Betti numbers of holomorphic symplectic quotients via arithmetic Fourier transform

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A Fourier transform technique is introduced for counting the number of solutions of holomorphic moment map equations over a finite field. This technique in turn gives information on Betti numbers of holomorphic symplectic quotients. As a consequence, simple unified proofs are obtained for formulas of Poincaré polynomials of toric hyperkähler varieties (recovering results of Bielawski–Dancer and Hausel–Sturmfels), Poincaré polynomials of Hilbert schemes of points and twisted Atiyah–Drinfeld–Hitchin–Manin (ADHM) spaces of instantons on $\mathbb{C}^2$ (recovering results of Nakajima–Yoshioka), and Poincaré polynomials of all Nakajima quiver varieties. As an application, a proof of a conjecture of Kac on the number of absolutely indecomposable representations of a quiver is announced.

quiver varieties | Weyl–Kac character formula

Let $\mathbb{K}$ be a field, which will be either the complex numbers $\mathbb{C}$ or the finite field $\mathbb{F}_q$ in this work. Let G be a reductive algebraic group over $\mathbb{K}$, $\mathfrak{g}$ its Lie algebra. Consider a representation $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{V})$ of G on a $\mathbb{K}$-vector space $\mathbb{V}$, inducing the Lie algebra representation $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$. The representation ρ induces an action $\rho : \mathrm{G} \to \mathrm{GL}(\mathbb{M})$ on $\mathbb{M} = \mathbb{V} \times \mathbb{V}^*$. The vector space $\mathbb{M}$ has a natural symplectic structure, defined by the natural pairing $\langle v, w\rangle = w(v)$, with $v \in \mathbb{V}$ and $w \in \mathbb{V}^*$. With respect to this symplectic form a moment map

$$\mu : \mathbb{V} \times \mathbb{V}^* \to \mathfrak{g}^*,$$

of ρ is given at $X \in \mathfrak{g}$ by

$$\langle \mu(v, w), X\rangle = \langle \varrho(X)v, w\rangle. \qquad [1]$$

Let now $\xi \in (\mathfrak{g}^*)^{\mathrm{G}}$ be a central element, then the holomorphic symplectic quotient is defined by the affine geometric invariant theory (GIT) quotient

$$\mathbb{M}////_{\xi}G := (\mu^{-1}(\xi))//G,$$

which is the affine algebraic geometric version of the hyperkähler quotient construction of ref. 1. In particular our varieties, in addition to the holomorphic symplectic structure, will carry a natural hyperkähler metric, although the latter will not feature in what follows.

Our main proposition counts rational points on the varieties $\mu^{-1}(\xi)$ over the finite fields $\mathbb{F}_q$, where $q = p^r$ is a prime power. For convenience we will use the same letters $\mathbb{V}$, G, $\mathfrak{g}$, $\mathbb{M}$, ξ for the corresponding vector spaces, groups, Lie algebras, and matrices over the finite field $\mathbb{F}_q$. We define the function $a_\varrho : \mathfrak{g} \to \mathbb{N} \subset \mathbb{C}$ at $X \in \mathfrak{g}$ as

$$a_\varrho(X) := |\ker(\varrho(X))|, \qquad [2]$$

where we used the notation $|S|$ for the number of elements in any set S. In particular $a_\varrho(X)$ is always a power of q. For an element $v \in V$ of any vector space we define the characteristic function $\delta_v : V \to \mathbb{C}$ by $\delta_v(x) = 0$ unless $x = v$ when $\delta_v(v) = 1$. We can now formulate our main proposition.

Proposition 1. *The number of solutions of the equation* $\mu(v, w) = \xi$ *over the finite field* $\mathbb{F}_q$ equals

$$\begin{aligned}\#\{(v, w) \in \mathbb{M} | \mu(v, w) = \xi\} &= |\mathfrak{g}|^{-1/2}|\mathbb{V}|\,\mathcal{F}(a_\varrho)(\xi)\\ &= |\mathfrak{g}|^{-1}|\mathbb{V}| \sum_{X\in\mathfrak{g}} a_\varrho(X)\Psi(\langle X, \xi\rangle).\end{aligned}$$

To explain the last two terms in the proposition above, we need to define Fourier transforms (2) of functions $f : \mathfrak{g} \to \mathbb{C}$ on the finite Lie algebra $\mathfrak{g}$, which here we think of as an abelian group with its additive structure. To define this fix $\Psi : \mathbb{F}_q \to \mathbb{C}^{\times}$ a nontrivial additive character, and then we define the Fourier transform $\mathcal{F}(f) : \mathfrak{g}^* \to \mathbb{C}$ at a $Y \in \mathfrak{g}^*$

$$\mathcal{F}(f)(Y) = |\mathfrak{g}|^{-1/2} \sum_{X\in\mathfrak{g}} f(X)\Psi(\langle X, Y\rangle).$$

Proof: Using two basic properties of Fourier transform

$$\mathcal{F}(\mathcal{F}(f))(X) = f(-X),$$

for $X \in \mathfrak{g}$ and

$$\sum_{w\in V^*} \Psi(\langle v, w\rangle) = |V|\delta_0(v), \qquad [3]$$

for $v \in V$ we get

$$\begin{aligned}\#\{(v, w) \in \mathbb{M} | \mu(v, w) = \xi\} &= \sum_{v\in\mathbb{V}}\sum_{w\in\mathbb{V}^*} \delta_\xi(\mu(v, w))\\
&= \sum_{v\in\mathbb{V}}\sum_{w\in\mathbb{V}^*} \mathcal{F}(\mathcal{F}(\delta_\xi))(-\mu(v, w))\\
&= \sum_{v\in\mathbb{V}}\sum_{w\in\mathbb{V}^*}\sum_{X\in\mathfrak{g}} |\mathfrak{g}|^{-1/2}\mathcal{F}(\delta_\xi)(X)\Psi(\langle X, -\mu(v, w)\rangle)\\
&= \sum_{v\in\mathbb{V}}\sum_{X\in\mathfrak{g}} |\mathfrak{g}|^{-1/2}\mathcal{F}(\delta_\xi)(X) \sum_{w\in\mathbb{V}^*} \Psi(-\langle \varrho(X)v, w\rangle)\\
&= \sum_{v\in\mathbb{V}}\sum_{X\in\mathfrak{g}} |\mathfrak{g}|^{-1/2}\mathcal{F}(\delta_\xi)(X)|\mathbb{V}|\delta_0(\varrho(X)v)\\
&= \sum_{X\in\mathfrak{g}} |\mathfrak{g}|^{-1/2}\mathcal{F}(\delta_\xi)(X)|\mathbb{V}|a_\varrho(X)\\
&= \sum_{X\in\mathfrak{g}} |\mathfrak{g}|^{-1}|\mathbb{V}|a_\varrho(X) \sum_{Y\in\mathfrak{g}^*} \delta_\xi(Y)\Psi(\langle X, Y\rangle)\end{aligned}$$

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Abbreviation: ADHM, Atiyah–Drinfeld–Hitchin–Manin.

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$$= |\mathfrak{g}|^{-1}|\mathbb{V}| \sum_{X\in\mathfrak{g}} a_\varrho(X)\Psi(\langle X, \xi\rangle)$$

$$= |\mathfrak{g}|^{-1/2}|\mathbb{V}| \mathcal{F}(a_\varrho)(\xi).$$

Affine Toric Hyperkähler Varieties

We take $G = \mathbb{T}^d \cong (\mathbb{C}^\times)^d$ a torus. A vector configuration $A = (a_1, \ldots, a_n) : \mathbb{Z}^n \to \mathbb{Z}^d$ gives a representation $\rho_A : \mathbb{T}^d \to \mathbb{T}^n \subset \mathrm{GL}(\mathbb{V})$, where $\mathbb{V} \cong \mathbb{C}^n$ is an n-dimensional vector space and $\mathbb{T}^n \subset \mathrm{GL}(\mathbb{V})$ is a fixed maximal torus. The corresponding map on the Lie algebras is $\varrho_A : \mathfrak{t}^d \to \mathfrak{t}^n$. The holomorphic moment map of this action $\mu_A : \mathbb{V} \times \mathbb{V}^* \to (\mathfrak{t}^d)^*$ is given by Eq. **1**, which in this case takes the explicit form

$$\mu_A(v, w) = \sum_{i=1}^{n} v_i w_i a_i.$$

We take a generic $\xi \in (\mathfrak{t}^d)^*$. The affine toric hyperkähler variety is then defined as the affine geometric invariant theory quotient: $\mathcal{M}(\xi, A) = \mu_A^{-1}(\xi)//\mathbb{T}^d$. To use our main result, we need to determine $a_\varrho(X)$. Note that the natural basis $e_1, \ldots, e_n \in (\mathfrak{t}^n)^*$ gives us a collection of hyperplanes $H_1, \ldots, H_n$ in $\mathfrak{t}^d$. Now for $X \in \mathfrak{t}^d$ we have that $a_\varrho(X) = q^{ca(X)}$, where $ca(X)$ is the number of hyperplanes that contain X. Finally, we take the intersection lattice $L(A)$ of this hyperplane arrangement, i.e., the set of all subspaces of $\mathfrak{t}^d$ that arise as the intersection of any collection of our hyperplanes, with partial ordering given by containment. The generic choice of ξ will ensure that ξ will not be trivial on any subspace in the lattice $L(A)$. Thus, for any subspace $V \in L(A)$, we have from Eq. **3** that $\Sigma_{X\in V}\, \Psi(\langle X, \xi\rangle) = 0$.

Now we can use *Proposition 1.* If we perform the sum we get a combinatorial expression

$$\#(\mathcal{M}(\xi, A)) = \frac{q^{n-d}}{(q-1)^d} \sum_{X\in\mathfrak{t}^d} a_\varrho(X)\Psi(\langle X, \xi\rangle)$$

$$= \frac{q^{n-d}}{(q-1)^d} \sum_{V\in L(A)} \mu_{L(A)}(V) q^{ca(V)},$$

where $\mu_{L(A)}$ is the Möbius function of the partially ordered set $L(A)$, while $ca(V)$ is the number of coatoms, i.e., hyperplanes containing V. Because the count above is polynomial in q and the mixed Hodge structure on $\mathcal{M}(\xi, A)$ is pure, we get that for the Poincaré polynomial we need to take the opposite of the count polynomial, i.e., substitute $q = 1/t^2$ and multiply by $t^{4(n-d)}$. This way we get:

Theorem 1. *The Poincaré polynomial of the toric hyperkähler variety is given by*

$$P_t(\mathcal{M}(\xi, A)) = \frac{1}{t^{2d}(1-t^2)^d} \sum_{V\in L(A)} \mu_{L(A)}(V)(t^2)^{n-ca(V)}.$$

One can prove by a simple deletion contraction argument (and it also follows from the second proof of proposition 6.3.26 of ref. 3) that for any matroid $\mathcal{M}_{\mathcal{A}}$ and its dual $\mathcal{M}_{\mathcal{B}}$

$$\frac{1}{q^d(1-q)^d} \sum_{V\in L(A)} \mu_{L(A)}(V)(q)^{n-ca(V)} = h(\mathcal{M}_{\mathcal{B}}),$$

where

$$h(\mathcal{M}_{\mathcal{B}}) = \sum_{i=0}^{n-d} h_i(\mathcal{M}_{\mathcal{B}}) q^i,$$

is the h-polynomial of the dual matroid $\mathcal{M}_{\mathcal{B}}$. This way we recover a result of refs. 4 and 5; for a more recent arithmetic proof see (6):

Corollary 1. *The Poincaré polynomial of the toric hyperkähler variety is given by*

$$P_t(\mathcal{M}(\xi, A)) = h(\mathcal{M}_{\mathrm{B}})(t^2),$$

where B is a Gale dual vector configuration of A.

Hilbert Scheme of *n*-Points on $\mathbb{C}^2$ and Atiyah–Drinfeld–Hitchin–Manin (ADHM) Spaces

Here $G = \mathrm{GL}(V)$, where V is an n-dimensional $\mathbb{K}$ vector space. We need three types of basic representations of G. The adjoint representation $\rho_{\mathrm{ad}} : \mathrm{GL}(V) \to \mathrm{GL}(\mathfrak{gl}(V))$, the defining representation $\rho_{\mathrm{def}} = Id : G \to \mathrm{GL}(V)$, and the trivial representations $\rho^k_{\mathrm{triv}} = 1 : G \to \mathrm{GL}(\mathbb{K}^k)$. Fix k and n. Define $\mathbb{V} = \mathfrak{gl}(V) \times V \otimes \mathbb{K}^k$, $\mathbb{M} = \mathbb{V} \times \mathbb{V}^*$ and $\rho : G \to \mathrm{GL}(\mathbb{V})$ by $\rho = \rho_{\mathrm{ad}} \times \rho_{\mathrm{def}} \otimes \rho^k_{\mathrm{triv}}$. Then we take the central element $\xi = Id_V \in \mathfrak{gl}(V)$ and define the twisted ADHM space as

$$\mathcal{M}(n, k) = \mathbb{M}////_\xi G = \mu^{-1}(\xi)//G,$$

where

$$\mu(A, B, I, J) = [A, B] + IJ,$$

with $A, B \in \mathfrak{gl}(V)$, $I \in \mathrm{Hom}(\mathbb{K}^k, V)$ and $J \in \mathrm{Hom}(V, \mathbb{K}^k)$.

The space $\mathcal{M}(n, k)$ is empty when $k = 0$ (the trace of a commutator is always zero), diffeomorphic with the Hilbert scheme of n-points on $\mathbb{C}^2$, when $k = 1$, and is the twisted version of the ADHM space (7) of $U(k)$ Yang–Mills instantons of charge n on $\mathbb{R}^4$ (cf. ref. 8). By our main *Proposition 1* the number of solutions over $\mathbb{K} = \mathbb{F}_q$ of the equation

$$[A, B] + IJ = Id_V,$$

is the Fourier transform on $\mathfrak{g}$ of the function $a_\varrho(X) = |\ker(\varrho(X))|$. First we determine $a_\varrho(X)$ for $X \in \mathfrak{g} = \mathfrak{gl}(V)$. By the definition of ϱ we have

$$\ker(\varrho(X)) = \ker(\varrho_{\mathrm{ad}}(X)) \times \ker(\varrho_{\mathrm{def}}) \otimes \mathbb{K}^k,$$

and so if $a_{\varrho_{\mathrm{ad}}}(X) = |\ker(\varrho_{\mathrm{ad}}(X))|$ and $a_{\varrho_{\mathrm{def}}} = |\ker(\varrho_{\mathrm{def}})|$, then we have

$$a_\varrho(X) = a_{\varrho_{\mathrm{ad}}}(X) a^k_{\varrho_{\mathrm{def}}}(X).$$

Now *Proposition 1* gives us

$$\#(\mathcal{M}(n, k)) = \frac{1}{|G|} \#\{(v, w) \in \mathbb{M} | \mu(v, w) = \xi\}$$

$$= \frac{|\mathbb{V}|}{|\mathfrak{g}||G|} \sum_{X\in\mathfrak{g}} a_\varrho(X)\Psi(\langle X, \xi\rangle)$$

$$= \frac{|\mathbb{V}|}{|\mathfrak{g}||G|} \sum_{X\in\mathfrak{g}} a_{\varrho_{\mathrm{ad}}}(X) a^k_{\varrho_{\mathrm{def}}}(X)\Psi(\langle X, \xi\rangle).$$

We will perform the sum adjoint orbit by adjoint orbit. The adjoint orbits of $\mathfrak{gl}(n)$, according to their Jordan normal forms, fall into types, labeled by $\mathcal{T}(n)$, which stands for the set of all possible Jordan normal forms of elements in $\mathfrak{gl}(n)$. We denote by $\mathcal{T}_{\mathrm{reg}}(t)$ the types of the regular (i.e., nonsingular) adjoint orbits, while $\mathcal{T}_{\mathrm{nil}}(s) = \mathcal{P}(s)$ denotes the types of the nilpotent adjoint orbits, which are just given by partitions of s. First we do the $k = 0$ case where we know *a priori*, that the count should be 0, because the commutator of any two matrix is always trace-free and thus

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cannot equal ξ (for almost all q). Additionally, if we separate the nilpotent and regular parts of our adjoint orbits we get

$$0 = \frac{1}{|\mathrm{G}|}\sum_{X\in\mathfrak{g}} a_{\varrho_{\mathrm{ad}}}(X)\Psi(\langle X,\ \xi\rangle)$$

$$= \sum_{n=s+t}\ \sum_{\lambda\in\mathcal{T}_{\mathrm{nil}}(s)} \frac{|\mathfrak{C}_\lambda|}{|C_\lambda|}\sum_{\tau\in\mathcal{T}_{\mathrm{reg}}(t)}\frac{|\mathfrak{C}_\tau|}{|C_\tau|}\Psi(\langle X_\tau,\ \xi\rangle),$$

where C_τ and, respectively, $\mathfrak{C}_\tau$ denotes the centralizer of an element X_τ of $\mathfrak{g}$ of type τ in the adjoint representation of G, respectively, $\mathfrak{g}$ on $\mathfrak{g}$.

So if we define the generating series

$$\Phi^0_{\mathrm{nil}}(T) = 1 + \sum_{s=1}^{\infty}\ \sum_{\lambda\in\mathcal{T}_{\mathrm{nil}}(s)}\frac{|\mathfrak{C}_\lambda|}{|C_\lambda|}T^s,$$

and

$$\Phi_{\mathrm{reg}}(T) = 1 + \sum_{t=1}^{\infty}\ \sum_{\tau\in\mathcal{T}_{\mathrm{reg}}(t)}\frac{|\mathfrak{C}_\tau|}{|C_\tau|}\Psi(\langle X_\tau,\ \xi\rangle)T^t,$$

then we have

$$\Phi^0_{\mathrm{nil}}(T)\Phi_{\mathrm{reg}}(T) = 1.$$

However, Φ^0_{nil} is easy to calculate (9)

$$\Phi^0_{\mathrm{nil}}(T) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{(1-T^iq^{1-j})}, \qquad [4]$$

thus we get

$$\Phi_{\mathrm{reg}}(T) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}(1-T^iq^{1-j}). \qquad [5]$$

Now the general case is easy to deal with:

$$\frac{\#(\mathcal{M}(n,k))}{q^{nk}} = \frac{1}{|\mathrm{G}|}\sum_{X\in\mathfrak{g}} a_{\varrho_{\mathrm{ad}}}(X)\Psi(\langle X,\ \xi\rangle)$$

$$= \sum_{n=s+t}\ \sum_{\lambda\in\mathcal{T}_{\mathrm{nil}}(s)}\frac{|\mathfrak{C}_\lambda| a^k_{\varrho_{\mathrm{def}}}(X_\lambda)}{|C_\lambda|}\sum_{\tau\in\mathcal{T}_{\mathrm{reg}}(t)}\frac{|\mathfrak{C}_\tau|}{|C_\tau|}\Psi(\langle X,\ \xi\rangle).$$

Thus, if we define the grand generating function by

$$\Phi^k(T) = 1 + \sum_{n=1}^{\infty}\#(\mathcal{M}(n,k))\frac{T^n}{q^{kn}}, \qquad [6]$$

and

$$\Phi^k_{\mathrm{nil}}(T) = 1 + \sum_{s=1}^{\infty}\ \sum_{\lambda\in\mathcal{T}_{\mathrm{nil}}(s)}\frac{|\mathfrak{C}_\lambda||\ker(X_\lambda)|^k}{|C_\lambda|}T^s,$$

then for the latter we get similarly to the argument for Eq. **4** in ref. 9 that

$$\Phi^k_{\mathrm{nil}} = \Phi^k_{\mathrm{nil}}(T) = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{(1-T^iq^{k+1-j})}.$$

For the grand generating function then we get

$$\Phi^k(T) = \Phi^k_{\mathrm{nil}}(T)\Phi_{\mathrm{reg}}(T) = \prod_{i=1}^{\infty}\prod_{l=1}^{k}\frac{1}{(1-T^iq^l)}.$$

Because the mixed Hodge structure is pure, and this count is polynomial, we get the compactly supported Poincaré polynomial. To get the ordinary Poincaré polynomial, we need to replace $q = 1/t^2$ and multiply the nth term in Eq. **6** by t^{4kn}. This way we get *Theorem 2.*

Theorem 2. *The generating function of the Poincaré polynomials of the twisted ADHM spaces are given by*

$$\sum_{n=0}^{\infty}P_t(\mathcal{M}(k,n))T^n = \prod_{i=1}^{\infty}\prod_{b=1}^{k}\frac{1}{(1-t^{2(k(i-1)+b-1)}T^i)}.$$

This result appeared as corollary 3.10 in ref. 10.

Quiver Varieties of Nakajima

Here we recall the definition of the affine version of Nakajima's quiver varieties (11). Let $Q = (\mathcal{V}, \mathcal{E})$ be a quiver, i.e., an oriented graph on a finite set $\mathcal{V} = \{1, \ldots, n\}$ with $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ a finite set of oriented (perhaps multiple and loop) edges. To each vertex i of the graph, we associate two finite dimensional $\mathbb{K}$ vector spaces V_i and W_i. We call $(\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_n) = (\mathbf{v}, \mathbf{w})$ the dimension vector, where $\mathbf{v}_i = \dim(V_i)$ and $\mathbf{w}_i = \dim(W_i)$. To these data we associate the grand vector space

$$\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{(i,j)\in\mathcal{E}}\mathrm{Hom}(V_i,\ V_j)\oplus\bigoplus_{i\in\mathcal{V}}\mathrm{Hom}(V_i,\ W_i),$$

the group and its Lie algebra

$$\mathrm{G}_{\mathbf{v}} = \bigtimes_{i\in\mathcal{V}}\mathrm{GL}(V_i)$$

$$\mathfrak{g}_{\mathbf{v}} = \bigoplus_{i\in\mathcal{V}}\mathfrak{gl}(V_i),$$

and the natural representation

$$\rho_{\mathbf{v},\mathbf{w}} : \mathrm{G}_{\mathbf{v}} \to \mathrm{GL}(\mathbb{V}_{\mathbf{v},\mathbf{w}}),$$

with derivative

$$\varrho_{\mathbf{v},\mathbf{w}} : \mathfrak{g}_{\mathbf{v}} \to \mathfrak{gl}(\mathbb{V}_{\mathbf{v},\mathbf{w}}).$$

The action is from both left and right on the first term and from the left on the second.

We now have $\mathrm{G}_{\mathbf{v}}$ acting on $\mathbb{M}_{\mathbf{v},\mathbf{w}} = \mathbb{V}_{\mathbf{v},\mathbf{w}}\times\mathbb{V}^*_{\mathbf{v},\mathbf{w}}$ preserving the symplectic form with moment map $\mu_{\mathbf{v},\mathbf{w}} : \mathbb{V}_{\mathbf{v},\mathbf{w}}\times\mathbb{V}^*_{\mathbf{v},\mathbf{w}}\to\mathfrak{g}^*_{\mathbf{v}}$ given by Eq. **1**. We take now $\xi_{\mathbf{v}} = (Id_{V_1}, \ldots, Id_{V_n}) \in (\mathfrak{g}^*_{\mathbf{v}})^{\mathrm{G}_{\mathbf{v}}}$, and define the affine Nakajima quiver variety (11) as

$$\mathcal{M}(\mathbf{v},\mathbf{w}) = \mu^{-1}_{\mathbf{v},\mathbf{w}}(\xi_{\mathbf{v}})//\mathrm{G}_{\mathbf{v}}.$$

Here we determine the Betti numbers of $\mathcal{M}(\mathbf{v},\mathbf{w})$ using our main *Proposition 1*, by calculating the Fourier transform of the function $a_{\varrho_{\mathbf{v},\mathbf{w}}}$ given in Eq. **2**.

First, we introduce, for a dimension vector $\mathbf{w}\in\mathcal{V}^{\mathbb{N}}$, the generating function

$$\Phi_{\mathrm{nil}}(\mathbf{w}) = \sum_{\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in\mathcal{V}^{\mathbb{N}}} \prod_{i\in V} T_i^{\mathbf{v}_i} \sum_{\lambda^1\in\mathcal{T}_{\mathrm{nil}}(\mathbf{v}_1)} \cdots \sum_{\lambda^n\in\mathcal{T}_{\mathrm{nil}}(\mathbf{v}_n)} \frac{a_{\varrho_{\mathbf{v},\mathbf{w}}}(X_{\lambda^1},\ldots,X_{\lambda^n})}{|C_{\lambda^1}|\cdots|C_{\lambda^n}|},$$

where $\mathcal{T}_{\mathrm{nil}}(s)$ is the set of types of nilpotent $s\times s$ matrices, where a type is given by a partition $\lambda\in\mathcal{P}(s)$ of s, X_λ denotes the typical $s\times s$ nilpotent matrix in $\mathfrak{gl}(s)$ in Jordan form of type λ, C_λ is the centralizer of X_λ under the adjoint action of $\mathrm{GL}(s)$ on $\mathfrak{gl}(s)$. We also introduce the generating function

$$\Phi_{\mathrm{reg}} = \sum_{\mathbf{v}=(\mathbf{v}_1,\ldots,\mathbf{v}_n)\in\mathcal{V}^{\mathbb{N}}} \prod_{i\in V} T_i^{\mathbf{v}_i} \sum_{\tau_1\in\mathcal{T}_{\mathrm{reg}}(\mathbf{v}_1)} \cdots \sum_{\tau_n\in\mathcal{T}_{\mathrm{reg}}(\mathbf{v}_n)} \frac{a_{\varrho_{\mathbf{v},0}}(X_{\tau_1},\ldots,X_{\tau_n})}{|C_{\tau_1}|\cdots|C_{\tau_n}|}\Psi(\langle X_\tau,\xi_{\mathbf{v}}\rangle),$$

where $\mathcal{T}_{\mathrm{reg}}(t)$ is the set of types τ, i.e., Jordan normal forms, of a regular $t\times t$ matrix X_τ in $\mathfrak{gl}(t)$, $C_\tau\subset\mathrm{GL}(t)$ its centralizer under the adjoint action. Note also that for a regular element $X\in\mathfrak{g}_{\mathbf{v}}$, $a_{\varrho_{\mathbf{v},\mathbf{w}}}(X) = a_{\varrho_{\mathbf{v},0}}(X)$ does not depend on $\mathbf{w}\in\mathcal{V}^{\mathbb{N}}$.

Now we introduce for $\mathbf{w}\in\mathcal{V}^{\mathbb{N}}$ the grand generating function

$$\Phi(\mathbf{w}) = \sum_{\mathbf{v}\in\mathcal{V}^{\mathbb{N}}} \#(\mathcal{M}(\mathbf{v},\mathbf{w}))\frac{|\mathfrak{g}_{\mathbf{v}}|}{|\mathbb{V}_{\mathbf{v},\mathbf{w}}|}T^{\mathbf{v}}. \qquad [7]$$

As in the previous section, our main *Proposition 1* implies

$$\Phi(\mathbf{w}) = \Phi_{\mathrm{nil}}(\mathbf{w})\Phi_{\mathrm{reg}}. \qquad [8]$$

Finally, we note that when $\mathbf{w}=0$ we have $\varrho_{\mathbf{v},0}(\xi_{\mathbf{v}}^*)=0$, where $\xi_{\mathbf{v}}^* = (Id_{V_1},\ldots,Id_{V_n})\in\mathfrak{g}$, thus by Eq. **1** $\langle\mu_{\mathbf{v},0}(v,0),\xi_{\mathbf{v}}^*\rangle = 0$. Because $\langle\xi_{\mathbf{v}},\xi_{\mathbf{v}}^*\rangle = \Sigma\,\mathbf{v}_i$, the equation $\mu_{\mathbf{v},\mathbf{0}}(v,w)=\xi_{\mathbf{v}}$ has no solutions (for almost all q). This way we get that $\Phi(0)=1$, and so Eq. **8** yields $\Phi_{\mathrm{reg}} = 1/\Phi_{\mathrm{nil}}(0)$, giving the result

$$\Phi(\mathbf{w}) = \frac{\Phi_{\mathrm{nil}}(\mathbf{w})}{\Phi_{\mathrm{nil}}(0)}.$$

Therefore, it is enough to understand $\Phi_{\mathrm{nil}}(\mathbf{w})$, which reduces to a simple linear algebra problem of determining $a_{\varrho_{\mathbf{v},\mathbf{w}}}(X_{\lambda^1},\ldots,X_{\lambda^n})$. Putting together everything yields the following:

Theorem 3. *Let $Q=(\mathcal{V},\mathcal{E})$ be a quiver, with $\mathcal{V}=\{1,\ldots,n\}$ and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$, with possibly multiple edges and loops. Fix a dimension vector $\mathbf{w}\in\mathbb{N}^{\mathcal{V}}$. The Poincaré polynomials $P_t(\mathcal{M}(\mathbf{v},\mathbf{w}))$ of the corresponding Nakajima quiver varieties are given by the generating function*

$$\sum_{\mathbf{v}\in\mathbb{N}^{\mathcal{V}}} P_t(\mathcal{M}(\mathbf{v},\mathbf{w}))t^{-d(\mathbf{v},\mathbf{w})}T^{\mathbf{v}} = \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\substack{\lambda^1\in\mathcal{P}(\mathbf{v}_1)\\ \vdots\\ \lambda^n\in\mathcal{P}(\mathbf{v}_n)}} \frac{\left(\prod_{(i,j)\in\mathcal{E}} t^{-2n(\lambda^i,\lambda^j)}\right)\left(\prod_{i\in\mathcal{V}} t^{-2n(\lambda^i,(1^{\mathbf{w}_i}))}\right)}{\prod_{i\in\mathcal{V}}\left(t^{-2n(\lambda^i,\lambda^i)}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\substack{\lambda^1\in\mathcal{P}(\mathbf{v}_1)\\ \vdots\\ \lambda^n\in\mathcal{P}(\mathbf{v}_n)}} \frac{\prod_{(i,j)\in\mathcal{E}} t^{-2n(\lambda^i,\lambda^j)}}{\prod_{i\in\mathcal{V}}\left(t^{-2n(\lambda^i,\lambda^i)}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}, \qquad [9]$$

where $d(\mathbf{v},\mathbf{w}) = 2\Sigma_{(i,j)\in\mathcal{E}}\mathbf{v}_i\mathbf{v}_j + 2\Sigma_{i\in\mathcal{V}}\mathbf{v}_i(\mathbf{w}_i-\mathbf{v}_i)$ is the dimension of $\mathcal{M}(\mathbf{v},\mathbf{w})$, and $T^{\mathbf{v}} = \Pi_{i\in\mathcal{V}}T_i^{\mathbf{v}_i}$. $\mathcal{P}(s)$ stands for the set of partitions[¶] *of $s\in\mathbb{N}$. For two partitions $\lambda=(\lambda_1,\ldots,\lambda_l)\in\mathcal{P}(s)$ and $\mu=(\mu_1,\ldots,\mu_m)\in\mathcal{P}(s)$ we define $n(\lambda,\mu)=\Sigma_{i,j}\min(\lambda_i,\mu_j)$, and if we write $\lambda=(1^{m_1}(\lambda),2^{m_2}(\lambda),\ldots)\in\mathcal{P}(s)$, then we can define $l(\lambda)=\Sigma\,m_i(\lambda)=l$ the number of parts in λ. With this notation $n(\lambda^i,(1^{\mathbf{w}_i}))=\mathbf{w}_i l(\lambda^i)$ in the above formula.*

Remark: This single formula encompasses a surprising amount of combinatorics and representation theory. When $\mathbf{v}=(1,\ldots,1)$, the Nakajima quiver variety is a toric hyperkähler variety, thus Eq. **9** gives a new formula for its Poincaré polynomial, which was given in *Corollary 1.* If additionally $\mathbf{w}=(1,0,\ldots,0)$ then $\mathcal{M}(\mathbf{v},\mathbf{w})$ is the toric quiver variety of ref. 5. Therefore, its Poincaré polynomial, which is the reliability polynomial[‖] of the graph underlying the quiver (5) also can be read off from the above formula **9**.

When the quiver is just a single loop on one vertex, our formula **9** reproduces *Theorem 2*. When the quiver is of type A_n Nakajima (14) showed, that the Poincaré polynomials of the quiver variety are related to the combinatorics of Young-tableaux, while in the general *ADE* case, Lusztig (15) conjectured a formula for the Poincaré polynomial, in terms of formulae arising in the representation theory of quantum groups. When the quiver is star-shaped recent work (T.H., unpublished work and T.H., E. Letellier, and F. Rodriguez–Villegas, unpublished work) calculates these Poincaré polynomials using the character theory of reductive Lie algebras over finite fields (2) and arrives at formulas determined by the Hall–Littlewood symmetric functions (12), which arose as the pure part of Macdonald symmetric polynomials (12). In the case when the quiver has no loops, Nakajima (16) gives a combinatorial algorithm for all Betti numbers of quiver varieties, motivated by the representation theory of quantum loop algebras. Finally, through the paper (17) of Crawley-Boevey and Van den Bergh, Poincaré polynomials of quiver varieties are related to the number of absolutely indecomposable representations of quivers in the work of Kac (18), which were eventually completely determined by Hua (19).

In particular, formula **9**, when combined with results in refs. 19 and 11 and the Weyl–Kac character formula in the representation theory of Kac–Moody algebras (20), yields a simple proof of conjecture 1 of Kac (18). Consequently, formula **9** can be viewed as a q-deformation of the Weyl–Kac character formula (20).

A detailed study of the above generating function **9**, its relationship to the wide variety of examples mentioned above, and details of the proofs of the results of this work will appear elsewhere.

¶The notation for partitions is that of ref. 12.

‖Incidentally, the reliability polynomial measures the probability of the graph remaining connected when each edge has the same probability of failure, a concept heavily used in the study of reliability of computer networks (13).

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