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TRANSILLUMINATION OF LATTICE PACKING OF BALLS

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The paper of J. Horváth [1, see Theorem 2] contains the following statement: If a lattice packing of balls is given in E^n $(n \geqq 3)$, then there exists an affine subspace of E^n of dimension $n-2$ which is disjoint to the balls. In the proof of this statement (see [1], pp. 424–425) he uses the following (for simplicity we only take the special case $\ell = 2$ and $b_3 =$ a shortest non-zero lattice vector of a lattice). Let $a_1, a_2, b_3, b_4, \dots, b_n$ be an arbitrary system of linearly independent vectors from a lattice in E^n, that contains b_3. Then he considers the orthogonal projection of E^n to an orthogonal complement Σ_n^2 of $\operatorname{lin}\{b_3, \dots, b_n\}$. This Σ_n^2 is a 2-plane in E^n, and can be supposed to contain 0. However, for arbitrary $a_1, a_2, b_4, \dots, b_n$ (of length at least $\|b_3\|$) Σ_n^2 is in general no subspace of $\operatorname{lin}\{a_1, a_2, b_3\}$, hence this projection has no restriction to a projection of $\operatorname{lin}\{a_1, a_2, b_3\}$ into itself, which is however used in [1] further. Namely [1] applies to this restricted projection $\operatorname{lin}\{a_1, a_2, b_3\} \to \Sigma_n^2$ a theorem of I. Hortobágyi, that necessitates $\Sigma_n^2 \subset \operatorname{lin}\{a_1, a_2, b_3\}$.

In fact, [1], Theorem 2 itself is invalid, and here we actually prove the following

THEOREM. *There exists a lattice packing of balls in* E^n *intersecting every affine subspace of* E^n *of dimension* $n - [c\sqrt{n}]$, *where* c *is a positive absolute constant.*

PROOF. Throughout the proof we use the terminology, notations and results of the paper of R. Kannan and L. Lovász [2], in particular $\lambda_1(\mathsf{L}_n)$ denotes the minimal length of a non-zero vector of a lattice L_n, and $\mu_j(K, \mathsf{L}_n)$ is the j-th covering minimum of a convex body K with respect to a lattice L_n.

According to the result of Conway and Thompson [3, Chapter II, Theorem 9.5] there exists a lattice L_n of rank n with $\mathsf{L}_n = \mathsf{L}_n^*$ in E^n for which

$$\lambda_1(\mathsf{L}_n)\lambda_1(\mathsf{L}_n^*) \geqq c_1 n, \tag{1}$$

where c_1 is a positive absolute constant. Let us draw balls around all points of L_n with diameter $\lambda_1(\mathsf{L}_n)$. We show that this lattice packing of balls possesses the property claimed in the Theorem. Let P be one of the points of L_n. Let us consider the ball B which is drawn around P. Since $1/\lambda_1(\mathsf{L}_n^*)$ is the maximum of the distances of two parallel and neighbouring lattice

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hyperplanes in $\mathbf{L}_n$, (1) implies that the lattice width of $\mathbf{B}$ is not less than $c_1 n$. Thus, $\mu_1(\mathbf{B},\mathbf{L}_n) \leqq 1/(c_1 n)$. Using Theorem (2.7) from [2] we get that

$$\mu_j(\mathbf{B},\mathbf{L}_n) < c' j^2 \mu_1(\mathbf{B},\mathbf{L}_n) \leqq \frac{c' j^2}{c_1 n}, \tag{2}$$

with a positive absolute constant c'. If we choose $j = [c\sqrt{n}]$ with $c^2 = c_1/c'$, then (2) proves our Theorem.

REMARK. If the conjecture $\mu_{j+1}(\mathbf{B}) \leqq \mu_j(\mathbf{B}) + \mu_1(\mathbf{B})$ (where $\mathbf{B}$ is ball) were true, see [2], then we could replace $n - [c\sqrt{n}]$ by $n - [cn]$ in the Theorem.

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