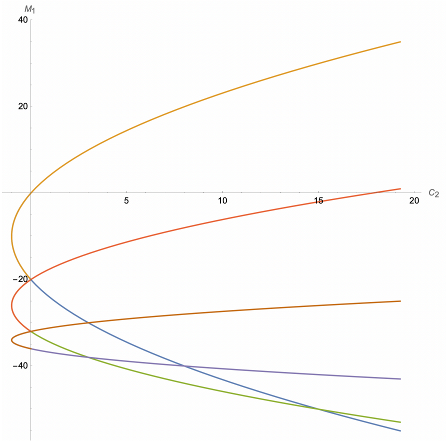

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.

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3. *$\mathcal{G}^\mu_{\chi_\lambda} \to \mathrm{End}(P_\lambda) \cong \mathrm{End}_{H^*(X_\lambda)}(IH^*(X_\lambda))$ induces a maximal commutative cyclic subalgebra and a ring structure on $IH^*(X_\lambda)$*

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4. $\mathcal{G}^\mu_{\chi_\lambda} \to \mathrm{End}(M_\lambda \otimes V^\mu)$ *encodes Kazhdan–Lusztig multiplicities*

Examples of infinitesimal characters in tensor product

$$M_{-1} \otimes V^{5\omega_1} = P_{-6} \oplus P_{-4} \oplus P_{-2}$$

$$M_0 \otimes V^{5\omega_1} = P_{-5} \oplus P_{-3} \oplus M_{-1} \oplus M_5$$

$$M_1 \otimes V^{5\omega_1} = P_{-4} \oplus P_{-2} \oplus M_4 \oplus M_6$$

$$M_2 \otimes V^{5\omega_1} = P_{-3} \oplus M_{-1} \oplus M_1 \oplus M_3 \oplus M_5 \oplus M_7$$

$$M_3 \otimes V^{5\omega_1} = P_{-2} \oplus M_0 \oplus M_2 \oplus M_4 \oplus M_6 \oplus M_8$$

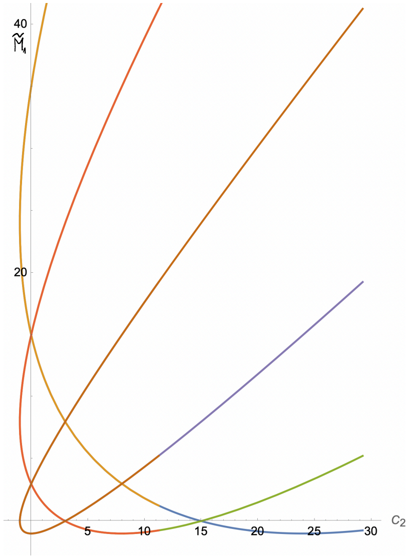

Figure: $\mathrm{Spec}(\mathcal{R}^{5\omega_1}(\mathfrak{sl}_2))$.