Geometry of the moduli space of Higgs bundles

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to Kati

Preface

The present dissertation is my own work, except where attributed to others. It is not the outcome of work done in collaboration, except Chapters 6 and 7.

Chapters 6 and 7 describe joint work with Michael Thaddeus. We started a correspondence in early 1997 about [Tha1] and related problems. The results appearing in these chapters were mostly achieved when we participated in the Research in Pairs program in the Mathematisches Forschungsinstitut Oberwolfach for three weeks in June, 1998.

I was advised by my supervisor that Chapters 1-5 are sufficient for a Ph.D. thesis. I added Chapters 6 and 7 to the thesis because I believe that they make the presented work more compact.

Contents

Acknowledgements

My principal debt of gratitude goes to my supervisor Nigel Hitchin, who first suggested the problem of this thesis and then helped in numerous ways during the course of the research. Many of the ideas appearing here were found in or after a supervision with him.

Collaboration with Michael Thaddeus has resulted in Chapters 6 and 7 of the present thesis. His influence is noticeable in the rest of the thesis as well. Working with him proved to be very stimulating.

A great amount of inspiration was provided by my college contact Michael Atiyah through exciting conversations and his own mathematical work.

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I was able to write this thesis because I was taught Mathematics by many excellent teachers throughout the years including Károly Bezdek, Balázs Csikós, Judit Köves, Endre Makai Jr., Gábor Moussong, József Pelikán, István Reimann, János Szenthe, András Szűcs, Imréné Thiry and Jánosné Tillman.

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I spent the academic year 1997-98 in Oxford, as a visiting student, where the hospitality of the Mathematical Institute and St. Catherine's College, Oxford helped a lot in finishing the thesis.

The joint results with Michael Thaddeus were mostly achieved in the Mathematisches Forschungsinstitut Oberwolfach, where we participated in the Research in Pairs program for three weeks in June, 1998. The RiP program is supported by Volkswagen-Stiftung. I thank the Institute for its hospitality which made our research there so fruitful.

For difficult calculations I used the mathematical software packages Mathematica, Maple and Macaulay in this reverse lexicographical order. In particular, Macaulay 2 proved to be essential in the calculation of the cohomology ring in Chapter 6.

Introduction

0.1 Motivation: Interaction with Physics

Traditional Mathematical Physics, the subject of mathematically rigorous results in Theoretical Physics, has always been a bridge between Mathematics and Physics. Since the late 70's we have been witnessing a profoundly new interaction between Geometry and Particle Physics, implying an intrinsic connection between them. Let us mention a few examples: Donaldson's theory of 4-manifolds and Seiberg-Witten theory, enumerative geometry and sigma models, the Jones polynomial and Chern-Simons theory, the Verlinde formula and Conformal Field Theory. A common feature in these pairs is that the theories are motivated by both Geometry and Particle Physics.

In the late 70's physicists started to use sophisticated geometrical and topological methods for the description of the non-perturbative aspects of Quantum Field Theories. It turned out that the independently developed geometrical methods could provide new insights into Physics. Indeed, these were used for testing physical theories in the absence of the available technology for experiments.

Geometers welcomed the renewed interest in their work – and tried to solve the new problems suggested by physicists. It resulted in an exchange of ideas – physicists revealed new directions in Geometry, and geometers delivered solutions in Physics.

By now links of this new kind are so numerous that one is tempted to hope that a new subject is emerging with mixed mathematical and physical motivations, that may be called Quantum Geometry.

While the present thesis is concerned only with Mathematics, the motivation of the original question comes from Physics. In the next section, by explaining the motivation for the problem of this thesis, we show a few examples of the different types of interaction between Geometry and Particle Physics.

0.2 Statement of the problem

Analyzing the conjectured S-duality in N=2 supersymmetric Yang-Mills theory, which is a proposed $SL(2,\mathbb{Z})$ symmetry of the theory, Sen in [Sen] could predict the dimension of the space of L^2 harmonic forms $\mathcal{H}_k$ on the universal cover of the moduli space of $SU(2)$ magnetic monopoles of charge k, by speculating that there must be an $SL(2,\mathbb{Z})$-action on the space $\bigoplus \mathcal{H}_k$, which represents bound electron states of the theory.

The moduli space of monopoles M_k of charge k is the parameter space of finite energy and charge k solutions to Bogomolny equations, which can be interpreted as the one dimensional reduction of the self-dual $SU(2)$ Yang-Mills equations on $\mathbb{R}^4$.

The parameter or moduli space M_k of magnetic monopoles of charge k is a non-compact manifold, with $\pi_1(M_k) = \mathbb{Z}_k$, and has a natural hyperkähler and complete metric on it, which comes from an abstract construction (the so-called hyperkähler quotient construction[1]) and known explicitly only in the case $k = 2$, when M_2 is called the Atiyah-Hitchin manifold[2].

When $k = 2$ Sen's conjecture says that $\dim(\mathcal{H}_2) = 1$. By knowing the metric of M_2 explicitly, Sen was able to find a non-trivial L^2 harmonic form on the universal cover $\tilde{M}_2$, giving some support for his conjecture and in turn for S-duality.

For higher k however Sen's conjecture says something about a metric which is not known explicitly. Nevertheless the statement is quite interesting from a mathematical point of view as the space of L^2 harmonic forms on a non-compact complete Riemannian manifold is not well understood.

Hodge theory tells us that in the compact case the space of L^2 harmonic forms is naturally isomorphic to the De-Rham cohomology of the manifold. However in the non-compact case there is no such theory, and indeed the harmonic space depends crucially on the metric.

Nevertheless some part of Hodge theory survives for complete Riemannian manifolds[3], such as the Hodge decomposition theorem which states that for a complete Riemannian manifold M the space $\Omega^*_{L^2}$ of L_2 forms on M has an orthogonal decomposition

$$\Omega^*_{L^2} = \overline{d(\Omega^*_{cpt})} \oplus \mathcal{H}^* \oplus \overline{\delta(\Omega^*_{cpt})},$$

also that $\mathcal{H}^* = \ker(d) \cap \ker(d^*)$. An easy corollary[4] of these results says that the composition

$$H^*_{cpt}(M) \to \mathcal{H}^* \to H^*(M)$$

is the forgetful map.

By calculating the image of $H^*_{cpt}(\tilde{M}_k)$ in $H^*(\tilde{M}_k)$ Segal and Selby could give a lower bound for the harmonic forms on the moduli space of magnetic monopoles which coincides with the dimension given by Sen's conjecture (see [Se,Se]). This purely mathematical result is thus a supporting evidence for the conjectured S-duality in $N = 2$ SYM of theoretical Physics.

In this thesis we will investigate the analogue of Sen's conjecture for Hitchin's moduli space $\mathcal{M}$ of Higgs bundles of fixed determinant of degree 1 over a Riemann surface Σ of genus $g > 1$. Hitchin introduced $\mathcal{M}$ in [Hit1] by considering the 2-dimensional reduction of the 4-dimensional self-dual Yang-Mills equation[5]. The moduli space $\mathcal{M}$ is a simply connected non-compact manifold of dimension $12g - 12$ with a complete hyperkähler metric on it. Led by the similarities between the spaces M_k and $\mathcal{M}$ and their origin, we address the following problem:

Problem 1 *What are the L^2 harmonic forms on $\mathcal{M}$?*

In Chapter 5 we solve the topological part of this problem by considering the so-called *virtual Dirac bundle*[6] $\mathbf{D}$. By examining the degeneration locus of $\mathbf{D}$ we prove:

Theorem 0.2.1 *The forgetful map*

$$j_{\mathcal{M}} : H^*_{cpt}(\mathcal{M}) \to H^*(\mathcal{M})$$

is 0.

[1] cf. [HKLR]
[2] For further details see Subsection 1.1.3.
[3] cf. [DeRh] Sect. 32 Theorem 24 and Sect. 35 Theorem 26
[4] Cf. [Se,Se]
[5] For details see Subsection 1.1.4.
[6] For its gauge theoretical construction see Section 1.1.5.

This says that unlike the case of $\tilde{M}_k$ the topology of $\mathcal{M}$ does not give the existence of L^2 harmonic forms. We can state this fact informally as: "There are no topological L^2 harmonic forms on Hitchin's moduli space of Higgs bundles".

Segal and Selby's result together with Sen's conjecture suggest that for $\tilde{M}_k$ the topology gives all the harmonic space. Led by this and supported by the discussion in Subsection 3.5 we can formulate the following conjecture:

Conjecture 1 *There are no non-trivial L^2 harmonic forms on Hitchin's moduli space of Higgs bundles.*

It would be interesting to see whether a physical argument could back this conjecture[7]. We know of one serious appearance of Hitchin's moduli space of Higgs bundles in the Physics literature. In [BJSV] a topological σ-model with target space $\mathcal{M}$ arises as certain limit of $N = 4$ supersymmetric Yang-Mills theory. However it is not clear whether L^2 harmonic forms on $\mathcal{M}$ have any physical interpretation in this theory.

The mathematical counterpart of a topological sigma model with target space $\mathcal{M}$ is the enumerative geometry of curves in $\mathcal{M}$. In order to consider enumerative problems in $\mathcal{M}$ one first has to understand $H^*(\mathcal{M})$, the ordinary cohomology ring of $\mathcal{M}$. The purpose of Chapter 6 is to describe the cohomology ring of $\mathcal{M}$, establishing the mathematical background of calculating the physical theory of [BJSV]. This is intended in a future work. Chapter 7 places the understanding of the cohomology ring of $\mathcal{M}$ into a more general framework, and finds intimate relations with the work of Atiyah-Bott, and Kirwan.

All in all the purpose of this thesis is to give a comprehensive description of the cohomology of Hitchin's moduli space of Higgs bundles with the aim of using it later for calculating physically interesting theories.

[7]Note that the conjecture does not hold for *parabolic* Higgs bundles, as the toy example after Theorem 4.6.13 shows. Note also that Dodziuk's vanishing theorem [Dod] shows that there are no non-trivial L^2 *holomorphic* forms on $\mathcal{M}$, since the Ricci tensor of a hyperkähler metric is zero.

0.3 Structure and Results

The thesis has two parts. The first part is a collection of results mainly from the literature which are needed in the second part. The second part has four chapters. The first two cover more or less the papers [Hau1] and [Hau2], respectively, while the last two are made up from [Ha,Th].

In Chapter 4 the emphasis is on the $\mathbb{C}^*$-action, and the approach is from Symplectic Geometry. In Chapter 5 we focus on the hypercohomology of Higgs bundles and the point of view is in Algebraic Geometry. In Chapter 6 the central feature is equivariant cohomology and the extra tool is Equivariant Topology. Finally in Chapter 7 the main object is the resolution tower and the new methods are from Mathematical Gauge theory and Homotopy theory.

In Chapter 4, using Lerman's construction of symplectic cutting, we consider a canonical compactification of $\mathcal{M}$, producing a projective variety $\overline{\mathcal{M}} = \mathcal{M} \cup Z$, with orbifold singularities. For this we thoroughly examine the $\mathbb{C}^*$-action on $\mathcal{M}$, given by scalar multiplication of the Higgs field. We find that the downward flows correspond to the components of the nilpotent cone while the upward flows correspond to the Shatz stratification. The former result will be exploited in Chapter 5, while the latter in Chapter 6 and in Chapter 7. Moreover we give a detailed study of the spaces Z and $\overline{\mathcal{M}}$. In doing so we reprove some assertions of Laumon and Thaddeus on the nilpotent cone.

In Chapter 5 we prove the physically motivated Theorem 0.2.1. For this we consider the virtual Dirac bundle on $\mathcal{M}$ which is the analogue of the virtual Mumford bundle on $\mathcal{N}$, the moduli space of rank 2 stable bundles of fixed determinant of odd degree over a projective curve Σ. Using results from Chapter 4 about the nilpotent cone we apply Porteous' theorem to the downward flows to obtain a proof of Theorem 0.2.1 that all intersection numbers in the compactly supported cohomology of $\mathcal{M}$ vanish, i.e. "there are no *topological* L^2 harmonic forms on $\mathcal{M}$". Our result generalizes two facts. One is the well known vanishing of the Euler characteristic of $\mathcal{N}$, which gives the vanishing of one intersection number on $\mathcal{M}$. The other is the vanishing of the ordinary cohomology class of the Prym variety, the generic fibre of the Hitchin map, which gives the vanishing of g intersection numbers on $\mathcal{M}$. We prove here that the rest of the g^2 intersection numbers also vanish. Our proof shows that the vanishing of all intersection numbers of $H^*_{cpt}(\mathcal{M})$ is given by relations analogous to the Mumford relations in the cohomology ring of $\mathcal{N}$.

In Chapter 6 we give a conjectured complete description of the cohomology ring of $\mathcal{M}$. We use an approach motivated by Kirwan's work on the proof of the Mumford conjecture, to prove that the equivariant cohomology ring of $\widetilde{\mathcal{M}}$ is generated by universal classes. However our proof is completely geometric in nature and rests on the degeneracy locus description of the upward flows. We conclude the chapter by explaining a – computer supported – conjectured complete description of $H^*_I(\mathcal{M})$, the subring of $H^*(\mathcal{M})$ generated by the universal classes α, β and γ. We support this conjecture by proving that the conjectured ring and $H^*_I(\mathcal{M})$ have the same Poincaré polynomial. Also we find the first two relations. The second of which turns out to be $\beta^g = 0$, showing that Newstead's conjecture is still true over $\mathcal{M}$.

In Chapter 7 we construct a *resolution tower* of smooth S^1-manifolds:

$$\widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}_0 \subset \widetilde{\mathcal{M}}_1 \subset ... \subset \widetilde{\mathcal{M}}_k \subset ...,$$

from the moduli spaces $\widetilde{\mathcal{M}}_k$ of stable Higgs bundle with a pole of order at most k at a fixed point, and of degree 1, and consider the direct limit

$$\widetilde{\mathcal{M}}_\infty = \lim_{k \to \infty} \widetilde{\mathcal{M}}_k.$$

We prove that its cohomology is a free graded algebra on the universal classes and that

$$i_0^* : H^*(\widetilde{\mathcal{M}}_\infty) \to H^*(\widetilde{\mathcal{M}})$$

is surjective, therefore a resolution of the cohomology ring of $\widetilde{\mathcal{M}}$. We show how $\widetilde{\mathcal{M}}_\infty$ can be used to provide a 'finite dimensional' and purely geometric proof of the Mumford conjecture. To shed light on many striking features of $\widetilde{\mathcal{M}}_\infty$ we show that $\widetilde{\mathcal{M}}_\infty$ is homotopy equivalent to $B\overline{\mathcal{G}}$, the classifying space of the gauge group modulo constant scalars, and that they are also homotopy equivalent as stratified spaces. We finish the chapter by proving that even the homotopy of the resolution tower stabilizes in the spirit of the Atiyah-Jones conjecture.

We conclude the thesis by summarizing the work in the previous chapters from the point of view of the compactification $\overline{\mathcal{M}}$.

Part I

General facts

Chapter 1

Moduli spaces

The subject of the thesis is the investigation of a particular moduli space, the moduli space of Higgs bundles. In this chapter we give an introduction to moduli spaces in general and collect results from the literature which will be needed in the second part. The chapter has two sections. In the first one we deal with gauge theory in the second with algebraic moduli spaces.

1.1 Mathematical gauge theory

The interests of mathematicians in gauge theory started in the late 70's with the appearance of a few pioneer papers, such as [ADHM] and [A,H,S]. The papers were concerned with the Yang-Mills equations in gauge theory, which for mathematicians meant a branch of differential geometry, namely connections on fibre bundles.

Yang and Mills introduced the Yang-Mills equations on Minkowski 4-space in 1954 as a non-Abelian generalization of Maxwell's equations. Later the same equations over Euclidean 4-space were also considered and these equations are our concern in the next subsection[1].

1.1.1 Yang-Mills equations in 4 dimensions

Let G be a Lie group (usually $U(2)$, $SU(2)$ or $SO(3)$) and P be a principal G-bundle over $\mathbb{R}^4$. If A is a connection on P, then its curvature $F(A) \in \Omega^2(\mathbb{R}^4; \mathrm{ad}(P))$ is a two-form with values in $\mathrm{ad}(P)$, where $\mathrm{ad}(P) = P \times_G \mathfrak{g}$ is the vector bundle associated to the adjoint representation.

In principle a physical theory is given by its Lagrangian. The *Yang-Mills Lagrangian* (or *functional* or *energy function* or *action*) is:

$$S(A) = -\int_{\mathbb{R}^4} \mathrm{tr}(F(A) \wedge *F(A)), \tag{1.1}$$

where

$$* : \Omega^2(\mathbb{R}^4; \mathrm{ad}(P)) \to \Omega^2(\mathbb{R}^4; \mathrm{ad}(P))$$

is the Euclidean Hodge star operator.

The corresponding Euler-Lagrange equations, which describe the critical points of the functional S, are called the *Yang-Mills equations*:

$$d_A * F(A) = 0, \tag{1.2}$$

Recall that the Bianchi identity says that

$$d_A F(A) = 0, \tag{1.3}$$

which is formally similar[2] to (1.2).

[1] For a more detailed introduction see [Ati1].

[2] As a matter of fact a possible duality between the two equations is the source of the conjectured S-duality, which was mentioned in the Introduction. In Maxwell's theory, which is a Yang-Mills theory on Minkowski 4 space with $G = U(1)$, this is the well known duality between electricity and magnetism.

The absolute minima of the Yang-Mills Lagrangian are given by the *self-dual*

$$F(A) = *F(A)$$

and *anti self-dual*

$$F(A) = - *F(A)$$

Yang-Mills equations. Physically a solution of finite energy to these equations represents a *multi-instanton.* Note that because of the Bianchi identity (1.3) it is immediate that solutions to the last two equations satisfy the original Yang-Mills equations (1.2). Mathematically the last two equations are equivalent by reversing the orientation of the underlying $\mathbb{R}^4$. We will only consider the self-dual equations here.

We will need later the form of the equations above in terms of a trivialization of P over $\mathbb{R}^4$. If the basic coordinates of $\mathbb{R}^4$ are (x_1, x_2, x_3, x_4) , then the connection A is described by a Lie algebra-valued 1-form:

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4. \tag{1.4}$$

and then its curvature is

$$F(A) = dA + A \wedge A.$$

Alternatively

$$F(A) = \sum_{i<j} F_{ij} dx_i \wedge \ dx_j,$$

where

$$F_{ij} = \left[\frac{\partial}{\partial x_i} + A_i, \frac{\partial}{\partial x_j} + A_j \right],$$

or if we write

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i,$$

then

$$F_{ij} = [\nabla_i, \nabla_j].$$

In this trivialization the self-dual Yang-Mills equations then take the form

$$\left.\begin{array}{l} F_{12} = F_{34}, \\ F_{13} = F_{42}, \\ F_{23} = F_{14}. \end{array}\right\} \tag{1.5}$$

An important aspect of the Yang-Mills equations is that they are *gauge invariant.* To explain this consider a C^∞ section g of the bundle of groups $P \times_{\mathrm{Ad}} G$. It is an automorphism of the principal bundle which leaves each fibre invariant. It is called a *gauge transformation*, and the group of all gauge transformations is called the *gauge group.* A gauge transformation transforms any connection A on P in a natural way to another connection $g(A)$. Moreover if A is a solution to the self-dual Yang-Mills equations then $g(A)$ is also a solution. Thus $\mathcal{G}$ acts naturally on the finite energy solution space of the self-dual Yang-Mills equations. Because we do not consider two gauge equivalent solutions different we define the *moduli space* (or *parameter space*) to be the quotient.

It can be shown that in nice cases the moduli space will inherit a natural structure of a finite dimensional smooth manifold. The study of these moduli spaces is the central problem of the mathematical gauge theory. The significance of the subject became more apparent in the early 80's with the seminal work of Donaldson and Atiyah-Bott.

Donaldson, analyzing the moduli space of solutions to the anti-self dual Yang-Mills equations over a closed oriented Riemannian 4-manifold M, could prove many fundamental theorems about the topology of the underlying differentiable manifold[3].

Atiyah and Bott, in [At,Bo], extensively studied the 2-dimensional Yang-Mills equations, relating the subject to stable vector bundles on algebraic curves and Morse theory. In the next subsection we explain how they related Yang-Mills theory to the algebraic geometry of vector bundles on curves.

[3] For more details consult [Do,Kr].

1.1.2 Yang-Mills equations in 2 dimensions

For the sake of simplicity we restrict our attention to $G = U(n)$, and let P be a principal $U(n)$-bundle over a Riemann surface Σ. We also fix a metric on Σ compatible with the complex structure. Now if A is a connection on P its curvature is:

$$F(A) \in \Omega^2(\mathbb{R}^4; \mathrm{ad}(P)).$$

The space $\mathcal{A}$ of all connections on P is naturally an affine space modelled on the infinite dimensional vector space $\Omega^{0,1}(\Sigma, \mathrm{ad}(P) \otimes \mathbb{C})$. As we explained above, the infinite dimensional gauge group $\mathcal{G} = \Gamma(\Sigma, P \times_{\mathrm{Ad}} U(n))$ acts naturally on the infinite dimensional affine space $\mathcal{A}$.

The strategy of [At,Bo] is to calculate $H^*_{\mathcal{G}}(\mathcal{A})$, the $\mathcal{G}$-equivariant cohomology[4] of $\mathcal{A}$, in two different ways. First is a direct way by noting that $\mathcal{A}$, being homeomorphic[5] to a topological vector space, is contractible, thus

$$H^*_{\mathcal{G}}(\mathcal{A}) \cong H^*(B\mathcal{G}),$$

and $H^*(B\mathcal{G})$ can be calculated directly[6].

The other approach for calculating $H^*_{\mathcal{G}}(\mathcal{A})$ is Morse theory. Morse theory gives information about the cohomology of the whole space in terms of the cohomology of the non-degenerate critical submanifolds of the Morse function. Recall that we have the Yang-Mills functional $S : \mathcal{A} \to \mathbb{R}$, defined by (1.1), which gives a function on $\mathcal{A}$.

The key idea of [At,Bo] is to think of S as a $\mathcal{G}$-equivariant Morse function on the infinite dimensional Banach manifold $\mathcal{A}$, and calculate $H^*_{\mathcal{G}}(\mathcal{A})$ from Morse theory. As in the previous subsection, critical points of S are given by solutions to the 2-dimensional Yang-Mills equations (the Euler-Lagrange equations of the Yang-Mills functional) of the form (1.2) with

$$* : \Omega^2(\Sigma; \mathrm{ad}(P)) \to \Omega^0(\Sigma; \mathrm{ad}(P))$$

now being the Hodge star operator corresponding to the fixed metric on Σ.

The solution space has infinitely many components: one component, corresponding to the absolute minimum of the Yang-Mills functional, contains irreducible connections, whereas the others contain only reducible ones. A famous result of Narasimhan and Seshadri identifies the moduli space of irreducible solutions to the Yang-Mills equations with the moduli space of stable vector bundles on Σ, a space, which had been investigated by algebraic geometers for decades.

Thus, provided that the Morse function is perfect, the Morse theory approach gives a method to calculate the cohomology of the moduli space of stable bundles, by knowing the cohomology of the moduli space of reducible connections, which can be inductively calculated via this process. Though they couldn't proceed with the analysis[7], they found an alternative way. Namely there is a one-to-one correspondence between unitary connections on P and holomorphic structures on the associated C^∞ complex vector bundle $\mathcal{V} = P \times_{\mathrm{ad}} \mathfrak{g}$. Thus one can identify $\mathcal{A}$ with the complex affine space $\mathcal{C}$ of holomorphic structures on $\mathcal{V}$. Moreover the latter space has the advantage of being independent of the metric on Σ, transferring the problem from differential geometry to holomorphic, and indeed algebraic geometry.

In Section 3.4 we will explain how Atiyah and Bott applied Morse theory to $\mathcal{C}$, motivated by the above heuristic argument of Morse theory over $\mathcal{A}$.

1.1.3 Bogomolny equations in 3 dimensions

In the 80's, further parts of gauge theory became the subject of study by mathematicians. A good example is the problem of magnetic monopoles. To explain the mathematical background, consider a connection A on a principal G-bundle P over $\mathbb{R}^4$ of the form (1.4). If we make the assumption that the Lie algebra-valued functions A_i are independent of x_4, then A_1,A_2 and A_3 define a connection:

$$A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

over $\mathbb{R}^3$, while A_4 becomes a function on $\mathbb{R}^3$ traditionally denoted by ϕ and called the *Higgs field*.

[4] For the definition of equivariant cohomology see Subsection 2.2.1.

[5] To get the correct topology on $\mathcal{A}$ as a Banach manifold, one has to consider Sobolev connections. Because we want to give only an intuitive picture we do not spell out the details here, but refer to §14 of [At,Bo]. For related remarks see the footnotes at the end of Subsection 1.2.3.

[6] For the result see Section 3.4.

[7] Later this was done by Daskalopoulos in [Das].

Under this procedure the self-dual Yang-Mills equation reduces to the so-called *Bogomolny equation*:

$$F(A) = *d_A\phi.$$

Moreover the reduced energy function takes the form

$$S(A) = -\int_{\mathbb{R}^3} \mathrm{tr}(F(A) \wedge *F(A)) + \mathrm{tr}(d_A\phi \wedge *d_A\phi).$$

This procedure is called *dimensional reduction*. Thus the Bogomolny equation is the 1-dimensional reduction of the self-dual Yang-Mills equation.

Physically a solution of finite energy to the Bogomolny equation represents a *magnetic monopole*. It can be shown that any solution to the Bogomolny equation with finite energy has energy $8\pi k$, where k is a positive integer, which is called the *charge* of the monopole.

As in the Introduction we denote by M_k the moduli space of charge k and $SU(2)$ magnetic monopoles, which is the charge k solution space to the $SU(2)$ Bogomolny equation modulo gauge transformations. This is a non-compact smooth manifold of dimension $4k-4$, with an inherited complete hyperkähler metric.

Atiyah and Hitchin could find the metric explicitly on the 4-dimensional manifold M_2 and by examining its geodesics they could provide[8] a description of the low energy scattering of $SU(2)$ magnetic monopoles of charge 2.

1.1.4 Hitchin's self-duality equations in 2 dimensions

In 1987 Hitchin in [Hit1] considered the 2 dimensional reduction of the self-dual Yang-Mills equations, whose moduli space is the central object of the present thesis. We now describe this construction.

If we assume (further) that the Lie algebra valued functions A_i in (1.4) are independent of both x_3 and x_4, then A_1 and A_2 define a connection

$$A = A_1 dx_1 + A_2 dx_2$$

on $\mathbb{R}^2$, while A_3 and A_4 become Lie algebra-valued functions on $\mathbb{R}^2$, which we relabel as ϕ_1 and ϕ_2. Furthermore we introduce $\phi = \phi_1 - i\phi_2$, called the *complex Higgs field*.

From a coordinate independent point of view, we have a connection A on a principal G-bundle P over $\mathbb{R}^2$ together with an auxiliary field

$$\phi \in \Omega^0(\mathbb{R}^2; \mathrm{ad}(P) \otimes \mathbb{C}).$$

If we moreover write $z = x_1 + ix_2$ and introduce

$$\Phi = \frac{1}{2}\phi dz \in \Omega^{1,0}(\mathbb{R}^2; \mathrm{ad}(P) \otimes \mathbb{C})$$

and

$$\Phi^* = \frac{1}{2}\phi^* d\bar{z} \in \Omega^{0,1}(\mathbb{R}^2; \mathrm{ad(P)} \otimes \mathbb{C})$$

then the 2-dimensional reduced self-dual Yang-Mills equation becomes

$$\left.\begin{array}{c} F(A) = -[\Phi, \Phi^*], \\ d''_A\Phi = 0. \end{array}\right\} \tag{1.6}$$

These equations are called *Hitchin's self-duality equations*.

Unfortunately there are no finite energy solutions to Hitchin's self-duality equations on $\mathbb{R}^2$. However by exploiting the conformal invariance of (1.6) we can write down Hitchin's self-duality equations over a compact Riemann surface Σ by demanding A to be a connection on a principal G-bundle P over Σ and the Higgs field to be

$$\Phi \in \Omega^{1,0}(\Sigma; \mathrm{ad}(P) \otimes \mathbb{C}).$$

Fortunately solutions to Hitchin's self-duality equations over a Riemann surface do exist and Hitchin made an extensive study of their parameter or moduli space $\mathcal{M}$ in [Hit1] for the case of $G = SU(2)$ and $G = SO(3)$. We list some of his results in Subsection 1.2.2.

[8] For further details see [At,Hi].

1.1.5 Dirac equations in 2 dimensions

To explain the gauge theoretic origin of the virtual Dirac bundle $\mathbf{D}$ of Chapter 5, following Hitchin [Hit4], we first consider the Dirac equation in $\mathbb{R}^4$.

Let ψ_1 and ψ_2 be scalar functions. The ordinary *Dirac equation* is of the form:

$$\left(\frac{\partial}{\partial x_1} + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{\partial}{\partial x_2} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial x_3} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_4}\right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

If A is a self-dual Yang-Mills connection on P, and $\mathcal{V}$ is the vector bundle associated to P in a vector representation, then the Dirac equation coupled to A is:

$$\not{D}_A(\psi) = 0,$$

where ψ is a twisted spinor and the Dirac operator coupled to A

$$\not{D}_A : \Gamma(S_+ \otimes \mathcal{V}) \to \Gamma(S_- \otimes \mathcal{V})$$

is given by

$$\not{D}_A = \nabla_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \nabla_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_3 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \nabla_4,$$

where

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i.$$

After the dimensional reduction we consider a connection on a principal G-bundle P over a Riemann surface Σ, the rank 2 vector bundle $\mathcal{V}$ is associated to P in a suitable vector representation and ϕ a complex Higgs field. Then the above coupled Dirac equation takes the following shape:

$$\left.\begin{aligned} \bar{\partial}_A \psi_1 + \phi \psi_2 = 0 \\ \partial_A \psi_2 + \phi^* \psi_1 = 0, \end{aligned}\right\} \tag{1.7}$$

now with

$$\psi_1 \in \Omega^{1,0}(\Sigma, \mathcal{V})$$

and

$$\psi_2 \in \Omega^{0,1}(\Sigma, \mathcal{V}).$$

Suppose that (A, ϕ) is a solution to Hitchin's self-duality equations (1.6) and consider the vector space of solutions to the equation (1.7). Using Hodge theory of elliptic complexes it can be shown that this vector space is canonically isomorphic to the Dolbeault definition of the hypercohomology[9] vector space $\mathbb{H}^1(\Sigma, E_A \xrightarrow{\Phi} E_A \otimes K)$, where E_A is the holomorphic vector bundle corresponding to the connection A on $\mathcal{V}$, and $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K)$ is the corresponding Higgs field. Thus we can assign to any point $(A, \Phi) \in \mathcal{M}$ the complex vector space $\mathbb{H}^1(\Sigma, E_A \xrightarrow{\Phi} E_A \otimes K)$. This will not be a vector bundle in general but only a coherent sheaf $\mathbf{D}$, which we call the *virtual Dirac bundle*. We will define $\mathbf{D}$ rigorously in Chapter 5 and use it to prove Theorem 0.2.1.

[9] For the Čech definition see Subsection 2.2.3

1.2 Algebraic Geometry of vector bundles on curves

As we already mentioned [At,Bo] established a link between Yang-Mills theory on Riemann surfaces and the algebraic geometry of vector bundles over projective algebraic curves via the theorem of Narasimhan and Seshadri. Recall that, in a suitable form, this theorem asserts that there is a one-to-one correspondence between rank n semi-stable vector bundles and those connections on a principal $U(n)$-bundle, which give absolute minima of the Yang-Mills functional (1.2).

In this section we recall some definitions and results from the theory of vector bundles over projective algebraic curves, which we will need later.

1.2.1 Moduli spaces of stable bundles

Algebraically a Riemann surface Σ is the same as a non-singular complex projective algebraic curve. Moreover holomorphic vector bundles over Σ correspond to complex algebraic vector bundles. We will not distinguish between them.

The main problem of the subject is to classify all vector bundles[10] over the curve Σ. This was done for genus 0 curves by Grothendieck in [Gro], and for genus 1 curves by Atiyah in [Ati2], both in 1957. However the genus at least 2 case has proved to be much more difficult. From now on we restrict our attention to the genus at least 2 case.

First recall that C^∞ vector bundles over Σ are classified by their ranks and degrees. Thus we will concentrate on $\mathcal{C}$, the complex affine space of holomorphic structures on a fixed C^∞ complex vector bundle $\mathcal{V}$ over Σ of rank n and degree d. Certainly we do not want to distinguish between isomorphic vector bundles, therefore we consider the complexified gauge group $\mathcal{G}^c = \mathrm{Aut}(\mathcal{V})$ of complex automorphisms of $\mathcal{V}$, which acts naturally on $\mathcal{C}$. Since an orbit of this action is clearly the set of vector bundles isomorphic to a given one, the main problem reduces to understand the orbits of this action. However the space $\mathcal{C}/\mathcal{G}^c$, though clearly parametrizes all vector bundles with underlying C^∞ bundle $\mathcal{V}$, is not even Hausdorff[11], hence its description is rather hopeless. One attempt of getting around this problem was provided by Mumford's Geometric Invariant Theory[12] in the 60's. The idea is to take a $\mathcal{G}^c$-invariant open subset $\mathcal{C}_s$ of $\mathcal{C}$ in order to get a Hausdorff space $\mathcal{C}_s/\mathcal{G}^c$, and indeed, as we will see later, a smooth algebraic variety, and in good cases a projective one.

Definition 1.2.1 *The* slope *of a vector bundle E on Σ is defined by:*

$$\mu(E) = \deg(E)/\operatorname{rank}(E).$$

Moreover E is semi-stable *(resp.* stable*) if it has at least as large (resp. strictly larger) slope than any of its proper subbundles. Finally $\mathcal{C}_{ss} \subset \mathcal{C}$ denotes the open set of semi-stable, $\mathcal{C}_s \subset \mathcal{C}$ the open set of stable bundles.*

In some sense semi-stable bundles are the analogues of simple finite groups in the classification of finite groups. For example one has the analogue of the Jordan-Hölder theorem:

Theorem 1.2.2 (Harder, Narasimhan) *Every holomorphic bundle E has a* canonical filtration*:*

$$0 = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_r = E, \tag{1.8}$$

with $D_i = E_i/E_{i-1}$ semi-stable and

$$\mu(D_1) > \mu(D_2) > \ldots > \mu(D_r).$$

Thus as in the classification of finite groups, one first starts with the classification of semi-stable vector bundles. As we mentioned above, Geometric Invariant Theory constructs a projective algebraic variety for the parameter space of semi-stable vector bundles:

Theorem 1.2.3 *When $(n,d) = 1$ then $\mathcal{C}_s = \mathcal{C}_{ss}$ and the moduli space $N(n,d) = \mathcal{C}_s/\mathcal{G}^c$ of rank n and degree d stable bundles over Σ is a smooth projective algebraic variety of dimension $n^2(g-1)+1$.*

[10] A vector bundle always means an algebraic vector bundle, unless otherwise stated.
[11] This is because of the so-called *jumping phenomenon*.
[12] Cf. [M,F,K]

Example. 1. Clearly every line bundle is stable thus $N(1,d)$ contains all line bundles of degree d. This space is the well-known Jacobian, which we relabel as $\mathcal{J}_d = N(1,d)$, and use $\mathcal{J}$ for $\mathcal{J}_1$. This is an Abelian variety of dimension g. In this case the classification problem is clearly settled.

2. The next moduli spaces in the list are the rank 2 moduli spaces $N(2,d)$, with d odd. These are all isomorphic, thus we restrict our attention to $N(2,1)$, which we rename as $\widetilde{\mathcal{N}}$. It is a smooth projective variety of dimension $4g-3$. The determinant gives a map $det_{\mathcal{N}} : \widetilde{\mathcal{N}} \to \mathcal{J}$. For any $\Lambda \in \mathcal{J}$ the fibre $det_{\mathcal{N}}^{-1}(\Lambda)$ will be denoted by $\mathcal{N}_\Lambda$, which is a smooth projective variety of dimension $3g-3$. The map $f : \mathcal{N}_{\Lambda_1} \to \mathcal{N}_{\Lambda_2}$ given by $f(E) = E \otimes (\Lambda_2 \otimes \Lambda_1^*)^{1/2}$, where $(\Lambda_2 \otimes \Lambda_1^*)^{1/2}$ is a fixed square root of $\Lambda_2 \otimes \Lambda_1^*$, is an isomorphism between $\mathcal{N}_{\Lambda_1}$ and $\mathcal{N}_{\Lambda_2}$. Hence we will write $\mathcal{N}$ for $\mathcal{N}_\Lambda$, when we do not want to emphasize the fixed line bundle Λ. In words $\mathcal{N}$ is the moduli space of stable rank 2 bundles of fixed determinant of degree 1 over the Riemann surface Σ.

The moduli spaces $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ will appear all along in this thesis. They have been much studied over the years. In particular their cohomology rings have been described completely. We explain some of the results in this direction in Section 3.4.

1.2.2 Moduli spaces of Higgs bundles

Hitchin in Theorem 4.3 of [Hit1] proved a generalization of the above mentioned theorem of Narasimhan and Seshadri, which linked the solutions of his self-duality equations (1.6) to algebro-geometric objects, so-called stable Higgs bundles[13]:

Definition 1.2.4 *The complex $E \xrightarrow{\Phi} E \otimes K$ with E a vector bundle on Σ, K the canonical bundle of Σ, and $\Phi \in H^0(\Sigma, \mathrm{Hom}(E, E \otimes K))$, is called a* Higgs bundle[14]*, while Φ is called the* Higgs field*.*

The slope $\mu(\mathcal{E})$ of a Higgs bundle $\mathcal{E} = E \xrightarrow{\Phi} E \otimes K$ is defined as the slope[15] $\mu(E)$ of its vector bundle E. A Higgs bundle is called semi-stable *(resp.* stable*) if it has at least as large (resp. strictly larger) slope than any of its proper Φ-invariant subbundles.*

Thus Hitchin's gauge theoretic construction of $\mathcal{M}$, the moduli space of solutions to Hitchin's self-duality equations, with fixed determinant connection as explained in [Hit1], in turn is isomorphic to the moduli space of rank 2 stable Higgs bundles with trace-free Higgs field and fixed determinant of odd degree. This space is the central moduli space of the present thesis: We fix a degree 1 line bundle Λ over Σ, and denote by $\mathcal{M}$ the moduli space of rank 2 stable Higgs bundles with trace free Higgs field and determinant Λ.

The moduli space $\mathcal{M}$ has many features which show its importance. Probably the most important is that the cotangent bundle of $\mathcal{N}$, (defined above) sits inside $\mathcal{M}$ as an open dense subset. Namely, $(T^*_{\mathcal{N}})_E$ is canonically isomorphic to $H^0(\Sigma, \mathrm{End}_0(E) \otimes K_\Sigma)$ thus the points of $T^*_{\mathcal{N}}$ are stable Higgs bundles.

After introducing the space $\mathcal{M}$, Hitchin gave its extensive description in [Hit1], [Hit2]. Here we restate those results, which we use later.

- $\mathcal{M}$ is a noncompact, smooth complex manifold of complex dimension $6g-6$ containing $T^*_{\mathcal{N}}$ as a dense open set.

- Furthermore $\mathcal{M}$ is canonically a Riemannian manifold with a complete hyperkähler metric. Thus $\mathcal{M}$ has complex structures parameterized by S^2. One of the complex structures, for which $T^*_{\mathcal{N}}$ is a complex submanifold, is distinguished, call it I. We will only be concerned with this complex structure here. The others (apart from $-I$) are biholomorphic to each other and give $\mathcal{M}$ the structure of a Stein manifold. From the corresponding Kähler forms one can build a holomorphic symplectic form ω_h on $(\mathcal{M}, I)$.

- There is a map, called the Hitchin map

$$\chi : \mathcal{M} \to H^0(\Sigma, K^2) = \mathbb{C}^{3g-3}$$

defined by

$$(E, \Phi) \mapsto \det \Phi.$$

---

[13] For a more thorough treatment see Subsection 5.1.

[14] The term *Higgs bundle* was first used by Simpson in [Sim1].

[15] Cf. Definition 1.2.1

The Hitchin map is proper and an algebraically completely integrable Hamiltonian system with respect to the holomorphic symplectic form ω_h, with generic fibre a Prym variety corresponding to the spectral cover of Σ at the image point.

- Let ω denote the Kähler form corresponding to the complex structure I. There is a holomorphic $\mathbb{C}^*$-action on $\mathcal{M}$ defined by $(E,\Phi)\mapsto(E,z\cdot\Phi)$. The restricted action of $U(1)$ defined by $(E,\Phi)\to(E,e^{i\theta}\Phi)$ is isometric and indeed Hamiltonian with proper moment map μ. The function μ is a perfect Morse function, moreover:

  μ has g critical values: an absolute minimum $c_0=0$ and $c_d=(d-\frac{1}{2})\pi$, where $d=1,...,g-1$.

  $\mu^{-1}(c_0)=\mu^{-1}(0)=F_0=\mathcal{N}$ is a non-degenerate critical manifold of index 0.

  $\mu^{-1}(c_d)=F_d$ is a non-degenerate critical manifold of index $2(g+2d-2)$ and is diffeomorphic to a 2^{2g}-fold cover of the symmetric product $\Sigma_{\bar{d}}$, where we used the notation $\bar{d}=2g-2d-1$.

- The fixed point set S of the involution $\sigma(E,\Phi)=(E,-\Phi)$ is the union of g complex submanifolds of $\mathcal{M}$ namely,

$$S=\mathcal{N}\cup\bigcup_{d=1}^{g-1}E_d^2,$$

  where E_d^2 is the total space of a vector bundle E_d^2 over Z_d. Moreover, E_d^2 is a complex submanifold of dimension $3g-3$.

1.2.3 The moduli space of Higgs k-bundles

Using Geometric Invariant Theory, Nitsure in [Nit] gave an algebraic construction of $\mathcal{M}$ and many other related spaces. As we will use some of them in Chapter 6, we define them here:

Definition 1.2.5 *Let $k\geq 0$ and $\widetilde{\mathcal{M}}_k$ denote the moduli space of stable rank 2 Higgs bundles of degree 1, with poles of order at most k at a fixed point $p\in\Sigma$. A Higgs bundle with pole is a complex $E\xrightarrow{\Phi}E\otimes K\otimes L_p^k$ where E is a rank 2 vector bundle over Σ, the line bundle L_p corresponds to the divisor $p\in\Sigma$ and the Higgs field with poles: $\Phi\in H^0(\Sigma,\mathrm{End}(E)\otimes K\otimes L_p^k)$. For convenience we call such a complex a* Higgs k-bundle *and Φ a* Higgs k-field. *Moreover we call $E\xrightarrow{\Phi}E\otimes K\otimes L_p^k$ stable if the slope of any Φ-invariant line subbundle of E is strictly smaller than $\mu(E)$.*

Proposition 7.4 of [Nit] then tells us:

Theorem 1.2.6 (Nitsure) *The space $\widetilde{\mathcal{M}}_k$ is a smooth quasi-projective variety of dimension*

$$8(g-1)+1+4k+\dim(H^1(\Sigma,K\otimes L_p^k)).$$

As in the case of $\widetilde{\mathcal{N}}$, the determinant map gives a map

$$det_{\mathcal{M}_k}:\widetilde{\mathcal{M}}_k\to\mathcal{J}\times H^0(\Sigma,K\otimes L_p^k),$$

defined by $det_{\mathcal{M}_k}(E,\Phi)=(\Lambda^2E,\mathrm{tr}(\Phi))$. For any $\mathcal{L}\in\mathcal{J}\times H^0(\Sigma,K\otimes L_p^k)$ the fibre $det^{-1}_{\mathcal{M}_k}(\mathcal{L})$ will be denoted by $\mathcal{M}^k_{\mathcal{L}}$. Just as in the stable vector bundle case any two fibres of $det_{\mathcal{M}_k}$ are isomorphic. Usually we will write $\mathcal{M}_k$ for $\mathcal{M}^k_{\mathcal{L}}$, when the Abelian Higgs bundle, with order k pole, $\mathcal{L}$ has zero Higgs field. The dimension of $\mathcal{M}_k$ is clearly

$$8(g-1)+1+4k+\dim(H^1(\Sigma,K\otimes L_p^k))-(g+\dim(H^0(\Sigma,K\otimes L_p^k)))=6(g-1)+3k.$$

By definition $\mathcal{M}_0=\mathcal{M}$, hence $\dim(\mathcal{M})=6(g-1)$, which checks up with the dimension calculated by Hitchin. We will also use $\widetilde{\mathcal{M}}$ for $\widetilde{\mathcal{M}}_0$.

A consequence of the Geometric Invariant Theory construction of Nitsure is the following[16]:

Corollary 1.2.7 *The spaces $\widetilde{\mathcal{M}}_k$ and consequently $\mathcal{M}_k$ are quasi-projective varieties.*

In particular Hitchin's moduli space $\mathcal{M}$ is also a quasi projective variety. We will examine in a certain sense the canonical compactification of $\mathcal{M}$ in Chapter 4.

Before we proceed to the next section we insert here a gauge theoretic construction of Nitsure's spaces $\widetilde{\mathcal{M}}_k$, which is more in the spirit of [At,Bo] and [Hit1] and shall be used throughout the thesis.

[16] Cf. Remark 5.12 in [Nit].

Gauge theoretic construction of $\widetilde{\mathcal{M}}_k$

We denote by $\mathcal{C}$ the complex affine space of holomorphic structures[17] on a fixed rank 2 smooth, complex vector bundle $\mathcal{V}$ of degree 1. For any integer k consider the infinite dimensional vector spaces[18]

$$\Omega_k^{q,r} = \Omega^{q,r}\left(\Sigma, \mathrm{End}(\mathcal{V}) \otimes \mathcal{L}_p^k\right),$$

where by $\mathcal{L}_p^k$ we denoted the smooth line bundle underlying the line bundle L_p^k, which is the line bundle of the divisor kp. Now fix $k \geq 0$ and define a map

$$\overline{\partial}_k : \mathcal{C} \times \Omega_k^{1,0} \to \Omega_k^{1,1} \tag{1.9}$$

by sending the pair (E, ϕ) to $\overline{\partial}_k^E \phi$, where

$$\overline{\partial}_k^E : \Omega_k^{q,r} \to \Omega_k^{q,r+1}$$

is the $\overline{\partial}$ operator associated to the holomorphic structure $\mathrm{End}(E) \otimes L_p^k$ on $\mathrm{End}(\mathcal{V}) \otimes \mathcal{L}_p^k$. It is characterized by the property that for a local section ϕ the equation $\overline{\partial}_k^E \phi = 0$ holds if and only if ϕ is holomorphic with respect to the holomorphic structure $\mathrm{End}(E) \otimes L_k^p$ on $\mathrm{End}(\mathcal{V}) \otimes \mathcal{L}_p^k$.

Now we define

$$\mathcal{B}_k = \overline{\partial}_k^{-1}(0) \subset \mathcal{C} \times \Omega_k$$

to be the subspace of pairs $(E, \phi) \in \mathcal{C} \times \Omega_k$ with ϕ being holomorphic. We denote by $\mathrm{pr}_k : \mathcal{B}_k \to \mathcal{C}$ the projection. Occasionally $\mathcal{B}$ will stand for $\mathcal{B}_0$ and pr for pr_0.

Now we have the complexified gauge group $\mathcal{G}^c = \Gamma(\mathrm{Aut}(\mathcal{V}))$, the group[19] of complex automorphisms of $\mathcal{V}$, acting on $\mathcal{C}$ and on $\Omega_k^{1,0}$, which induces an action on $\mathcal{B}_k$. Let us denote by $(\mathcal{B}_k)^s \subset \mathcal{B}_k$ the subspace of Higgs k-bundles which are stable. The subset $(\mathcal{B}_k)^s \subset \mathcal{B}_k$ is clearly invariant under the complex gauge group $\mathcal{G}^c$. Then we can form the quotient $(\mathcal{B}_k)^0/\mathcal{G}^c$, which is exactly the moduli space of stable Higgs k-bundles $\widetilde{\mathcal{M}}_k$, we are after.

[17]To make later heuristic arguments about infinite dimensional manifolds precise we need to choose holomorphic structures of Sobolev class L_1^2.

[18]To be precise for $(q, r) = (0, 1)$ and $(1, 0)$ we consider sections of Sobolev class L_1^2 for $(1, 1)$ of class L^2.

[19]To make later arguments precise we have to consider gauge transformations of Sobolev class L_2^2

Chapter 2

Geometry of manifolds

This chapter deals with the symplectic geometry and topology of manifolds in general. The results appearing here will be used in the second part for the moduli space of Higgs bundles.

2.1 $\mathbb{C}^*$-actions on Kähler manifolds

In this section we collect the results from the literature concerning $\mathbb{C}^*$-actions on Kähler manifolds. At the same time we sketch the structure of Chapter 4.

2.1.1 Stratifications

Suppose that we are given a Kähler manifold (M, I, ω) with complex structure I and Kähler form ω. Suppose also that $\mathbb{C}^*$ acts on M biholomorphically with respect to I and such that the Kähler form is invariant under the induced action of $U(1) \subset \mathbb{C}^*$. Suppose furthermore that this latter action is Hamiltonian with proper moment map $\mu : M \to \mathbb{R}$, with finitely many critical points and 0 being the absolute minimum of μ. Let $\{F_\lambda\}_{\lambda \in A}$ be the set of the components of the fixed point set of the $\mathbb{C}^*$-action.

We list some results of [Kir1] extended to our case. Namely, Kirwan's results are stated for compact Kähler manifolds, but one can usually modify the proofs for non-compact manifolds, with proper moment maps as above[1].

There exist two stratifications in such a situation. The first one is called the *Morse stratification* and can be defined as follows. The stratum U_λ^M, the so-called *upward Morse flow from* F_λ, is the set of points of M whose path of steepest descent for the Morse function μ and the Kähler metric have limit points in F_λ. One can also define the sets D_λ^M, the so-called *downward Morse flow of* F_λ, as the points of M whose path of steepest descent for the Morse function $-\mu$ and the Kähler metric have limit points in F_λ. U_λ^M gives a stratification even in the non-compact case, however the set $\bigcup_\lambda D_\lambda^M$ is not the whole space but a deformation retract of it. The set $\bigcup_\lambda D_\lambda^M$ is called the *downward Morse flow*.

The other stratification is the *Bialynicki-Birula stratification*, where the stratum U_λ^B is the set of points $p \in M$ for which $\lim_{t \to 0} tp \in F_\lambda$. Similarly, as above, we can define D_λ^B as the points $p \in M$ for which $\lim_{t \to \infty} tp \in F_\lambda$.

One of Kirwan's important results in [Kir1] Theorem 6.16 asserts that the stratifications U_λ^M and U_λ^B coincide, and similarly $D_\lambda^M = D_\lambda^B = D_\lambda$. This result is important because it shows that the strata $U_\lambda = U_\lambda^M = U_\lambda^B$ of the stratifications are total spaces of affine bundles (so-called β-fibrations) on F_λ (this follows from the Bialynicki-Birula picture) and moreover this stratification is responsible for the topology of the space M (this follows from the Morse picture). Thus we have the following theorem[2]:

Theorem 2.1.1 *U_λ and D_λ are complex submanifolds of M. They are isomorphic to total spaces of some β-fibrations over F_λ, such that the normal bundles of F_λ in these β-fibrations are E_λ^+ and E_λ^-, respectively, where E_λ^+ is the positive and E_λ^- is the negative subbundle of $D_M \mid_{F_\lambda}$ with respect to the $U(1)$-action. Moreover, the downward Morse flow $\bigcup_\lambda D_\lambda$ is a deformation retract of M.*

[1] cf. Chapter 9 in [Kir1].

[2] Cf. Theorem 4.1 of [Bia] and also Theorem 1.12 of [Tha3], for the statement about the downward Morse flow cf. § 3 of [Sim4].

Recall that a β-fibration in our case is a fibration $E \to B^n$ with a $\mathbb{C}^*$-action on the total space which is locally like $\mathbb{C}^n \times V$, where V is the $\mathbb{C}^*$-module $\beta : \mathbb{C}^* \to GL(V)$. Note that such a fibration is not a vector bundle in general, but it is if β is the sum of isomorphic, one-dimensional non-trivial $\mathbb{C}^*$-modules.

2.1.2 Kähler quotients

Whenever we are given a Hamiltonian $U(1)$-action on a Kähler manifold with a proper moment map, we can form the Kähler quotients $Q_t = \mu^{-1}(t)/U(1)$, which are compact Kähler orbifolds at a regular value t of μ.

If this $U(1)$-action is induced from an action of $\mathbb{C}^*$ on M as above, then we can relate the Kähler quotients to the quotients $M/\mathbb{C}^*$ as follows. First we define $M_t^{min} \subset M$ as the set of points in M whose $\mathbb{C}^*$ orbit intersects $\mu^{-1}(t)$. Now Theorem 7.4 of [Kir1] states that it is possible to define a complex structure on the orbit space $M_t^{min}/\mathbb{C}^*$, and she also proves that this space is homeomorphic to Q_t, defining the complex structure for the Kähler quotient Q_t. (Here again we used the results of Kirwan for non-compact manifolds, but as above, these results can be easily modified for our situation.) It now simply follows that M_t^{min} only depends on that connected component of the regular values of μ in which t lies, and as a consequence of this we can see that the complex structure on Q_t is the same as on $Q_{t'}$ if the interval $[t, t']$ does not contain any critical value of μ. We have as a conclusion the following theorem:

Theorem 2.1.2 *At a regular level $t \in \mathbb{R}$ of the moment map μ, we have the Kähler quotient $Q_t = \mu^{-1}(t)/U(1)$ which is a compact Kähler orbifold with M_t^{min} as a holomorphic $\mathbb{C}^*$-principal orbibundle on it. Moreover M_t^{min} and the complex structure on Q_t only depend on that connected component of the regular values of μ where t lies.*

It follows from the above theorem that there is a discrete family of complex orbifolds which arise from the above construction. Moreover, at each level we get a Kähler form on the corresponding complex orbifold. The evolution of the different Kähler quotients has been well investigated[3]. We can summarize these results in the following theorem:

Theorem 2.1.3 *The Kähler quotients Q_t and $Q_{t'}$ are biholomorphic if the interval $[t, t']$ does not contain a critical value of the moment map. They are related by a blowup followed by a blow-down if the interval $[t, t']$ contains exactly one critical point c different from the endpoints. To be more precise, Q_t blown up along the union of submanifolds $\bigcup_{\mu(F_\lambda)=c} P_w(E_\lambda^-)$ is isomorphic to $Q_{t'}$ blown up along $\bigcup_{\mu(F_\lambda)=c} P_w(E_\lambda^+)$ and in both cases the exceptional divisor is $\bigcup_{\mu(F_\lambda)=c} P_w(E_\lambda^+) \times_{F_\lambda} P_w(E_\lambda^-)$ the fibre product of weighted projective bundles over F_λ.*

Moreover, in a connected component of the regular values of μ the cohomology classes of the Kähler forms $\omega_t(Q_t)$ depend linearly on t according to the formula:

$$[\omega_t(Q_t)] - [\omega_{t'}(Q_{t'})] = (t - t')c_1(M_t^{min}) = (t - t')c_1(M_{t'}^{min}),$$

where c_1 is the first Chern class of the $\mathbb{C}^*$-principal bundle.

2.1.3 Symplectic cuts

Now let us recall the construction of Lerman's symplectic cut[4], first in the symplectic and then in the Kähler category.

If (M, ω) is a symplectic manifold with a Hamiltonian $U(1)$-action and proper moment map μ with absolute minimum 0, then we can define the symplectic cut of M at the regular level t by a symplectic quotient construction as follows.

We let $U(1)$ act on the symplectic manifold $M \times \mathbb{C}$ (where the symplectic structure is the product of the symplectic structure on M and the standard symplectic structure on $\mathbb{C}$) by acting on the first factor according to the above $U(1)$-action and on the second factor by the standard multiplication. This action is clearly Hamiltonian with proper moment map $\mu + \mu_{\mathbb{C}}$, where $\mu_{\mathbb{C}}$ is the standard moment map on $\mathbb{C}$: $\mu_C(z) = |z|^2$.

[3] E.g. in the papers [Du,He], [Gu,St], cf. also [Tha3] and [Br,Pr].
[4] Cf. [Ler] and also [EdGr1] for the algebraic case.

Now if t is a regular value of the moment map $\mu + \mu_{\mathbb{C}}$, such that $U(1)$ acts with finite stabilizers on $M_t = \mu^{-1}(t)$ (i.e. $M_t/U(1)$ gives a symplectic orbifold), then the symplectic quotient $\overline{M}_{\mu<t}$ defined by

$$\overline{M}_{\mu<t} = \{(m,w) \in M \times \mathbb{C} : \mu(m) + |w|^2 = t\}$$

will be a symplectic compactification of the symplectic manifold $M_{\mu<t}$ in the sense that

$$\overline{M}_{\mu<t} = M_{\mu<t} \cup Q_t,$$

and the inherited symplectic structure on $\overline{M}_{\mu<t}$ restricted to $M_{\mu<t}$ coincides with its original symplectic structure. Moreover, if we restrict this structure onto Q_t, it coincides with its quotient symplectic structure.

Now suppose that we are given a Kähler manifold (M, I, ω) and a holomorphic $\mathbb{C}^*$-action on it, such that the induced $U(1) \subset \mathbb{C}^*$-action preserves the Kähler form and is Hamiltonian with proper moment map. With these extra structures the symplectic cut construction will give us $\overline{M}_{\mu<t}$ a compact Kähler orbifold with a $\mathbb{C}^*$-action, such that $\overline{M}_{\mu<t} \setminus Q_t$ is symplectomorphic to $M_{\mu<t}$ as above and furthermore is biholomorphic to $\mathbb{C}^*(M_{\mu<t})$, the union of $\mathbb{C}^*$-orbits intersecting $M_{\mu<t}$. This is actually an important point[5], as it shows that $M_{\mu<t}$ is *not* Kähler isomorphic to $\overline{M}_{\mu<t} \setminus Q_t$. We can collect all these results into the next theorem:

Theorem 2.1.4 *The symplectic cut $\overline{M}_{\mu<t} = M_{\mu<t} \cup Q_t$ as a symplectic manifold compactifies the symplectic manifold $M_{\mu<t}$, such that the restricted symplectic structure on Q_t coincides with the quotient symplectic structure.*

Furthermore, if M is a Kähler manifold with a $\mathbb{C}^*$-action as above, then $\overline{M}_{\mu<t}$ will be a Kähler orbifold with a $\mathbb{C}^*$-action, such that Q_t with its quotient complex structure is a codimension 1 *complex suborbifold of $\overline{M}_{\mu<t}$ whose complement is equivariantly biholomorphic to $\mathbb{C}^*(M_{\mu<t})$ with its canonical $\mathbb{C}^*$-action.*

Remark. Note that if t is higher than the highest critical value (this assumes that we have finitely many of them), then $\mathbb{C}^*(M_{\mu<t}) = M$ is the whole space, therefore the symplectic cutting in this case gives a holomorphic compactification of M itself. The compactification is $\overline{M}_{\mu<t}$, which is equal to the quotient of $(M \times \mathbb{C} - N \times \{0\})$ by the action of $\mathbb{C}^*$, where N is the downward Morse flow. This is the compactification we shall examine in Chapter 4 for the case of $\mathcal{M}$, the moduli space of stable Higgs bundles with fixed determinant of degree 1.

[5] Cf. [Ler].

2.2 Cohomologies

In this section we explain two –not so well known– cohomology theories, which will appear in the thesis: Equivariant cohomology of stratified spaces and at the end hypercohomology of complexes. We have also inserted some generalities about the Porteous' theorem, which will enable us to geometrically present equivariant cohomology classes.

2.2.1 Equivariant cohomology of stratified spaces

Let G be a topological group. Let its universal fibration be $EG \to BG$. For any G-space X we define $X_G := (X \times EG)/G$, where G acts on $X \times EG$ with the diagonal action. The G-equivariant cohomology of X is then defined as $H^*_G(X) = H^*(X_G)$. Since G acts freely on EG we have the fibration

$$X \to X_G \to BG. \tag{2.1}$$

An immediate consequence of this is that if X is contractible, then the fibration (2.1) has contractible fibres, thus for a contractible space $H^*_G(X) = H^*(BG)$. In particular $H^*_G(pt) = H^*(BG)$ is not trivial. Since H^*_G is a contravariant functor the map $X \to pt$ gives rise to a ring homomorphism $H^*(BG) \to H^*_G(X)$. Thus we see that $H^*_G(X)$ is a module over $H^*(BG)$. Because of this we see that for a nontrivial G, equivariant cohomology is richer than ordinary cohomology.

In case G acts freely on X we have another fibration

$$EG \to X_G \to X/G, \tag{2.2}$$

which shows that for a free action:

$$H^*_G(X) \cong H^*(X/G).$$

Equivariant cohomology can be calculated particularly well for G stratified spaces. Following §1 of [At,Bo], we now explain this.

Let M be a manifold. A disjoint set $\{M_\lambda\}_{\lambda\in I}$ of locally closed G-invariant submanifolds of M indexed by a partially ordered set I, with minimal element 0, defines a *stratification* of M if

$$M = \bigcup_{\lambda\in I} M_\lambda$$

and

$$\overline{M}_\lambda \subset \bigcup_{\mu\geq\lambda} M_\mu, \tag{2.3}$$

moreover we assume that $M_0 \neq \emptyset$, consequently it is the unique open stratum. Because we will encounter stratifications of Banach manifolds with strata of *finite codimension*, such that I is countable infinite, we make two extra finiteness conditions on such stratifications:

Condition 1 *For every finite subset $A \subset I$ there are a positive, finite number of minimal elements in the complement $I \setminus A$.*

This will ensure that our inductive arguments still apply. Although the inductions never terminate for our purposes the following condition will do:

Condition 2 *For each integer q there are only finitely many indices $\lambda \in I$ such that* $\operatorname{codim}(M_\lambda) < q$.

Given a G-equivariant stratification on M, as defined above, one can use Morse theory type arguments to get information about the G-equivariant cohomology of M. To explain this define a subset $J \subset I$ of the indices to be *open* if $\lambda \in J$ and $\mu \leq \lambda$ yields $\mu \in J$. It follows from (2.3) that if J is open then

$$M_J = \bigcup_{\lambda\in J} M_\lambda$$

is open in M. Now if J is open and $\lambda \in I \setminus J$ is minimal then $J' = J \cup \lambda$ will be open. From (2.3) it follows that $M_\lambda = M_{J'} \setminus M_J$ is a closed submanifold of $M_{J'}$ of index, say, k_λ. Using the Thom isomorphism we get the long exact sequence of the pair $(M_{J'}, M_J)$ in the form:

$$\to H^{q-k_\lambda}_G(M_\lambda) \xrightarrow{(i_\lambda)_*} H^q_G(M_{J'}) \xrightarrow{i_J^*} H^q_G(M_J) \to, \tag{2.4}$$

Now we say that the stratification is *G-perfect* if

$$i_J^* : H_G^*(M_{J'}) \to H_G^*(M_J) \tag{2.5}$$

is a surjection for all open J and λ minimal in $I \setminus J$. In this case it follows that the G-equivariant Poincaré polynomial of M

$$GP_t(M) = \sum_{\lambda \in I} t^{k_\lambda} GP_t(M_\lambda) \tag{2.6}$$

can be calculated in terms of the G-equivariant Poincaré polynomials of the strata.

Thus a G-perfect stratification provides a good understanding of the equivariant cohomology of the space. Consequently useful sufficient conditions for G-perfectness are important. One such criterion is due to [At,Bo]. To explain this consider the long exact sequence (2.4) and let $a \in H_G^*(M_\lambda)$. Observe that if $a \neq 0$ implies $i_\lambda^*((i_\lambda)_*(a)) \neq 0$ then clearly $(i_\lambda)_*$ is an injection. But $i_\lambda^*((i_\lambda)_*(a)) = ae_\lambda$, where e_λ denotes the equivariant Euler class of the normal bundle of M_λ in $M_{J\cup\lambda}$. Therefore if e_λ is not a zero divisor in $H_G^*(M_\lambda)$ then $(i_\lambda)_*$ is an injection. Thus a sufficient condition for perfectness is to demand $e_\lambda \in H_G^*(M_\lambda)$ to be a non zero divisor for each $\lambda \in I$. We call such a stratification a *strongly G-perfect* stratification.

In the case of a strongly G-perfect stratification we have an even better understanding of the cohomology of M. Following the paper [Kir2] of Kirwan[6], we now explain this.

Let $J \subset I$ be an open subset, $i_J : M_J \to M$ be the embedding, let $\lambda \in I \setminus J$ be minimal and $J' = J \cup \lambda$. In the case of a strongly G-perfect stratification (2.4) gives that the restriction map

$$H_G^*(M_{J'}) \to H_G^*(M_\lambda) \oplus H_G^*(M_J)$$

is injective. An inductive argument then shows that the restriction map

$$H_G^*(M_{J'}) \to \bigoplus_{\mu \in J'} H_G^*(M_\mu) \tag{2.7}$$

is injective.

Now define for any subset $A \subset I$:

$$\mathcal{K}_A = \ker\left(i_A^* : H_G^*(M) \to H_G^*(M_A)\right),$$

where

$$M_A = \bigcup_{\lambda \in A} M_\lambda$$

and we set $\mathcal{K}_\emptyset = H_G^*(M)$. Then $\mathcal{K}_A$ is an ideal of $H_G^*(M)$. Moreover for an open $J \subset I$ we have

$$\mathcal{K}_J = \bigcap_{\lambda \in J} \mathcal{K}_\lambda$$

from (2.7). If now J and λ are as above then the long exact sequence (2.4) gives that

$$i_\lambda(\mathcal{K}_J) \subset \langle e_\lambda \rangle \subset H_G^*(M_\lambda),$$

where $\langle e_\lambda \rangle$ is the ideal of $H_G^*(M_\lambda)$ generated by e_λ. Since the stratification is G-perfect the map $i_{J'}^* : H_G^*(M) \to H_G^*(M_{J'})$ is surjective consequently

$$i_\lambda^*(\mathcal{K}_J) = \langle e_\lambda \rangle \subset H_G^*(M_\lambda). \tag{2.8}$$

The following proposition shows that, in the case of a strongly G-perfect stratification, in some sense $\mathcal{K}_J$ is unique with respect to this property.

[6]Though [Kir2] only works with the Shatz stratification, its methods are general enough to extend them to our general setting.

Proposition 2.2.1 (Kirwan) *Let $M = \bigcup_{\lambda \in I} M_\lambda$ be a strongly G-perfect stratification of M. Suppose we are given a subset $\mathcal{R}_\lambda \subset H^*_G(M)$ for each $\lambda \in I$, with the following property:*

$$i^*_\mu(\mathcal{R}_\lambda) = 0 \text{ if } \mu \ngeq \lambda, \text{ and } i^*_\lambda(\mathcal{R}_\lambda) = \langle e_\lambda \rangle \subset H^*_G(M_\lambda), \tag{2.9}$$

where $\langle e_\lambda \rangle$ is the ideal of $H^*_G(M_\lambda)$ generated by the equivariant Euler class of the normal bundle[7] *of M_λ in M. Then for any open subset $J \subset I$ we have*

$$\langle \mathcal{R}_{\bar{J}} \rangle_{\mathbb{Q}} = \mathcal{K}_J = \ker\left(i^*_J : H^*_G(M) \to H^*_G(M_J)\right),$$

where $\langle \mathcal{R}_{\bar{J}} \rangle_{\mathbb{Q}}$ denotes the $\mathbb{Q}$ vector subspace of $H^*_G(M)$, generated additively by $\mathcal{R}_{\bar{J}} = \bigcup_{\mu \notin J} \mathcal{R}_\mu$.

Proof. Though the proof is essentially the same as the proof[8] of Proposition 1 of [Kir1], we give it here for the sake of completeness.

Let us suppose that $a \in H^*_G(M)$ such that $i^*_J(a) = 0$, i.e. $a \in \mathcal{K}_J$. Let λ be a smallest element of $I \setminus J$ and set $J' = J \cup \lambda$. By the assumption (2.9) we have $a_\lambda \in \mathcal{R}_\lambda$ such that $i^*_\lambda(a_\lambda) = i^*_\lambda(a)$. It follows that $i^*_\mu(a - a_\lambda) = 0$ for every $\mu \in J'$, hence $i^*_{J'}(a - a_\lambda) = 0$ from (2.7). An inductive argument[9] now gives $a_\mu \in R_\mu$ for each $\mu \notin J$ such that $a = \sum_{\mu \notin J} a_\mu$.

The result follows. □

Remarks. 1. If $J = \{0\}$ then the above result gives that

$$\left\langle \bigcup_{\mu \neq 0} \mathcal{R}_\mu \right\rangle_{\mathbb{Q}} = \mathcal{K}_{\{0\}} = \ker\left(i^*_0 : H^*_G(M) \to H^*_G(M_0)\right),$$

thus if one can choose $\mathcal{R}$ in a simple form, satisfying the condition (2.9), then one has information about the relations in $H^*_G(M_0)$. This form was actually stated and used by Kirwan in [Kir2] in the proof of the Mumford conjecture[10]. In Section 7.3 we provide a purely geometric proof of the Mumford conjecture using this special case.

2. We will also use another special case of the above proposition in Chapter 6. Namely if $J = \emptyset$, then the proposition says that

$$\left\langle \bigcup_{\mu \in I} \mathcal{R}_\mu \right\rangle_{\mathbb{Q}} = \mathcal{K}_\emptyset = H^*_G(M).$$

Thus if one can find the sets $\mathcal{R}_\mu$ being generated by a subset of $H^*_G(M)$, then the whole $H^*_G(M)$ is generated by this subset. We now give another, closely related, application of this special case of Proposition 2.2.1:

Corollary 2.2.2 *Let $M = \bigcup_{\lambda \in I} M_\lambda$ and $M' = \bigcup_{\lambda \in I} M'_\lambda$ be two strongly G-perfect stratified spaces. Suppose further that a map $f : M' \to M$ is such that $f^{-1}(M_\lambda) = M'_\lambda$ and*

$$f^*_\lambda : H^*_G(M_\lambda) \to H^*_G(M'_\lambda)$$

is surjective for each $\lambda \in I$. Then

$$f^* : H^*_G(M) \to H^*_G(M')$$

is surjective.

Proof. For each $\lambda \in I$ set $\mathcal{R}'_\lambda = f^*(\mathcal{K}_\lambda) \subset H^*_G(M')$, which is an additively closed subspace of $H^*(M')$. From $f^*_\lambda(e_\lambda) = e'_\lambda$ and (2.8) it follows that the sets $\{\mathcal{R}'_\lambda\}_{\lambda \in I}$ satisfy the conditions of Proposition 2.2.1. In particular

$$f^*(H^*_G(M)) = \langle \mathcal{R}'_I \rangle_{\mathbb{Q}} = \mathcal{K}'_\emptyset = H^*_G(M').$$

The result follows. □

Many examples of G-perfect stratifications arise in symplectic geometry. We explain one special case in detail:

[7] We set $e_0 = 1$.
[8] Cf. also Proposition 11 of [Earl].
[9] That this works even in the infinite dimensional case is ensured by Condition 1 and 2 above.
[10] Cf. Section 3.4.

The special case $G = U(1)$.

First note that $BU(1) \sim \mathbb{C}P^\infty$ thus $H^*(BU(1)) \cong \mathbb{Q}[u]$ is a free polynomial algebra on a degree 2 generator u. For convenience we will write $H^*_\circ$ instead of $H^*_{U(1)}$.

Assume that M is a symplectic manifold and $U(1)$ acts on M in a Hamiltonian way with proper moment map as in Subsection 2.1.1. Recall from Subsection 2.1.1 that in this case we get a stratification $M = \bigcup_{\lambda \in I} U_\lambda$, which is a stratification in the sense we defined above, as Kirwan proves in [Kir1]. Moreover an important result of Kirwan in [Kir1] shows that this stratification is always strongly $U(1)$-perfect. Thus we can calculate the $U(1)$-equivariant cohomology of M from (2.6), and even say something about the ring $H^*_\circ(M)$ from Proposition 2.2.1. However in the present case we have an even better understanding of the ring $H^*_\circ(M)$ through the so called *Localization Theorem*:

Theorem 2.2.3 (Localization Theorem) *For any $\psi \in H^*_\circ(M)$ one gets the following equation*[11] *in the localized ring $H^*_\circ(M) \otimes_{\mathbb{Q}[u]} \mathbb{Q}(u)$:*

$$\psi = \sum_{\lambda \in I} \frac{(i_{F_\lambda})_* i^*_{F_\lambda} \psi}{E_\circ(\nu_{F_\lambda})},$$

where $E_\circ(\nu_{F_\lambda}) \in H^*_\circ(F_\lambda) \cong H^*(F_\lambda) \otimes \mathbb{Q}[u]$ is the equivariant Euler class of the normal bundle of F_λ in M.

An easy consequence of this theorem is the following[12]

Corollary 2.2.4 (Kirwan) *The restriction map*

$$H^*_\circ(M) \to \bigoplus_{\lambda \in I} H^*_\circ(F_\lambda) \tag{2.10}$$

is an injection.

Thus in order to understand $H^*_\circ(M)$ it is enough to understand the restriction maps $H^*_\circ(M) \to H^*_\circ(F_\lambda)$ and the ring structures[13] of $H^*(F_\lambda)$. This will be our method of describing the cohomology ring of the moduli space of Higgs bundles in Chapter 6.

2.2.2 Porteous' theorem

First we introduce some notation concerning cohomology classes of subvarieties, which will be used throughout this thesis:

Notation 2.2.5 *If X is an irreducible locally closed subvariety of a smooth algebraic variety Y of codimension d, then $\eta^Y_X \in H^{2d}(Y)$ denotes the cohomology class of $\overline{X}$ in Y. Moreover if G acts on Y and X is G-invariant, then $\eta^{G,Y}_X \in H^{2d}_G(Y)$ denotes the G-equivariant cohomology class of $\overline{X}$.*

If X is an irreducible locally closed and relatively complete[14] *subvariety of Y then $\overline{\eta}^Y_X \in H^{2d}_{cpt}(Y)$ denotes the compactly supported cohomology class of $\overline{X}$ in Y.*

A frequently used technique of the present thesis will be to find some geometric way to calculate the cohomology class η^Y_X. The prototype of such a presentation is when a closed subvariety X is the zero locus of a section s of a vector bundle W of rank $d = \mathrm{codim} X$. In differential topology if the section s is transversal, then we have the geometric formula

$$c_d(W) = \eta^X_Y. \tag{2.11}$$

In algebraic geometry the transversality assumption is replaced by demanding X to have codimension $d = \mathrm{rank}(W)$ in order the geometric formula (2.11) to be true. In this thesis we will need some ramification of this idea.

[11] Recall from Subsection 2.1.1 that $\{F_\lambda\}_{\lambda \in I}$ denotes the fixed point set of the circle action.
[12] Note that it follows also from (2.7).
[13] Since $U(1)$ acts trivially on F_λ, we have $H^*_\circ(F_\lambda) \cong H^*(F_\lambda) \otimes H^*(BU(1))$.
[14] It means that $\overline{X}$ is complete i.e. compact.

To explain it let us reformulate the above construction, in a manner more suitable for this thesis, as follows: Let V be the trivial line bundle on Y. Then s is equivalent to a map $s : V \to W$. Let us denote by $\mathbf{F}$ the virtual bundle $W - V \in K(Y)$. Then we get that $\mathbf{F}$ outside X is an honest vector bundle.

More generally suppose that we are given a virtual bundle $\mathbf{F}$ in $K(Y)$ of rank[15] $d-1 > 0$. Suppose also that $\mathbf{F}\mid_{Y\setminus X}$ is an honest vector bundle outside a closed smooth subvariety $X \subset Y$ of complex codimension d. Then consider the following bit of the cohomology long exact sequence of the pair $(Y, Y \setminus X)$:

$$H^0(X) \xrightarrow{\tau} H^{2d}(Y) \to H^{2d}(Y \setminus X), \tag{2.12}$$

where τ is the Thom map. Since $\mathbf{F}\mid_{Y\setminus X}$ is an honest vector bundle of rank $d-1$, we have that $c_d(\mathbf{F})\mid_{Y\setminus X}= 0$ vanishes. From the exactness of (2.12), we have that $c_d(\mathbf{F}) = \tau(q) = q \cdot \eta_X^Y$ for some $q \in H^0(X) \cong \mathbb{Q}$. If q was non-zero, or especially 1, we would find a nice geometric way expressing η_X^Y as $c_d(\mathbf{F})$. The following special case of Porteous' theorem[16] states that if $\mathbf{F} = W - V$ is a difference of two vector bundles then $q = 1$, choosing X to be the degeneration locus of a homomorphism $f : V \to W$.

Theorem 2.2.6 (Porteous) *Let $f : V \to W$ be a homomorphism of vector bundles over a smooth algebraic variety Y. Let X be the degeneration locus of f, i.e. the subvariety of Y consisting of points, where f fails to be an injection*[17] *Then we have*

$$\eta_X^Y = c_d(W - V),$$

if the codimension of X coincides with $d = \operatorname{rank}(W) - \operatorname{rank}(V) + 1$.

Since (2.12) exists in the equivariant cohomology too, the next corollary is an easy consequence of this Theorem 2.2.6.

Corollary 2.2.7 *Let G act on Y, and V and W be G-equivariant vector bundles, with f a G-equivariant homomorphism. We have*

$$\eta_X^{G,Y} = c_d^G(W - V),$$

if X, as in the above Theorem 2.2.6, has the expected codimension $d = \operatorname{rank}(W) - \operatorname{rank}(V) + 1$.

[15] I.e. 0th Chern class.

[16] Cf. (4.2) of [ACGH].

[17] For a rigorous construction of degeneracy loci cf. [ACGH] p.83. Our degeneracy locus is the k-th degeneracy locus of [ACGH], where $k = \operatorname{rank}(V) - 1$.

2.2.3 Hypercohomology of complexes

In this subsection we recall the notion of hypercohomology of a complex from [Gr,Ha], and list some properties of it, which we will use in Chapter 5 to describe rigorously the virtual Dirac bundle, already mentioned in Subsection 1.7.

Definition 2.2.8 *Let*

$$\mathcal{A} = (A_0 \xrightarrow{d} A_1 \xrightarrow{d} A_2 \longrightarrow ...)$$

be a complex of coherent sheaves A_i over an algebraic variety X. For a covering $\underline{U} = \{U_\lambda\}$ of X and each A_i we get the Čech cochain complex with boundary operator δ:

$$(C^0(\underline{U}, A_i) \xrightarrow{\delta} C^1(\underline{U}, A_i) \xrightarrow{\delta} ...).$$

Clearly d induces operators

$$(C^j(\underline{U}, A_i) \xrightarrow{d} C^j(\underline{U}, A_j)),$$

satisfying $\delta^2 = d^2 = d\delta + \delta d = 0$: and hence gives rise to a double complex

$$\{C^{p,q} = C^p(\underline{U}, A_q); \delta, d\}.$$

The hypercohomology of the complex $\mathcal{A}$ is given by the cohomology of the total complex of the double complex $C^{p,q}$:

$$\mathbb{H}^*(X, \mathcal{A}) = \lim_{\underline{U}} H^*(C^*(\underline{U}), D).$$

Moreover if $\mathcal{A}$ is a complex over X and $f : X \to Y$ is a projective morphism then for every non-negative integer i define the sheaf $\mathbb{R}^i f_*(\mathcal{A})$ over Y by

$$\mathbb{R}^i f_*(\mathcal{A})(U) = \mathbb{H}^i(f^{-1}(U), \mathcal{A}).$$

Finally, define the pushforward of a complex to be:

$$f_!(\mathcal{A}) = \mathbb{R}^0 f_*(\mathcal{A}) - \mathbb{R}^1 f_*(\mathcal{A}) + \mathbb{R}^2 f_*(\mathcal{A}) - \ldots \in K(Y).$$

Remark. In the present thesis we will work only with two-term complexes.

There is one important property of hypercohomology which we will make constant use of. If

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is a short exact sequence of complexes then there is a long exact sequence of hypercohomology vector spaces:

$$0 \to \mathbb{H}^0(X, \mathcal{A}) \to \mathbb{H}^0(X, \mathcal{B}) \to \mathbb{H}^0(X, \mathcal{C}) \to \mathbb{H}^1(X, \mathcal{A}) \to \ldots \tag{2.13}$$

As an example consider the short exact sequence of two-term complexes:

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ 0 & \xrightarrow{0} & A_2 \\ \uparrow & & \uparrow \cong \\ A_1 & \xrightarrow{d} & A_2 \\ \cong \uparrow & & \uparrow \\ A_1 & \xrightarrow{0} & 0 \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 \end{array}$$

The long exact sequence in this case is:

$$0 \to \mathbb{H}^0(X, \mathcal{A}) \to H^0(X, A_1) \to H^0(X, A_2) \to \mathbb{H}^1(X, \mathcal{A}) \to \ldots \tag{2.14}$$

which we will call the hypercohomology long exact sequence of the two-term complex $\mathcal{A} = A_1 \xrightarrow{d} A_2$.

Consequently if $\mathcal{A} = A_1 \xrightarrow{d} A_2$ is a two-term complex over X and $f : X \to Y$ is a projective morphism then we have:

$$0 \to \mathbb{R}^0 f_*(X, \mathcal{A}) \to R^0 f_*(X, A_1) \to R^0 f_*(X, A_2) \to \mathbb{R}^1 f_*(X, \mathcal{A}) \to \dots,$$

a long exact sequence of sheaves over Y.

Chapter 3

Cohomology of moduli spaces

In general the cohomology of moduli spaces obviously plays an important role in the algebraic geometric understanding of the moduli problem. As was recently discovered by physicists[1] the cohomology of the moduli spaces of some physical theories contain physically relevant information about the global aspects of the theory. For an example intersection numbers on these moduli spaces sometimes can be identified with correlation functions, which are the experimentally measurable quantities of the theory.

The main subject of this thesis is the determination of the rational cohomology of $\mathcal{M}$, the moduli space of Higgs bundles. Since we will use the cohomological properties of other moduli spaces, and we would like to convey a general view of the cohomology of moduli spaces of objects on an algebraic curve Σ, we describe in detail the cohomology of some such moduli spaces here.

First we summarize the general picture which is emerging from the rest of the section. Let M be a smooth manifold (not necessarily compact), the moduli space of a moduli problem. In each of the following cases we will find an infinite tower of spaces M_k, (which usually will be the moduli spaces of some deformed moduli problem):

$$M = M_0 \subset M_1 \subset \ldots \subset M_k \subset \ldots,$$

with inclusions $i_k : M_k \to M_{k+1}$, such that $i_k^* : H^*(M_{k+1}) \to H^*(M_k)$ is surjective. From this data we have the direct limit:

$$M_\infty = \varinjlim M_k,$$

and the inverse limit:

$$H^*(M_\infty) = \varprojlim H^*(M_k),$$

since H^* is a contravariant functor. It will turn out that $H^*(M_\infty)$ will be a free graded commutative[2] algebra on a finite set of universal generators. Furthermore, it follows that the map

$$i_M^* : H^*(M_\infty) \to H^*(M)$$

is surjective. Putting everything together, the cohomology ring $H^*(M)$ is generated by the images of the universal generators and the kernel of i_M^* provides the relations, in other words i_M^* is a resolution of the cohomology ring of M. Because of this we call a tower having the above properties a *resolution tower* of M.

In Chapter 7 we will show that the cohomology ring of $\mathcal{M}$, the moduli space of Higgs bundles, can be understood in this general framework too.

[1] Cf. Section 0.1

[2] It means that even degree classes commute with any other class, while odd degree classes anticommute among each other. Note that the cohomology ring of any space is automatically graded commutative.

3.1 The curve Σ

The basic object of this thesis is a fixed, smooth and complex projective curve Σ of genus $g \geq 2$. We also fix a point $p \in \Sigma$.

An additive basis of $H^*(\Sigma)$: $1 \in H^0(\Sigma)$, $\xi_i^\Sigma \in H^1(\Sigma)$, $i = 1, .., 2g$ and the fundamental cohomology class $\sigma^\Sigma \in H^2(\Sigma)$ with the properties that $\xi_i^\Sigma \cdot \xi_{i+g}^\Sigma = -\xi_{i+g}^\Sigma \cdot \xi_i^\Sigma = \sigma^\Sigma$ for $i = 1, .., g$ and otherwise $\xi_i^\Sigma \cdot \xi_j^\Sigma = 0$, will be fixed throughout the thesis.

As a matter of fact the curve Σ is itself a moduli space, the moduli space of its points, or in a more sophisticated way: the moduli space of degree one divisors on Σ, or in other words Σ_1 the first symmetric product.

3.2 The Jacobian $\mathcal{J}$

The moduli space of line bundles of degree k over Σ is the Jacobian $\mathcal{J}_k$. This is an Abelian variety of dimension g. Tensoring by a fixed line bundle of degree $k-l$ gives an isomorphism between $\mathcal{J}_l$ and $\mathcal{J}_k$. We will write $\mathcal{J}$ for $\mathcal{J}_1$.

Being a torus $H^*(\mathcal{J}_k)$ is a free exterior algebra –in particular a free graded commutative algebra– on $2g$ classes $\tau_i \in H^1(\mathcal{J}_k)$ defined by the formula

$$c_1(\mathbb{P}_k) = k \otimes \sigma^\Sigma + \sum_{i=1}^{g} (\tau_i \otimes \xi_{i+g}^\Sigma - \tau_{i+g} \otimes \xi_i^\Sigma) \in H^2(\mathcal{J}_k \times \Sigma) \cong \sum_{r=0}^{2} H^r(\mathcal{J}_k) \otimes H^{2-r}(\Sigma).$$

Here $\mathbb{P}_k$ is the normalized Poincaré bundle, or universal line bundle over $\mathcal{J}_k \times \Sigma$. Universal means that for any $L \in \mathcal{J}_k$:

$$\mathbb{P}_k \mid_{\{L\}\times\Sigma} \cong L$$

and normalized means that $\mathbb{P}_k \mid_{\mathcal{J}_k \times \{p\}}$ is trivial[3].

Being a free exterior algebra, the Poincaré polynomial of $H^*(\mathcal{J})$ is given by

$$P_t(\mathcal{J}_k) = (1+t)^{2g}.$$

[3]Cf. p.166 [ACGH].

3.3 The symmetric product Σ_n

The n-th symmetric product Σ_n is the moduli space of degree n effective divisors. It is a smooth projective variety of dimension n. Clearly $\Sigma_1 = \Sigma$.

The cohomology ring $H^*(\Sigma_n)$ is multiplicatively generated by $\xi_i \in H^1(\Sigma_n)$, for $i = 1 \dots 2g$ and $\eta \in H^2(\Sigma_n)$ defined by the formula:

$$c_1(\Delta_k) = k \otimes \sigma^\Sigma + \sum_{i=1}^{g} (\xi_i \otimes \xi_{i+g}^\Sigma - \xi_{i+g} \otimes \xi_i^\Sigma) + \eta \otimes 1 \tag{3.1}$$

in the decomposition

$$H^2(\Sigma_n \times \Sigma) \cong \sum_{r=0}^{2} H^r(\Sigma_n) \otimes H^{2-r}(\Sigma).$$

Here $\Delta_n \in Div(\Sigma_n \times \Sigma)$ is the universal divisor[4], i.e. $\Delta_n \mid_{\{D\}\times\Sigma} = D$ for every divisor $D \in \Sigma_n$. The relation set of the ring $H^*(\Sigma_n)$ and thus a complete description of it is given in [Macd]. We let $\sigma_i = \xi_i \xi_{i+g}$ and $\sigma = \sum_{i=1}^{g} \sigma_i \in H^2(\Sigma_n)$.

The Poincaré polynomial of $H^*(\Sigma_n)$ was calculated in [Macd] in the form:

$$P_t(\Sigma_n) = \operatorname*{Coeff}_{x^n} \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right). \tag{3.2}$$

For the case $n > 2g-2$ we have the Abel-Jacobi map $\Sigma_n \to \mathcal{J}_n$ being a locally trivial fibration with fibre $\mathbb{P}^{n-g}$, which for the Poincaré polynomial gives

$$P_t(\Sigma_n) = \frac{(1+t)^{2g}(1-t^{2(n-g+1)})}{(1-t^2)}.$$

3.3.1 The resolution tower of Σ

There are embeddings $i_n : \Sigma_n \to \Sigma_{n+1}$ given by $i_n(D) = D + p$, yielding the tower

$$\Sigma_1 \subset \Sigma_2 \subset \dots \subset \Sigma_n \subset \dots. \tag{3.3}$$

We consider the direct limit of them:

$$\Sigma_\infty = \lim_{n\to\infty} \Sigma_n.$$

It is a $\mathbb{P}^\infty$ bundle over $\mathcal{J}$, thus its Poincaré polynomial is

$$P_t(\Sigma^\infty) = \frac{(1+t)^{2g}}{(1-t^2)}. \tag{3.4}$$

The pullback map

$$i_n^* : H^*(\Sigma_{n+1}) \to H^*(\Sigma_n) \tag{3.5}$$

is surjective because these rings are generated by universal classes. Since the cohomology ring of Σ_∞ is the inverse limit of the cohomology rings of the Σ_n's, i.e.

$$H^*(\Sigma_\infty) = \lim_{\infty\leftarrow n} H^*(\Sigma_n),$$

it is also generated by the tautological classes ξ_i and η. In fact it is a free graded commutative algebra on these generators, as (3.4) shows.

[4] Cf. [ACGH].

3.3.2 Some results about $H^*(\Sigma_n)$

It is convenient to insert the following lemmata here, which will be needed in Section 6.5. They are taken from [Ha,Th].

Lemma 3.3.1 *Let us denote by $H^*_I(\Sigma_n)$ the subring of $H^*(\Sigma)$ generated by η and σ. For $n \leq 2g-2$ its Poincaré polynomial $P^I_T(\Sigma_n)$ can be written in the form:*

$$P^I_T(\Sigma_n) = \sum_{\substack{q,s\geq 0\\ q+2s\leq n}} T^{q+s} \tag{3.6}$$

Proof. It follows from B-1 and B-2 on p. 328-329 of [ACGH] that the classes $\eta^q \cdot \sigma^s$ for $q, s \geq 0$ and $q+2s \leq n$ form an additive basis for $H^*_I(\Sigma_n)$. The result follows. □

Lemma 3.3.2 *Let n, k, m and l be non-negative integers. If $n-g+m \leq l$ and $g+k-m < l$ then*

$$\left(\frac{\exp(\sigma)\eta^k}{(1+\eta)^m}\right)_l = 0$$

over Σ_n. Here $(\cdots)_l$ denotes the part of total degree $2l$.

Proof. First we show that Poincaré duality still holds in the subring generated by η and σ. By Poincaré duality in $H^*(\Sigma_n)$, the vanishing of a polynomial in η and σ of total degree $2d$ is equivalent to the vanishing of its product with every monomial in $\eta \in H^2$ and $\xi_i \in H^1(\Sigma_n)$ of total degree $2(n-d)$. If the power of ξ_i is greater than 1, this certainly vanishes; if the power of ξ_i is 1 but the power of $\xi_{i\pm g}$ is 0, this again vanishes since $H^*(\Sigma_n) \cong H^*(\Sigma^n)^{S_n} \subset H^*(\Sigma^n)$ and the corresponding expression in $H^*(\Sigma^n)$ is certainly zero. So it suffices to consider monomials in η and σ_i with degree 1 in the latter; then by symmetry it suffices to consider monomials in η and σ only.

As pointed out by Zagier in (7.2) of [Tha2], for any power series $A(x)$ and $B(x)$,

$$A(\eta)\exp(B(\eta)\sigma)[\Sigma_n] = \operatorname*{Res}_{\eta=0}\left(\frac{A(\eta)(1+\eta B(\eta))^g}{\eta^{n+1}}\right).$$

We multiply our expression by the generating function $\exp(s\sigma)/(1+t\eta)$ for the monomials in η and σ, and ask the coefficient of $s^i t^j$ to be 0 whenever $i+j = n-l$:

$$\begin{aligned}
\operatorname*{Coeff}_{s^it^j}\left(\frac{\exp((s+1)\sigma)\eta^k}{(1+\eta)^m(1+t\eta)}[\Sigma_n]\right) &= \operatorname*{Coeff}_{s^it^j}\operatorname*{Res}_{\eta=0}\left(\frac{\eta^k((\eta+1)+s\eta)^g}{(1+\eta)^m(1+t\eta)\eta^{n+1}}\right)\\
&= \text{const.}\operatorname*{Res}_{\eta=0}\left(\frac{\eta^k(1+\eta)^{g-i}\eta^i\eta^j}{(1+\eta)^m\eta^{n+1}}\right)\\
&= \text{const.}\operatorname*{Res}_{\eta=0}\left(\eta^{k+i+j-n-1}(1+\eta)^{g-i-m}\right).
\end{aligned}$$

But this is 0 because

$$g-i-m \geq g-m-(n-l) \geq 0,$$

from the first condition, thus $(1+\eta)^{g-i-m}$ is a polynomial of degree $g-i-m$, therefore the highest degree term of $\eta^{k+i+j-n-1}(1+\eta)^{g-i-m}$ has degree

$$(k+i+j-n-1)+(g-i-m) \leq k+(n-l)-n-1+g-m = k-l-1+g-m < -1$$

from the second condition.

The result follows. □

3.4 The moduli space of rank 2 stable bundles $\mathcal{N}$

Collecting results from the literature, in this section we describe $H^*(\mathcal{N})$. First we explain how Atiyah and Bott calculated the Poincaré polynomial of $\mathcal{N}$ and how they found generators for the ring. Then we describe the method of Kirwan for proving Mumford's conjecture and thus providing in principle a complete description of the cohomology ring $H^*(\mathcal{N})$. Then we state a calculationally useful description of the ring structure of $H^*(\mathcal{N})$ and cite some formulae of Zagier.

3.4.1 The Poincaré polynomial and generators

Here we explain how the idea of Subsection 1.1.2 was made rigorous in [At,Bo] in the rank 2 case.

The Shatz stratification

Let $\mathcal{V}$ be a fixed rank 2 smooth complex vector bundle of degree 1. Let $\mathcal{C}$ denote the infinite dimensional complex affine space of holomorphic structures on $\mathcal{V}$. Fixing a hermitian structure on $\mathcal{V}$, we have the gauge group $\mathcal{G}$ of smooth unitary automorphisms of $\mathcal{V}$ acting naturally on $\mathcal{C}$. Its complexification $\mathcal{G}^c = \Gamma(\Sigma, \mathrm{Aut}(\mathcal{V}))$, the group of complex automorphisms of $\mathcal{V}$, also acts on $\mathcal{C}$. Moreover $\mathcal{G}^c/\mathcal{G}$ is the contractible space of Hermitian structures on $\mathcal{V}$, thus $B\mathcal{G} \sim B\mathcal{G}^c$ and so for the purpose of equivariant cohomology they are equivalent: $H^*_{\mathcal{G}^c} = H^*_{\mathcal{G}}$. For convenience we will always use $\mathcal{G}$-equivariant cohomology, even where $\mathcal{G}^c$-equivariant cohomology is understood.

We also let $\mathcal{C}_s \subset \mathcal{C}$ (and frequently $\mathcal{C}_0$) denote the open subspace of stable bundles. By Theorem 1.2.2 of Harder and Narasimhan, if $E \in \mathcal{C}$ is not stable it has a unique destabilizing subbundle of degree $d > 0$. For $d > 0$ we let $\mathcal{C}_d \subset \mathcal{C}$ denote the subspace of unstable vector bundles with destabilizing bundle of degree d. Since the Harder-Narashiman filtration is canonical, $\mathcal{C}_d$ is invariant under $\mathcal{G}^c$, thus each $\mathcal{C}_d$ is a union of orbits. Atiyah and Bott prove that for $d > 0$, the space $\mathcal{C}_d$ is locally a Banach submanifold of $\mathcal{C}$ of finite codimension $2g + 4d - 4$ and that

$$\mathcal{C} = \bigcup_{d=0}^{\infty} \mathcal{C}_d \tag{3.7}$$

is a $\mathcal{G}^c$-equivariant stratification in the sense of Subsection 2.2.1. It is called the *Shatz stratification*.

The paper [At,Bo] uses Morse theory, in a manner as we explained in Subsection 2.2.1, for the Shatz stratification in order to calculate the Poincaré polynomial of $\widetilde{\mathcal{N}} = \mathcal{C}_0/\mathcal{G}^c$. They prove that the stratification is strongly $\mathcal{G}^c$-perfect[5]. It follows from (2.6) that for the $\mathcal{G}$-equivariant Poincaré polynomials we have:

$$\mathcal{G}P_t(\mathcal{C}) = \mathcal{G}P_t(\mathcal{C}_0) + \sum_{i=1}^{\infty} t^{2g+4i-4} \mathcal{G}P_t(\mathcal{C}_i),$$

or in a more suitable form

$$\mathcal{G}P_t(\mathcal{C}_0) = \mathcal{G}P_t(\mathcal{C}) - \sum_{i=1}^{\infty} t^{2g+4i-4} \mathcal{G}P_t(\mathcal{C}_i). \tag{3.8}$$

Now $\mathcal{C}$ is contractible and so $H^*_{\mathcal{G}}(\mathcal{C}) \cong H^*(B\mathcal{G})$. The cohomology ring of $B\mathcal{G}$ is described in §2 of [At,Bo]:

The cohomology of $B\mathcal{G}$

The ring $H^*(B\mathcal{G})$ is freely generated as a graded commutative algebra by classes

$$a_r \in H^{2r}(B\mathcal{G}), \qquad b_r^j \in H^{2r-1}(B\mathcal{G}), \qquad f_2 \in H^2(B\mathcal{G}),$$

for $1 \leq r \leq 2$ and $1 \leq j \leq 2g$. These classes appear as the Künneth components of a certain (universal) rank 2 vector bundle $\mathbb{U}$ on $B\mathcal{G} \times \Sigma$. Namely

$$c_1(\mathbb{U}) = a_1 \otimes 1 + \sum_{j=1}^{2g} b_1^j \otimes \xi_j^{\Sigma},$$

[5] Cf. Subsection 2.2.1.

and

$$c_2(\mathbb{U}) = a_2 \otimes 1 + \sum_{j=1}^{2g} b_2^j \otimes \xi_j^\Sigma + f_2 \otimes \sigma.$$

Since $H^*(B\mathcal{G})$ is a freely generated graded commutative algebra on the above mentioned classes, its Poincaré polynomial is

$$P_t(B\mathcal{G}) = \frac{\left\{(1+t)(1+t^3)\right\}^{2g}}{(1-t^2)^2(1-t^4)}. \tag{3.9}$$

Consider the constant central $U(1) \subset \mathcal{G}$ subgroup of $\mathcal{G}$. Let $\overline{\mathcal{G}} = \mathcal{G}/U(1)$ denote the quotient group. Then we have the fibration

$$BU(1) \to B\mathcal{G} \to B\overline{\mathcal{G}}. \tag{3.10}$$

It is shown in §9 of [At,Bo] that this fibration is actually a product:

$$B\mathcal{G} \sim BU(1) \times B\overline{\mathcal{G}}, \tag{3.11}$$

and so the generators of $H^*(B\mathcal{G})$ give generators for $H^*(B\overline{\mathcal{G}})$, with a_1 being redundant, they are: degree 1 generators b_1^j, a degree 2 generator f_2, degree 3 generators b_3^j and a degree 4 generator a_2. Now $B\overline{\mathcal{G}}$ is a free graded commutative algebra on these generators and consequently (or equivalently from (3.11)) we have:

$$P_t(B\overline{\mathcal{G}}) = (1-t^2)P_t(B\mathcal{G}) = \frac{\left\{(1+t)(1+t^3)\right\}^{2g}}{(1-t^2)(1-t^4)}. \tag{3.12}$$

The $\mathcal{G}$-equivariant cohomology of $\mathcal{C}_d$

For $d > 0$ the ring $H^*_{\mathcal{G}}(\mathcal{C}_d)$ can be described explicitly[6] as a freely generated graded commutative algebra by degree 1 elements b_1^j and b_2^j for $1 \leq j \leq 2g$ and degree 2 elements a_1^1 and a_1^2. Thus the $\mathcal{G}$-equivariant Poincaré polynomial of $\mathcal{C}_d$ is

$$\mathcal{G}P_t(\mathcal{C}_d) = \left(\frac{(1+t)^{2g}}{(1-t^2)}\right)^2. \tag{3.13}$$

The $\mathcal{G}$-equivariant cohomology of $\mathcal{C}_s$

Since the constant central gauge transformations act trivially on $\mathcal{C}$ the factor group $\overline{\mathcal{G}}^c = \mathcal{G}^c/\mathbb{C}^*$ acts on $\mathcal{C}$ and moreover it acts freely on $\mathcal{C}_s$. Now since (3.11) is a product we have that as rings:

$$\begin{aligned} H^*_{\mathcal{G}}(\mathcal{C}_s) &\cong H^*(BU(1)) \otimes H^*_{\overline{\mathcal{G}}}(\mathcal{C}_s) \\ &\cong H^*(BU(1)) \otimes H^*(\mathcal{C}_s/\overline{\mathcal{G}}) \\ &\cong H^*(BU(1)) \otimes H^*(\widetilde{\mathcal{N}}), \end{aligned}$$

which for the Poincaré polynomials gives:

$$P_t(\widetilde{\mathcal{N}}) = (1-t^2)\mathcal{G}P_t(\mathcal{C}_s). \tag{3.14}$$

Now putting (3.8), (3.9), (3.13) and (3.14) together yields the desired formula for the Poincaré polynomial of $\widetilde{\mathcal{N}}$:

$$\begin{aligned} P_t(\widetilde{\mathcal{N}}) &= P_t(B\overline{\mathcal{G}}) - \sum_{d=1}^{\infty} t^{2(g+2d-2)} \frac{(1+t)^{4g}}{(1-t^2)} \\ &= \frac{\left\{(1+t)(1+t^3)\right\}^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{2g}(1+t)^{4g}}{(1-t^2)(1-t^4)} \\ &= (1+t)^{2g}\left(\frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}\right) \end{aligned} \tag{3.15}$$

[6]For more details see [At,Bo]. For the homotopy type of $(\mathcal{C}_d)_{\mathcal{G}}$ see Corollary 7.5.5.

Finally, since the Shatz stratification is $\mathcal{G}$-perfect, the map

$$H^*(B\mathcal{G}) \cong H^*_{\mathcal{G}}(\mathcal{C}) \to H^*_{\mathcal{G}}(\mathcal{C})$$

is surjective, thus the images of the generators of $H^*(B\mathcal{G})$ give generators in $H^*_{\mathcal{G}}(\mathcal{C}_s)$. As we saw above

$$H^*_{\mathcal{G}}(\mathcal{C}_s) \cong H^*(BU(1)) \otimes H^*(\widetilde{\mathcal{N}}),$$

thus the images of the generators of $H^*(B\mathcal{G})$ give generators in $H^*(\widetilde{\mathcal{N}})$. Since one of them a_1 is redundant, we have the following list of generators for the cohomology of $\widetilde{\mathcal{N}}$: degree 1 generators b_1^j, a degree 2 generator f_2, degree 3 generators b_3^j and a degree 4 generator a_2.

In the next paragraph we explain the consequences of these result for the cohomology of $\mathcal{N}$.

The cohomology of $\mathcal{N}$

Let $\Gamma := H^1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g} \cong \ker(\sigma_2)$, where $\sigma_2 : \mathcal{J}_0 \to \mathcal{J}_0$ is given by $\sigma_2(L) = L^2$. Now Γ acts on $\mathcal{N}$ and $\mathcal{J}$ by tensoring with the corresponding line bundle in $\ker(\sigma_2)$ and also on $\mathcal{N} \times \mathcal{J}$ by the diagonal action. Then we have

$$\widetilde{\mathcal{N}} = (\mathcal{N} \times \mathcal{J})/\Gamma. \tag{3.16}$$

Because Γ acts trivially on $H^*(\mathcal{J})$ and on $H^*(\mathcal{N})$ (the latter was first proved in [Ha,Na]) we see that as rings

$$H^*(\widetilde{\mathcal{N}}) \cong (H^*(\mathcal{N}) \otimes H^*(\mathcal{J}))^\Gamma \cong H^*(\mathcal{N}) \otimes H^*(\mathcal{J}). \tag{3.17}$$

Thus for understanding the cohomology ring $H^*(\widetilde{\mathcal{N}})$ it is enough to know the cohomology ring $H^*(\mathcal{N})$. Since we know a generator set for the ring $H^*(\widetilde{\mathcal{N}})$, it gives one for $H^*(\mathcal{N})$. However the classes b_1^j will go to 0. Thus a generator set is provided by: a degree 2 generator f_2, degree 3 generators b_2^s and a degree 4 generator a_2.

The generators we have just described differ from the original generator set defined by Newstead in [New2]. He defined cohomology classes: $\alpha \in H^2(\mathcal{N})$, $\psi_i \in H^3(\mathcal{N})$ and $\beta \in H^4(\mathcal{N})$, which appear in the Künneth decomposition of $c_2(\mathrm{End}(\mathbb{E}_{\mathcal{N}}))$:

$$c_2(\mathrm{End}(\mathbb{E}_{\mathcal{N}})) = 2\alpha \otimes \sigma^\Sigma + \sum_{i=1}^{2g} 4\psi_i \otimes \xi_i^\Sigma - \beta \otimes 1 \tag{3.18}$$

in $H^4(\mathcal{N} \times \Sigma) \cong \sum_{r=0}^4 H^r(\mathcal{N}) \otimes H^{4-r}(\Sigma)$. Here $\mathbb{E}_{\mathcal{N}}$ is the normalized rank 2 universal bundle over $\mathcal{N} \times \Sigma$, i.e. $c_1(\mathbb{E}_{\mathcal{N}}) = \alpha$ and $\mathbb{E}_{\mathcal{N}} \mid_{\{E\}\times\Sigma} \cong E$ for every $E \in \mathcal{N}$. We will later use the following notations: $\gamma = -2\sum_{i=1}^g \psi_i\psi_{i+g} \in H^6(\mathcal{N})$, $\gamma^* = 2\gamma + \alpha\beta \in H^6(\mathcal{N})$.

The relation between the Newstead and Atiyah-Bott generators can be traced back by using the fact[7] that the universal bundle $\mathbb{U}$ over $B\mathcal{G}$ restricts to $\mathbb{E}_{\mathcal{N}}$. The correspondence is given[8] by the formulae:

$$\alpha = 2f_2 - a_1, \qquad \beta = (a_1)^2 - 4a_2, \qquad \psi_i = b_2^i.$$

Finally, another consequence of (3.17) and (3.15) is the so-called Harder-Narasimhan formula for the Poincaré polynomial of $\mathcal{N}$:

$$P_t(\mathcal{N}) = \frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)} \tag{3.19}$$

3.4.2 The resolution tower of $\widetilde{\mathcal{N}}$

We promised the existence of a resolution tower for $\widetilde{\mathcal{N}}$ at the beginning of the present section. It exists up to homotopy equivalence. Namely $\widetilde{\mathcal{N}} \sim (\mathcal{C}_0)_{\overline{\mathcal{G}}}$ and let us choose $\widetilde{\mathcal{N}}_k \sim (\mathcal{C}_{\leq k})_{\overline{\mathcal{G}}}$ for $k > 0$. Thus homotopically we have the tower:

$$\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 \subset \widetilde{\mathcal{N}}_1 \subset \ldots \subset \widetilde{\mathcal{N}}_k \subset \ldots,$$

[7] Cf. [At,Bo] p. 579-580

[8] Cf. [Earl].

which has the property that

$$i_k^* : H^*(\widetilde{\mathcal{N}}_{k+1}) \to H^*(\widetilde{\mathcal{N}}_k)$$

is surjective[9]. Thus if we consider

$$\widetilde{\mathcal{N}}_\infty = \varinjlim \widetilde{\mathcal{N}}_k \sim \varinjlim (\mathcal{C}_{\leq k})_{\overline{\mathcal{G}}} \sim (\mathcal{C})_{\overline{\mathcal{G}}} \sim B\overline{\mathcal{G}},$$

then we have that

$$H^*(\widetilde{\mathcal{N}}_\infty) \to H^*(\widetilde{\mathcal{N}})$$

is surjective, and moreover $H^*(\widetilde{\mathcal{N}}_\infty) \cong H^*(B\overline{\mathcal{G}})$ is a free commutative graded algebra on universal classes, providing the picture we described at the beginning of the present chapter.

In the next section we explain how Kirwan used this resolution tower, or more generally Proposition 2.2.1, in order to settle the Mumford conjecture, providing a complete set of relations for $\widetilde{\mathcal{N}}$.

3.4.3 Complete set of relations

The ring structure of $H^*(\mathcal{N})$ is described in terms of the so-called *Mumford relations* of the generators α, β and the ψ_i's. To explain this consider the *virtual Mumford bundle* over $\widetilde{\mathcal{N}}$:

$$\begin{aligned} \mathbf{M} &= -\mathrm{pr}_{\widetilde{\mathcal{N}}!}(\mathbb{E}_{\widetilde{\mathcal{N}}} \otimes \mathrm{pr}_\Sigma^*(L_p^{-1})) \\ &= -R^0\mathrm{pr}_{\widetilde{\mathcal{N}}*}(\mathbb{E}_{\widetilde{\mathcal{N}}} \otimes \mathrm{pr}_\Sigma^*(L_p^{-1})) + R^1\mathrm{pr}_{\widetilde{\mathcal{N}}*}(\mathbb{E}_{\widetilde{\mathcal{N}}} \otimes \mathrm{pr}_\Sigma^*(L_p^{-1})) \in K(\widetilde{\mathcal{N}}). \end{aligned}$$

Using standard properties of stable bundles it can be shown that R^0 vanishes. Thus $\mathbf{M}$ is a vector bundle of rank $2g-1$. Its total Chern class is a complicated[10] polynomial of the universal classes. Since $\mathrm{rank}(\mathbf{M}) = 2g-1$, the Chern class $c_{2g+r}(\mathbf{M}) \in H^{4g+2r}(\widetilde{\mathcal{N}})$ vanishes for $r \geq 0$. According to (3.17), the cohomology of $\widetilde{\mathcal{N}}$ is the tensor product of $H^*(\mathcal{J})$ and $H^*(\mathcal{N})$. Thus if we write $\tau_S = \prod_{i\in S} \tau_i$ for $S \subset \{1 \dots 2g\}$ and

$$c_{2g+r}(\mathbf{M}) = \sum_{S\subset\{1\dots 2g\}} \zeta_S^r \otimes \tau_S$$

in the Künneth decomposition of (3.17) then we get the vanishing of each ζ_S^r. Thus for every $r \geq 0$ and $S \subset \{1 \dots 2g\}$ we get a relation

$$\zeta_S^r \in \mathbb{Q}[\alpha, \beta, \psi_i] \tag{3.20}$$

of degree $d = 4g + 2r - \deg(\tau_S)$. The classes ζ_S^r are called the *Mumford relations*.

Mumford conjectured[11] that the Mumford relations constitute a complete set of relations of the cohomology ring of $\mathcal{N}$. Mumford's conjecture was first settled by Kirwan in [Kir2], by using the method of Remark 1 after Proposition 2.2.1 for the Shatz stratification. The proof of [Kir2] goes by building $\mathcal{R}_d$ from the classes ζ_S^r for $d > 0$, and for $d = 0$ setting $\mathcal{R}_0 = H_{\mathcal{G}}^*(\mathcal{C})$. The heart of the proof is to show that $\mathcal{R}_d$ satisfies the conditions of Proposition 2.2.1, which takes some pages of calculation. Now Proposition 2.2.1 proves the Mumford conjecture as explained in Remark 1 after it. We will give a purely geometric proof of the Mumford conjecture in Section 7.3.

Now we explain an explicit description of the cohomology ring.

3.4.4 Explicit description

Recently, the following very explicit characterization of the ring $H^*(\mathcal{N})$ has been obtained by several authors [Bar],[Ki,Ne],[Si,Ti] and [Zag]. To explain it first note that there is a natural action of $Sp(2g,\mathbb{Z})$ on $H^*(\mathcal{N})$ induced by the obvious action on $H^3(\mathcal{N})$. The above mentioned sources prove that as an $Sp(2g,\mathbb{Z})$-algebra

$$H^*(\mathcal{N}) \cong \bigoplus_{k=0}^{g} \Lambda_0^k H^3(\mathcal{N}) \otimes \mathbb{Q}[\alpha,\beta,\gamma]/I_{g-k},$$

[9] It is a consequence of the $\mathcal{G}$-perfectness of the Shatz stratification.
[10] It was calculated by Zagier in [Zag].
[11] Cf. [At,Bo] p. 582.

where

$$\Lambda_0^k = \ker\left(\gamma^{g-k+1} : \Lambda^k H^3(\mathcal{N}) \to \Lambda^{2g-k+2} H^3(\mathcal{N})\right)$$

and I_g is the relation ideal of the $Sp(2g,\mathbb{Z})$-invariant part $H_I^*(\mathcal{N})$, i.e.

$$H_I^*(\mathcal{N}) \cong \mathbb{Q}[\alpha,\beta,\gamma]/I_g.$$

The ideal I_{g-k} is moreover described as being generated by polynomials

$$\zeta_{g-k}, \zeta_{g-k+1}, \zeta_{g-k+2} \in \mathbb{Q}[\alpha,\beta,\gamma],$$

which are given recursively by the following rule:

$$(r+1)\zeta_{r+1} = \alpha\zeta_r + r\beta\zeta_{r-1} + 2\gamma\zeta_{r-2}, \tag{3.21}$$

with initial conditions $\zeta_0 = 1, \zeta_r = 0$ for $r < 0$.

Moreover an additive basis for I_g is given by Zagier in [Zag] of the form

$$\zeta_{r,s,t} \text{ for all } \; r,s,t \geq 0 \text{ and } \; r+s+t \leq g-1, \tag{3.22}$$

where $\zeta_{r,s,t} = \zeta_{r,s}(2\gamma)^t/t!$, and the polynomials $\zeta_{r,s}$ are given by a generating function:

$$\sum_{r,s\geq 0} \zeta_{r,s} x^r y^s = \frac{e^{-2\gamma x}/\beta}{\sqrt{(1-\beta y)^2 - \beta x^2}} \left(\frac{1+x\sqrt{\beta}-\beta y}{1-x\sqrt{\beta}-\beta y}\right)^{\gamma^*/2\beta\sqrt{\beta}}. \tag{3.23}$$

It follows from (3.22) that the Poincaré polynomial of $H_I^*(\mathcal{N})$ equals:

$$P_t^I(\mathcal{N}) = \sum_{\substack{r,s,t\geq 0\\ r+s+t\leq g-1}} T^{r+2s+3t}. \tag{3.24}$$

3.5 The moduli space of Abelian Higgs bundles $T^*_{\mathcal{J}}$

As an instructive example for the discussions in Section 0.2, we consider here the moduli space of Abelian Higgs bundles.

The tangent bundle of $\mathcal{J}$ is canonically isomorphic to $\mathcal{J} \times H^1(\Sigma, \mathcal{O}_\Sigma)$. Thus by Serre duality $T^*_{\mathcal{J}} \cong \mathcal{J} \times H^0(\Sigma, K)$ canonically. An element $\Phi \in (T^*_{\mathcal{J}})_L \cong H^0(\Sigma, K)$, can be thought of as a rank 1 Higgs bundle[12]: $\mathcal{L} = L \xrightarrow{\Phi} LK$. Thus we can think of $T^*_{\mathcal{J}}$ as the moduli space of rank 1 Higgs bundles.

The cohomology of $T^*_{\mathcal{J}}$ is isomorphic to that of $\mathcal{J}$. However there is an extra piece of cohomological information, namely the intersection numbers in the compactly supported cohomology or in other words the map:

$$j_{\mathcal{J}} : H^*_{cpt}(T^*_{\mathcal{J}}) \to H^*(T^*_{\mathcal{J}}).$$

Clearly this map is interesting only in the middle dimension, where both $H^{2g}_{cpt}(T^*_{\mathcal{J}})$ and $H^{2g}(T^*_{\mathcal{J}})$ are one-dimensional. However the Euler characteristic of $\mathcal{J}$ is clearly 0, thus the self-intersection number of the zero section of $T^*_{\mathcal{J}}$ is 0, which shows that $j_{\mathcal{J}}$ vanishes.

We can also determine the L^2-cohomology of $T^*_{\mathcal{J}}$ for the Riemann metric on $T^*_{\mathcal{J}} \cong \mathcal{J} \times H^0(\Sigma, K)$ which is the product of the flat metrics on the two terms (this is the metric which we get if we perform Hitchin's work in [Hit1] for the Abelian case). From the Weitzenböck decomposition of the Hodge Laplacian and the L^2-vanishing theorem of Dodziuk [Dod], since the metric is flat there are no non-trivial L^2 harmonic forms on $T^*_{\mathcal{J}}$. Thus in the Abelian Higgs case the topology gives the harmonic space, as conjectured for the rank 2 Higgs moduli space in Conjecture 1 and the moduli space of magnetic monopoles in [Sen].

[12]Cf. Definition 5.1.1.

3.6 Moduli space of rank 2 stable Higgs bundles $\mathcal{M}$

Recall from Section 1.2.2 that $\widetilde{\mathcal{M}}$ denotes the moduli space of rank 2 stable Higgs bundles of degree 1. The determinant gives a map $det_{\mathcal{M}} : \widetilde{\mathcal{M}} \to T^*_{\mathcal{J}}$, defined by $det_{\mathcal{M}}(E,\Phi) = (\Lambda^2 E, \mathrm{tr}(\Phi))$. For any $\mathcal{L} \in T^*_{\mathcal{J}}$ the fibre $det^{-1}_{\mathcal{M}}(\mathcal{L})$ will be denoted by $\mathcal{M}_{\mathcal{L}}$. Just as in the case of $\widetilde{\mathcal{N}}$ any two fibres of $det_{\mathcal{M}}$ are isomorphic. Usually we will write $\mathcal{M}$ for $\mathcal{M}_{\mathcal{L}}$, when the Abelian Higgs bundle $\mathcal{L}$ has zero Higgs field.

Our main concern in the present thesis is $\mathcal{M}$. Recall from Subsection 1.2.2 that it is a non-projective, smooth quasi-projective variety of dimension $6g-6$.

Similarly to (3.16) we have a Γ-action on $\widetilde{\mathcal{M}}$ and on $T^*_{\mathcal{J}}$ such that:

$$\widetilde{\mathcal{M}} = (\mathcal{M} \times T^*_{\mathcal{J}})/\Gamma.$$

This on the level of cohomology gives

$$H^*(\widetilde{\mathcal{M}}) \cong (H^*(\mathcal{M}) \otimes H^*(T^*_{\mathcal{J}}))^{\Gamma} \cong (H^*(\mathcal{M}))^{\Gamma} \otimes H^*(\mathcal{J}). \tag{3.25}$$

In the case of $\mathcal{M}$ however we do not have the triviality of the action of Γ on $H^*(\mathcal{M})$, but nevertheless the cohomology ring of $\widetilde{\mathcal{M}}$ is determined by the ring $(H^*(\mathcal{M}))^{\Gamma}$.

There is quite little known about the ring $H^*(\mathcal{M})$. The Poincaré polynomial of it is calculated in [Hit1] using Morse theory. We now explain this:

3.6.1 The Poincaré polynomial of $\mathcal{M}$

We can outline Hitchin's Morse theory calculation in the language of stratifications of Subsection 2.2.1 as follows. As we explained in Subsection 2.1.1 the Morse function μ defines an upward stratification of $\mathcal{M} = \bigcup_{d=0}^{g-1} U_d$, where $U_d = \{x \in \mathcal{M} : \lim_{z\to 0} z \cdot x \in F_d\}$, with F_d denoting a component of the fixed point set of the $U(1)$-action. We call this the *Hitchin stratification*, because Hitchin used the perfectness of this stratification to calculate the Poincaré polynomial of $\mathcal{M}$ in Theorem 7.6 of [Hit1]. In particular he proved in Proposition 7.1 of [Hit1] that for $d > 0$ the index of F_d in $\mathcal{M}$ is $2(g+2d-2)$. On the other hand the index of F_d in $\mathcal{M}$ is the same as the real codimension of U_d in $\mathcal{M}$. The perfectness of the stratification follows from [Kir1]. Now [Hit1] calculates the Poincaré polynomial of $\mathcal{M}$ using the short exact sequences (2.4):

$$0 \to H^*(U_d) \overset{(i_d)_*}{\to} H^*(\mathcal{M}_{\leq d}) \overset{i^*_{<d}}{\to} H^*(\mathcal{M}_{<d}) \to 0, \tag{3.26}$$

and in turn the formula (2.6) giving:

$$P_t(\mathcal{M}) = P_t(\mathcal{N}) + \sum_{d=1}^{g-1} t^{2(g+2d-2)} P_t(U_d) = P_t(\mathcal{N}) + \sum_{d=1}^{g-1} t^{2(g+2d-2)} P_t(F_d), \tag{3.27}$$

as F_d is a deformation retract of U_d. For the Γ-invariant part we have:

$$\begin{aligned}(P_t(\mathcal{M}))^{\Gamma} &= (P_t(\mathcal{N}))^{\Gamma} + \sum_{d=1}^{g-1} t^{2(g+2d-2)} (P_t(U_d))^{\Gamma} \\ &= (P_t(\mathcal{N}))^{\Gamma} + \sum_{d=1}^{g-1} t^{2(g+2d-2)} (P_t(F_d))^{\Gamma}\end{aligned} \tag{3.28}$$

from the short exact sequences

$$0 \to H^*(U_d)^{\Gamma} \overset{(i_d)_*}{\to} H^*(\mathcal{M}_{\leq d})^{\Gamma} \overset{i^*_{<d}}{\to} H^*(\mathcal{M}_{<d})^{\Gamma} \to 0, \tag{3.29}$$

which can be obtained by noticing that the Hitchin stratification is Γ-invariant and in turn that the short exact sequence (3.26) is in fact a sequence of Γ-modules. The next step is thus to understand $H^*(F_d)$ and the action of Γ on it:

3.6.2 The cohomology of F_d

We can construct F_d as the moduli space of rank 2 degree 1 Higgs bundles of the form : $E \xrightarrow{\Phi} E \otimes K$, where

$$E = L \oplus L^{-1}\Lambda,$$

and

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}, \tag{3.30}$$

with $\deg(L) = d$ and $0 \neq \phi \in H^0(\Sigma; L^{-2}\Lambda K)$. Note that $\deg(L^{-2}\Lambda K) = 2g - 2d - 1$, thus modulo non-zero scalars ϕ is a point in $\Sigma_{\bar{d}}$, where we used the notation $\bar{d} = 2g - 2d - 1$.

Now we determine $H^*(F_d)$. From the description (3.30) it follows that F_d is isomorphic to the moduli space of complexes: $L \xrightarrow{\phi} L^{-1}\Lambda K$, with $\deg(L) = d$ and $\phi \in H^0(\Sigma; L^{-2}\Lambda K)$. Thus if we define maps $\mathcal{J}_d \to \mathcal{J}_{\bar{d}}$ by sending $L \mapsto L^{-2}\Lambda K$ and the Abel-Jacobi map $\Sigma_{\bar{d}} \to \mathcal{J}_{\bar{d}}$ by sending $D \mapsto L(D)$, then the fibred product of the these maps is:

$$F_d = \Sigma_{\bar{d}} \times_{\mathcal{J}_{\bar{d}}} \mathcal{J}_d, \tag{3.31}$$

We have the two projections $\mathrm{pr}_{\mathcal{J}_d} : F_d \to \mathcal{J}_d$ and $\mathrm{pr}_{\Sigma_{\bar{d}}} : F_d \to \Sigma_{\bar{d}}$. This last one is a 2^{2g}-fold cover, and it is induced by the action of $\Gamma \cong \mathbb{Z}_2^{2g}$ on $F_d \subset \mathcal{M}$. Now Hitchin calculates the Poincaré polynomial of F_d by understanding the action Γ on $H^*(F_d)$. He finds in (7.13) of [Hit1] that this action is not trivial and finds the cohomology of F_d in the form:

$$H^*(F_d) = (H^*(F_d))^\Gamma \oplus V^{\bar{d}} = H^*(\Sigma_{\bar{d}}) \oplus V^{\bar{d}},$$

where $V^{\bar{d}}$ is a faithful representation of Γ in the middle $\bar{d}$ degree. Moreover its dimension is $(2^{2g}-1)\binom{2g-2}{\bar{d}}$. Thus for the Poincaré polynomial he finds

$$\begin{aligned} P_t(F_d) &= (P_t(F_d))^\Gamma + (2^{2g}-1)\binom{2g-2}{\bar{d}} \\ &= P_t(\Sigma_{\bar{d}}) + (2^{2g}-1)\binom{2g-2}{\bar{d}}. \end{aligned} \tag{3.32}$$

Now Hitchin calculates[13] the Poincaré polynomial of $\mathcal{M}$ from (3.27), (3.32) and (3.2). By subtracting the contributions of the non Γ-invariant parts it follows[14] from the formula of Hitchin that:

$$(P_t(\mathcal{M}))^\Gamma = \frac{(1+t^3)^{2g} - t^{4g-2}(P(t))}{(1-t^2)(1-t^4)},$$

where $P(t)$ is some complicated polynomial of t. Consequently from (3.25) we have

$$P_t(\widetilde{\mathcal{M}}) = \frac{(1+t)^{2g}(1+t^3)^{2g} - t^{4g-2}\widetilde{P}(t)}{(1-t^2)(1-t^4)} = P_t(B\overline{\mathcal{G}}) - t^{4g-2}\frac{\widetilde{P}(t)}{(1-t^2)(1-t^4)}, \tag{3.33}$$

where $\widetilde{P}(t) = (1+t)^{2g}P(t)$ is some polynomial. Comparing this result to (3.15) we see that the Poincaré polynomial of $\widetilde{\mathcal{M}}$ approximates the Poincaré polynomial of the classifying space of $\overline{\mathcal{G}}$ roughly twice better than the Poincaré polynomial of $\widetilde{\mathcal{N}}$! An explanation of this phenomenon will be provided in Chapter 7.

3.6.3 Contribution of the present thesis

In Chapter 4 we make a detailed study of the $\mathbb{C}^*$-action, compactify $\mathcal{M}$, and calculate the Poincaré polynomial of the compactification. By doing so we establish the geometric background for the cohomological calculations of the following chapters, where we deal with two aspects of the cohomology of $\mathcal{M}$: the intersection numbers and the cohomology ring structure of $\mathcal{M}$.

[13] There is, however, a small calculational mistake in (7.16) and (7.17) of [Hit1], namely they should be multiplied by t^2 in order to get the correct residue.

[14] Keeping in mind the above Footnote 13.

In the non-compact case the intersection numbers are in the compactly supported cohomology. We will calculate all intersection numbers of $H^*_{cpt}(\mathcal{M})$ in Chapter 5, by proving Theorem 0.2.1.

As we already mentioned in Section 0.2, for the calculation of the sigma model of [BJSV] one needs to have a good understanding of the ring structure of $(H^*(\mathcal{M}))^\Gamma$. There is, however, no result about the cohomology ring $H^*(\widetilde{\mathcal{M}})$ in the literature. We will attempt to fill this gap in Chapters 6 and 7: We find generators for $H^*(\widetilde{\mathcal{M}})$, and conjecture a complete set of relations for $H^*_I(\mathcal{M})$ in Chapter 6. In Chapter 7, we approach $H^*(\widetilde{\mathcal{M}})$ in the general framework, described at the beginning of the present chapter. Namely we show that the tower of $U(1)$-manifolds

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_0 \subset \widetilde{\mathcal{M}}_1 \subset \ldots \subset \widetilde{\mathcal{M}}_k \subset \ldots$$

of moduli spaces[15] of Higgs k-bundles with poles is the right candidate for resolving the cohomology ring $H^*(\widetilde{\mathcal{M}})$.

As byproducts of our considerations in Chapter 7, we give a simple geometric proof of Mumford's conjecture, following ideas of Kirwan, and show that the homotopy type of the above tower of Higgs k-bundle moduli spaces shares the Atiyah-Jones property appearing in the theory of instantons.

[15] Cf. Definition 1.2.5

Part II

New results

Chapter 4

Compactification

The main aim of this chapter is to investigate a canonical compactification of $\mathcal{M}$: among other things we show that the compactification is projective, calculate its Picard group, and calculate the Poincaré polynomial for the cohomology.

We use a simple method to compactify non-compact Kähler manifolds with a nice proper Hamiltonian $U(1)$-action via Lerman's construction of symplectic cutting [Ler]. We use this method to compactify $\mathcal{M}$. Our approach is symplectic in nature and eventually produces some fundamental results about the spaces occurring, using existing techniques from the theory of symplectic quotients.

We show that the compactification described here is a good example of Yau's problem of finding a complete Ricci flat metric on the complement of a nef anticanonical divisor in a projective variety.

Many of the results of this chapter can be easily generalized to other Higgs bundle moduli spaces, which have been extensively investigated (see e.g. [Nit] and [Sim1]). As a matter of fact Simpson gave a definition of a similar compactificitation for these more general Higgs bundle moduli spaces in Theorems 11.2 and 11.1 of [Sim2] and in Proposition 17 of [Sim3], without investigating it in detail. For example, the projectiveness of the compactification is not clear from these definitions. One novelty of this chapter is the proof of the projectiveness of the compactification in our case.

Since the compactification method used here is fairly general it is possible to apply it to other Kähler manifolds with the above properties. It could be interesting for instance to see how this method works for the toric hyperkähler manifolds of Goto [Goto] and Bielawski and Dancer [Bi,Da].

Finally we note that the compactification described in the subsequent sections solves one half of the problem of compactifying the moduli space $\mathcal{M}$, namely the 'outer' half, i.e. shows what the resulting spaces look like. The other half of the problem the 'inner' part, i.e. how this fits into the moduli space description of $\mathcal{M}$, is treated in the recent paper of Schmitt [Schm]. Schmitt's approach is algebro-geometric in nature, and concerns mainly the construction of the right notion of moduli to produce $\overline{\mathcal{M}}$, thus complements our results.

4.1 Statement of results

In this section we describe the structure of the chapter and list the results.

In Section 4.2 we describe $\mathcal{M}_{toy}$ the moduli space of parabolic Higgs bundle on $\mathbb{P}^1_4$, which will serve as a toy example throughout this chapter.

In Section 4.3 we start to apply the ideas of Section 2.1. Here we show that the Hitchin stratification, constructed from the upward flows of Subsection 2.1.1 coincides with the Shatz stratification coming from the stratification (3.7) on $\mathcal{C}$.

In Section 4.4 (following ideas of Subsection 2.1.1) we describe the nilpotent cone after Thaddeus [Tha1] and show that it coincides with the downward Morse flow (Theorem 4.4.2). We reprove Laumon's theorem in our case, that the nilpotent cone is Lagrangian (Corollary 4.4.3).

In Section 4.5 we describe Z, the highest level Kähler quotient of $\mathcal{M}$, while in 4.6 we analyse $\overline{\mathcal{M}} = \mathcal{M} \cup Z$. Here we follow the approaches of Subsection 2.1.2 and Subsection 2.1.3, respectively. Among others, we prove the following statements:

- $\overline{\mathcal{M}}$ is a compactification of $\mathcal{M}$, the moduli space of stable Higgs bundles with fixed determinant and degree 1 (Theorem 4.6.2).
- Z is a symplectic quotient of $\mathcal{M}$ by the circle action $(E, \Phi) \mapsto (E, e^{i\theta} \cdot \Phi)$. $\overline{\mathcal{M}}$ is a symplectic quotient of $\mathcal{M} \times \mathbf{C}$ with respect to the circle action, which is the usual one on $\mathcal{M}$ and multiplication on $\mathbf{C}$.
- While $\mathcal{M}$ is a smooth manifold, Z is an orbifold, with only $\mathbf{Z}_2$ singularities corresponding to the fixed point set of the map $(E, \Phi) \mapsto (E, -\Phi)$ on $\mathcal{M}$ (Theorem 4.5.2), while similarly $\overline{\mathcal{M}}$ is an orbifold with only $\mathbf{Z}_2$ singularities, and the singular locus of $\overline{\mathcal{M}}$ coincides with that of Z (Theorem 4.6.3).
- The Hitchin map
$$\chi : \mathcal{M} \to \mathbf{C}^{3g-3}$$
extends to a map
$$\overline{\chi} : \overline{\mathcal{M}} \to \mathbf{P}^{3g-3}$$
which when restricted to Z gives a map
$$\overline{\chi} : Z \to \mathbf{P}^{3g-4}$$
whose generic fibre is a Kummer variety corresponding to the Prym variety of the generic fibre of the Hitchin map (Theorem 4.5.10, Theorem 4.6.8).
- $\overline{\mathcal{M}}$ is a projective variety (Theorem 4.6.11), with divisor Z such that
$$(3g-2)Z = -K_{\overline{\mathcal{M}}},$$
the anticanonical divisor of $\overline{\mathcal{M}}$ (Corollary 4.6.7).
- Moreover, Z itself is a projective variety (Theorem 4.5.16) with an inherited holomorphic contact structure with contact line bundle L_Z (Theorem 4.5.9) and a one-parameter family of Kähler forms $\omega_t(Z)$ (Theorem 4.5.15). The Picard group of Z is described in Corollary 4.5.7. Moreover, the normal bundle of Z in $\overline{\mathcal{M}}$ is L_Z which is nef by Corollary 4.5.14.
- Furthermore, $\overline{\mathcal{M}}$ has a one-parameter family of Kähler forms $\omega_t(\overline{\mathcal{M}})$, which when restricted Z gives the above $\omega_t(Z)$.
- Z is birationally equivalent to $P(T^*_{\mathcal{N}})$ the projectivized cotangent bundle of the moduli space of rank 2 stable bundles with fixed determinant and odd degree (Corollary 4.5.4). $\overline{\mathcal{M}}$ is birationally equivalent to $P(T^*_{\mathcal{N}} \oplus \mathcal{O}_{\mathcal{N}})$, the canonical compactification of $T^*_{\mathcal{N}}$ (Corollary 4.6.4).
- We calculate certain sheaf cohomology groups in Corollaries 4.5.12 and 4.5.13 and interpret some of these results as the equality of certain infinitesimal deformation spaces.
- The Poincaré polynomial of Z is described in Corollary 4.5.5, the Poincaré polynomial of $\overline{\mathcal{M}}$ is described in Theorem 4.6.12.
- We end Section 4.6 by showing an interesting isomorphism between two vector spaces: one contains information about the intersection of the components of the nilpotent cone, the other says something about the contact line bundle L_Z on Z.

4.2 A toy model $\mathcal{M}_{toy}$

Unfortunately, even when $g = 2$ the moduli space $\mathcal{M}$ is already 6 dimensional, too big to serve as an instructive example. We rather choose $\mathcal{M}_{toy}$, the moduli space[1] of stable parabolic Higgs bundles on $\mathbb{P}^1$, with four marked points, in order to show how our later constructions work. We choose this example because it is a complex surface, and can be constructed explicitly.

We fix four distinct points on $\mathbb{P}^1$ and denote by $\mathbb{P}^1_4$ the corresponding complex orbifold. Let P be the elliptic curve corresponding to $\mathbb{P}^1_4$. Let σ_P be the involution $\sigma_P(x) = -x$ on P. Thus, P/σ_P is just the complex orbifold $\mathbb{P}^1_4$. The four fixed points of the involution $x_1, x_2, x_3, x_4 \in P$ correspond to the four marked points on $\mathbb{P}^1_4$. Furthermore, let τ be the involution $\tau(z) = -z$ on $\mathbb{C}$.

Consider now the quotient space $(P \times \mathbb{C})/(\sigma_P \times \tau)$. This is a complex orbifold of dimension 2 with four isolated $\mathbb{Z}_2$ quotient singularities at the points $x_i \times 0$. Blowing up these singularities we get a smooth complex surface $\mathcal{M}_{toy}$ with four exceptional divisors D_1, D_2, D_3 and D_4. Moreover the map $\chi : (P \times \mathbb{C}) \to \mathbb{C}$ sending $(x, z) \mapsto z^2$, descends to the quotient $(P \times \mathbb{C})/(\sigma_P \times \tau)$ and sending the exceptional divisors to zero one obtains a map $\chi_{toy} : \mathcal{M}_{toy} \to \mathbb{C}$, with generic fibre P. The map χ_{toy} will serve as our toy Hitchin map.

There is a $\mathbb{C}^*$-action on $\mathcal{M}_{toy}$, coming from the standard action on $\mathbb{C}$. The fixed point set of $U(1) \subset \mathbb{C}^*$ has five components: one is $\mathcal{N}_{toy} \subset \mathcal{M}_{toy}$ (the moduli space of stable parabolic bundles on $\mathbb{P}^1_4$) which is the proper transform of $(P \times 0)/(\tau \times \sigma_P) = P^1_4 \subset (P \times \mathbb{C})/(\sigma_P \times \tau)$ in $\mathcal{M}_{toy}$. The other four components consist of single points $\tilde{x_i} \in D_i,\ i = 1, 2, 3, 4$.

The fixed point set of the involution $\sigma : \mathcal{M}_{toy} \to \mathcal{M}_{toy}$ has five components, one of which is $\mathcal{N}_{toy}$, the others E^2_i are the proper transforms of the sets $(x_i \times \mathbb{C})/(\sigma_P \times \tau) \subset (P \times \mathbb{C})/(\sigma_P \times \tau)$.

[1]These moduli spaces were considered by Yokogawa [Yoko].

4.3 The Shatz stratification on $\mathcal{M}$

The results in Section 1.2.2 show that the Kähler manifold $(\mathcal{M}, I, \omega)$ is equipped with a $\mathbb{C}^*$-action which restricts to an $U(1) \subset \mathbb{C}^*$-action which is Hamiltonian with proper moment map μ. Moreover, 0 is an absolute minimum for μ. Therefore we are in the situation described in Section 2.1. In the following sections we will apply the ideas developed there to our situation and deduce important properties of the spaces $\mathcal{M}$, Z and $\overline{\mathcal{M}}$.

In the present and the following section we apply the general results of Section 2.1.1 to $\mathcal{M}$. Moreover we identify the downward and upward flows with important objects in the algebraic geometric understanding of the Higgs moduli problem. We show that the Hitchin stratification coincides with the Shatz stratification and that the downward Morse flow coincides with the nilpotent cone. First we deal with the Hitchin stratification given by the upward flows:

Recall from Subsection 2.1.1 that the upward flows give a stratification on $\mathcal{M}$:

$$\mathcal{M} = \bigcup_{d=0}^{g-1} U_d,$$

we call this the *Hitchin stratification* because Hitchin calculated[2] the Poincaré polynomial of $\mathcal{M}$ using the perfectness of this stratification.

We show in this section that the Hitchin stratification coincides with the Shatz stratification on $\mathcal{M}$. First we define the latter:

Definition 4.3.1 *Let $U_0' \subset \mathcal{M}$ be the locus of points $(E, \Phi) \in \mathcal{M}$ such that E is stable, and moreover for $d > 0$ let $U_d' \subset \mathcal{M}$ be the locus of points $(E, \Phi) \in \mathcal{M}$ such that the destabilizing line bundle of E is of degree d.*

Remark. 1. This stratification can be easily constructed in the gauge theory setting of Subsection 1.2.3. Namely one can pullback the Shatz stratification

$$\mathcal{C} = \bigcup_{d=0}^{\infty} \mathcal{C}_d,$$

by $\mathrm{pr}_0 : \mathcal{B}_0 \to \mathcal{C}$ from $\mathcal{C}$ to $\mathcal{B}_0$, restrict it to the open subset $(\mathcal{B}_0)^s$, which, being a $\overline{\mathcal{G}}^c$-invariant stratification induces a stratification on the quotient $\widetilde{\mathcal{M}}$ and in turn on $\mathcal{M}$. This is exactly the same what we defined above.

2. From (2) of Proposition 3.2 of [Nit] it follows that if $d > g-1$ the locus U_d' is empty.

3. The gauge theoretic construction in Remark 1 above implies that $\mathcal{M} = \bigcup_{d=0}^{g-1} U_d'$ is a (perfect) stratification in the sense of Subsection 2.2.1 and the stratum U_d' has real codimension $2g + 4d - 4$ for $d > 0$. Moreover from the description (3.30) it easily follows that $F_d \subset U_d'$.

It follows that U_d has the same codimension as U_d'. Moreover if $(E, \Phi) \in U_d'$ then clearly its entire $\mathbb{C}^*$-orbit is contained in U_d' thus it follows from (2.3) that

$$\lim_{z \to 0} (E, z \cdot \Phi) \in F \cap \overline{U_d'} \subset F \cap \bigcup_{i \geq d} U_i' = \bigcup_{i \geq d} F_d,$$

and in turn that

$$\overline{U_d'} \subset \bigcup_{i \geq d} U_d.$$

It follows that $U_{g-1}' \subset U_{g-1}$ and hence $U_{g-1}' = U_{g-1}$ because they are closed submanifolds of $\mathcal{M}$ with the same codimension and U_{g-1} is connected. An inductive argument proves the following

Proposition 4.3.2 *The Shatz stratification coincides with the Hitchin stratification i.e. $U_d' = U_d$.*

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[2] Cf. Subsection 3.6.

Remark. Thus the perfectness of the Shatz stratification implies the perfectness of the Hitchin stratification. Putting it in other words: Hitchin's calculation[3] of the Poincaré polynomial of $\mathcal{M}$ is analogous to the Atiyah-Bott's calculation[4] of the Poincaré polynomial of $\mathcal{N}$! In Section 7.5 we will show why the two calculations are profoundly related.

[3] Cf. Section 3.6.
[4] Cf. Section 3.4.

4.4 The nilpotent cone N

We saw in Theorem 2.1.1 that the downward Morse flow is a deformation retract of $\mathcal{M}$, so it is responsible for the topology, and as such it is an important object. On the other hand we will prove that the downward Morse flow coincides with the nilpotent cone.

Definition 4.4.1 *The nilpotent cone is the preimage of zero of the Hitchin map $N = \chi^{-1}(0)$.*

The name 'nilpotent cone' was given by Laumon, to emphasize the analogy with the nilpotent cone in a Lie algebra. In our context this is the most important fibre of the Hitchin map, and the most singular one at the same time. We will show below that the nilpotent cone is a central notion in our considerations.

Laumon in [Lau] investigated the nilpotent cone in a much more general context and showed its importance in the Geometric Langlands Correspondence. Thaddeus in [Tha1] concentrated on our case, and gave the exact description of the nilpotent cone. In what follows we will reprove some of their results.

The following assertion was already stated in [Tha1] which will turn out to be crucial in some of our considerations.

Theorem 4.4.2 *The downward Morse flow coincides with the nilpotent cone.*

Proof. As we saw in Theorem 2.1.1 the downward Morse flow can be identified with the set of points in $\mathcal{M}$ whose $\mathbb{C}^*$-orbit is relatively compact in $\mathcal{M}$.

Since the nilpotent cone is invariant under the $\mathbb{C}^*$-action and compact (χ is proper) we immediately get that the nilpotent cone is a subset of the downward Morse flow.

On the other hand if a point in $\mathcal{M}$ is not in the nilpotent cone then the image of its $\mathbb{C}^*$-orbit by the Hitchin map is a line in $\mathbb{C}^{3g-3}$, therefore cannot be relatively compact. □

Laumon's main result is the following assertion[5], which we prove in our case:

Corollary 4.4.3 (Laumon) *The nilpotent cone is a Lagrangian subvariety of $\mathcal{M}$ with respect to the holomorphic symplectic form ω_h.*

Proof. The Hitchin map is a completely integrable Hamiltonian system, and the nilpotent cone is a fibre of this map, so it is coisotropic. Therefore it is Lagrangian if and only if its dimension is $3g-3$.

On the other hand the nilpotent cone is exactly the downward Morse flow and we can use Hitchin's description of the critical submanifolds in [Hit1], giving that the sum of the index and the real dimension of any critical submanifold is $6g-6$. We therefore conclude that the complex dimension of the downward Morse flow (i.e. the nilpotent cone) is $3g-3$. □

Remark. Nakajima's Proposition 7.1 in [Nak] states that if X is a Kähler manifold with a $\mathbb{C}^*$-action and a holomorphic symplectic form ω_h of homogeneity 1 then the downward Morse flow of X is Lagrangian with respect to ω_h. Thus Nakajima's result and Theorem 4.4.2 together give an alternative proof of the theorem. We preferred the one above for it concentrates on the specific properties of $\mathcal{M}$.

From the above proof we can see that for higher rank Higgs bundles Laumon's theorem is equivalent to the assertion that every critical submanifold contributes to the middle dimensional cohomology, i.e the sum of the index and the real dimension of any critical submanifold should always be half of the real dimension of the corresponding moduli space.

Using the results of [Goth1] one easily shows that the above statement also holds for the rank 3 case. Gothen could show directly the above statement for any rank and therefore gave an alternative proof of Laumon's theorem in these cases [Goth2].

Corollary 4.4.4 *The middle dimensional homology $H_{6g-6}(\mathcal{M})$ of $\mathcal{M}$ is freely generated by the homology classes of irreducible components of the nilpotent cone and therefore has dimension g.*

[5] Cf. Theorem 3.1 in [Lau].

Proof. We know that each component of N is a projective variety of dimension $3g-3$. N is a deformation retract of $\mathcal{M}$, therefore the middle dimensional homology of $\mathcal{M}$ is generated by the homology classes of the components of N. Furthermore, from the Morse picture, components of N are in a one to one correspondence with the critical manifolds of $\mathcal{M}$, so there are g of them. The result follows. □

We finish this section with Thaddeus's description[6] of the nilpotent cone.

Theorem 4.4.5 *The nilpotent cone is the union of $D_0 = \mathcal{N}$ and the downward flows D_d, which are total spaces of vector bundles E_d^- over F_d, where E_d^- is the negative subbundle of $T_{\mathcal{M}} \mid_{F_d}$.*

Moreover, the restricted action of $\mathbb{C}^*$ on N is just the inverse multiplication on the fibres.

Proof. This follows directly from Theorem 2.1.1 and Theorem 4.4.2, noting that by Hitchin's description of the weights of the circle action on $T_{\mathcal{M}} \mid F_d$ in the proof of Proposition 7.1 of [Hit1], we have that there is only one negative weight. Therefore the β-fibration of Theorem 2.1.1 is a vector bundle in this case. The result follows. □

Remark. From the description of E_d^- in [Tha1] and that of E_d^2, a component of the fixed point set of the involution $\sigma(E,\Phi) = (E,-\Phi)$, in [Hit1], one obtains the remarkable fact that the vector bundle E_d^- is actually dual to E_d^2. This is not so surprising if we observe that E_d^- is the weight -1 and E_d^2 is the weight 2 component of the $\mathbb{C}^*$-equivariant bundle $T_{\mathcal{M}} \mid_{F_d}$ and these are naturally dual, because of the homogeneity 1 holomorphic symplectic structure on $\mathcal{M}$!

Example. In our toy example we have the elliptic fibration $\chi_{toy} : \mathcal{M}_{toy} \to \mathbb{C}$, with the only singular fibre $N_{toy} = \chi_{toy}^{-1}(0)$, the toy nilpotent cone. We have now the decomposition

$$N_{toy} = \mathcal{N}_{toy} \cup \bigcup_{i=1}^{4} D_i,$$

where we think of D_i as the closure of E_i, the total space of the trivial line bundle on $\tilde{x}_i$.

The possible singular fibres of elliptic fibrations have been classified by Kodaira[7]. According to this classification N_{toy} is of type $I_0^*(\tilde{D}_4)$.

[6] See [Tha1] and Proposition 5.4.2, cf. also [Lau].

[7] Cf. [B,P,V], p. 150.

4.5 The highest level Kähler quotient Z

In this section we apply the ideas of Subsection 2.1.2 to our situation.

Definition 4.5.1 *Define for every non negative t the Kähler quotient*

$$Q_t = \mu^{-1}(t)/U(1).$$

As the complex structure of the Kähler quotient depends only on the connected component of the regular values of μ, we can define $Z_d = Q_t$ for $c_d < t < c_{d+1}$ as a complex orbifold[8]. Similarly, we define X_{Z_d} to be $\mathcal{M}_t^{min}$ for $c_d < t < c_{d+1}$.

For simplicity let the highest level quotient Z_{g-1} be denoted by Z and the corresponding principal $\mathbb{C}^*$-bundle $X_{Z_{g-1}}$ by X_Z.

In the spirit of Theorem 2.1.3 we have the following

Theorem 4.5.2 *Z_d is a complex orbifold with only $\mathbb{Z}_2$-singularities, the singular locus is diffeomorphic to some union of projectivized vector bundles $P(E_i^2)$:*

$$\mathrm{Sing}(Z_d) = \bigcup_{0<i\leq d} P(E_i^2),$$

where $E_i^2 \subset \mathcal{M}$ is the total space of a vector bundle over F_i and is a component of the fixed point set of the involution $\sigma(E,\Phi) = (E,-\Phi)$.

Proof. The induced action of $U(1)$ on $\mathbb{C}^{3g-3}$ by the Hitchin map is multiplication by $e^{2i\theta}$ so an orbit of $U(1)$ on $\mathcal{M}\setminus N$ is a non trivial double cover of the image orbit on $\mathbb{C}^{3g-3}$. On the other hand by Thaddeus' description of N (Theorem 4.4.5) it is clear that if a point of N is not a fixed point of the circle action, then the stabilizer is trivial at that point.

Summarizing these two observations we obtain that if a point of $\mathcal{M}$ is not fixed by $U(1)$, then its stabilizer is either trivial or $\mathbb{Z}_2$. The latter case occurs exactly at the fixed point set of the involution σ. The statement now follows from Theorem 2.1.3. □

Proposition 4.5.3 *Z_d and Z_{d+1} are related by a blowup followed by a blowdown. Namely, Z_d blown up along $P(E_d^-)$ is the same as the singular quotient Q_{c_d} blown up along F_d (its singular locus), which in turn gives Z_{d+1} blown up at $P(E_d^+)$. Moreover, this birational equivalence is an isomorphism outside an analytic set of codimension at least* 3.

Proof. The first bit is just the restatement of Theorem 2.1.3 in our setting.

The second part follows because

$$\dim(P(E_d^-)) = 3g-3-1 < 6g-6-2$$

and

$$\dim(P(E_d^+)) = 3g-3+2g-2d-1-1 < 6g-6-2$$

for $g > 1$. □

Corollary 4.5.4 *$Z = Z_{g-1}$ is birationally equivalent to $P(T_{\mathcal{N}}^*) = Z_0$. Moreover this gives an isomorphism in codimension > 2.*

Proof. Obviously X_{Z_0} is $T_{\mathcal{N}}^*$, and therefore by Theorem 2.1.2 Z_0 is isomorphic to the projectivized cotangent bundle $P(T_{\mathcal{N}}^*)$. The statement follows from the previous theorem. □

Corollary 4.5.5 *Z has Poincaré polynomial*

$$P_t(Z) = \frac{t^{6g-6}-1}{t^2-1}P_t(\mathcal{N}) + \sum_{d=1}^{g-1}\frac{t^{6g-6}-t^{2g-4+4d}}{t^2-1}P_t(F_d),$$

where F_d is a 2^{2g}-fold cover of $\Sigma_{\bar{d}}$.

[8] We set $c_g = \infty$.

Proof. One way to derive this formula is through Kirwan's formula in [Kir1]. We use the above blowup, blowdown picture instead. This approach[9] is due to Thaddeus.

Applying the formula in [Gr,Ha],p.605 twice we get that

$$P_t(Z_{d+1}) - P_t(Z_d) = P_t(PE_d^+) - P_t(PE_d^-).$$

On the other hand for a projective bundle on a manifold $P \to M$ with fibre $\mathbb{P}^n$ one has[10]

$$P_t(P) = \frac{t^{2n+2}-1}{t^2-1} P_t(M).$$

Hence the formula follows. □

Remark. All the Poincaré polynomials on the right hand side of the above formula were calculated[11] in Chapter 3.

We will determine the Picard group of Z exactly. First we define some line bundles on several spaces.

Notation 4.5.6 *Let*

- $\mathcal{L}_{\mathcal{N}}$ *denote the ample generator*[12] *of the Picard group of* $\mathcal{N}$,
- $\mathcal{L}_{PT^*_{\mathcal{N}}}$ *be its pullback to* $PT^*_{\mathcal{N}}$,
- $\mathcal{L}_Z$ *denote the corresponding line bundle*[13] *on* Z,
- $L_{PT^*_{\mathcal{N}}}$ *be the dual of the tautological line bundle on* $PT^*_{\mathcal{N}}$,
- $L_Z = X_Z^* \times_{\mathbb{C}^*} \mathbb{C}$ *denote the corresponding line orbibundle on* Z.

Corollary 4.5.7 $\mathrm{Pic}(Z)$*, the Picard group of* Z*, is of rank* 2 *over* $\mathbb{Z}$ *and is freely generated by* $\mathcal{L}_Z$ *and* L_Z.

Remark. The Picard group of Z is the group of invertible sheaves on Z. As the singular locus of Z has codimension ≥ 2, this group can be thought of as the group of holomorphic line orbibundles on Z. Namely, in this case the restriction of a holomorphic line orbibundle to $Z \setminus \mathrm{Sing}(Z)$ gives a one-to-one correspondence between holomorphic line orbibundles on Z and holomorphic line bundles on $Z \setminus \mathrm{Sing}(Z)$, by the appropriate version of Hartog's theorem.

Proof. It is well known[14] that $\mathrm{Pic}(\mathcal{N})$ is freely generated by one ample line bundle $\mathcal{L}_{\mathcal{N}}$ therefore is of rank 1. Thus $\mathrm{Pic}(P(T^*_{\mathcal{N}}))$ is of rank 2 and freely generated by $\mathcal{L}_{PT^*_{\mathcal{N}}}$ the pullback of $\mathcal{L}_{\mathcal{N}}$ and the dual of the tautological line bundle $L_{PT^*_{\mathcal{N}}}$. From Corollary 4.5.4 $\mathrm{Pic}(Z)$ is isomorphic with $\mathrm{Pic}(P(T^*_{\mathcal{N}}))$ therefore is of rank 2, and freely generated by $\mathcal{L}_Z$ and L_Z, where $\mathcal{L}_Z$ is isomorphic to $\mathcal{L}_{PT^*_{\mathcal{N}}}$ and L_Z is isomorphic to $L_{PT^*_{\mathcal{N}}}$ outside the codimension 2 subset of Corollary 4.5.4. □

Definition 4.5.8 *A contact structure on a compact complex orbifold* Z *of complex dimension* $2n-1$ *is given by the following data:*

1. *a contact line orbibundle* L_Z *such that* $L_Z^n = K_Z^{-1}$*, where* K_Z *is the line orbibundle of the canonical divisor of* Z,
2. *a complex contact form* $\theta \in H^0(Z, \Omega^1(Z) \otimes L_Z)$ *a holomorphic* L_Z *valued* 1*-form, such that*

$$0 \neq \theta \wedge (d\theta)^{n-1} \in H^0(Z, \Omega^{2n-1}(Z) \otimes K_Z^{-1}) = H^0(Z, \mathcal{O}_Z) = \mathbb{C} \tag{4.1}$$

is a nonzero constant.

Theorem 4.5.9 *There is a canonical holomorphic contact structure on* Z *with contact line orbibundle* L_Z.

[9] Cf. [Tha2].
[10] Cf. [Gr,Ha] p.606.
[11] For $P_t(\mathcal{N})$ see (3.19), for $P_t(F_d)$ see (3.32).
[12] Cf. [Dr,Na].
[13] Cf. Corollary 4.5.4.
[14] Cf. [Dr,Na].

Proof. This contact structure can be defined by the construction of Lebrun as in [Leb] Remark 2.2. We only have to note that the holomorphic symplectic form ω_h on $\mathcal{M}$ is of homogeneity 1.

The construction goes as follows. If $\mathrm{pr}_Z : X_Z^* \to Z$ denotes the canonical projection of the principal $\mathbb{C}^*$-orbibundle X_Z^* the dual of X_Z, then $\mathrm{pr}_Z^*(L_Z)$ is canonically trivial with the canonical section having homogeneity 1. Thus in order to give a complex contact form $\theta \in H^0(Z, \Omega^1(Z) \otimes L_Z)$ it is sufficient to give a 1-form $\mathrm{pr}_Z^* \theta$ on X^* of homogeneity 1. This can be defined by $\mathrm{pr}_Z^* \theta = i(\xi)\omega_h$, where $\xi \in H^0(\mathcal{M}, T_\mathcal{M})$ is the holomorphic vector field generated by the $\mathbb{C}^*$-action. The non-degeneracy condition (4.1) is exactly equivalent to requiring that the closed holomorphic 2 form ω_h satisfy $\omega_h^n \neq 0$. This is the case as ω_h is a holomorphic symplectic form.

The result follows. □

We will be able to determine the line orbibundle L_Z explicitly. For this, consider the Hitchin map $\chi : \mathcal{M} \to \mathbb{C}^{3g-3}$. As it is equivariant with respect to the $\mathbb{C}^*$-action, χ induces a map

$$\overline{\chi} : Z \to \mathbb{P}^{3g-4}$$

on Z. The generic fibre of this map is easily seen to be the Kummer variety corresponding to the Prym variety (the Kummer variety of an Abelian variety is the quotient of the Abelian variety by the involution $x \to -x$), the generic fibre of the Hitchin map. Thus we have proved

Lemma 4.5.10 *There exists a map $\chi : Z \to \mathbb{P}^{3g-4}$ the reduction of the Hitchin map onto Z, for which the generic fibre is a Kummer variety.*

Remark. This observation was already implicit in Oxbury's thesis[15].

The following theorem determines the line bundle L_Z in terms of the Hitchin map.

Theorem 4.5.11 $L_Z^2 = \overline{\chi}^* \mathcal{H}_{3g-4}$ *where $\mathcal{H}_{3g-4}$ is the hyperplane bundle on $\mathbb{P}^{3g-4}$.*

Proof. We know from Corollary 4.5.7 that $\overline{\chi}^* \mathcal{H}_{3g-4} = \mathcal{L}_Z^k \otimes L_Z^l$ for some integers k and l.

We show that $k = 0$. For this consider the pullback of $\mathcal{L}_Z$ onto $\mathcal{M} \setminus N$ the total space of the principal $\mathbb{C}^*$-orbibundle X_Z^*. This line orbibundle extends to $\mathcal{M}$ as $\mathcal{L}_\mathcal{M}$ and restricts to $T_\mathcal{N}^*$ as the pullback of $\mathcal{L}_{PT_\mathcal{N}^*}$ by construction. $c_1(\mathcal{L}_\mathcal{M})$ is not trivial when restricted to $\mathcal{N}$ (namely it is $c_1(\mathcal{L}_\mathcal{N})$, since this bundle is ample) therefore is not trivial when restricted to a generic fibre of the Hitchin map. We can deduce that $c_1(\mathcal{L}_Z)$ is not trivial on the generic fibre of $\overline{\chi}$.

On the other hand L_Z restricted to a generic fibre of $\overline{\chi}$ can be described as follows. Let this Kummer variety be denoted by K, the corresponding Prym variety by P. Form the space $P \times \mathbb{C}^*$, the trivial principal $\mathbb{C}^*$-bundle on P and quotient it out by the involution $\tau(p, z) = (-p, -z)$. The resulting space is easily seen to be the $\mathbb{C}^*$-orbit of the Prym P in $\mathcal{M}$, therefore the total space of the principal $\mathbb{C}^*$-orbibundle $L_Z^* \setminus (L_Z^*)_0$ on K. Hence L_Z^2 is the trivial line orbibundle on K. Thus $c_1(L_Z \mid_K) = 0$.

Now $\overline{\chi}^* \mathcal{H}_{3g-4}$ is trivial on the Kummer variety. Hence the assertion $k = 0$.

The rest of the proof will follow the lines of Hitchin's proof of Theorem 6.2 in [Hit2]. We show that $l = 2$.

The sections of L_Z can be identified with holomorphic functions homogeneous of degree 2 on the principal $\mathbb{C}^*$-orbibundle $X_Z = L_Z^* \setminus (L_Z^*)_0$. As N is of codimension ≥ 2 such functions extend to $\mathcal{M}$. Since the Hitchin map is proper, these functions are constant on the fibres of the Hitchin map, therefore are the pullbacks of holomorphic functions on $\mathbb{C}^{3g-3}$ of homogeneity 1 which can be identified with the holomorphic sections of the hyperplane bundle $\mathcal{H}_{3g-4}$ on $P(\mathbb{C}^{3g-3}) = \mathbb{P}^{3g-4}$. □

Corollary 4.5.12 *If n is odd, there are natural isomorphisms*

$$H^0(Z, L_Z^n) \cong H^0(\mathcal{N}, S^n T_\mathcal{N}) \cong 0,$$

whereas if n is even, then

$$H^0(Z, L_Z^n) \cong H^0(\mathcal{N}, S^n T_\mathcal{N}) \cong H^0(\mathbb{P}^{3g-4}, \mathcal{H}_{3g-4}^{\frac{n}{2}}).$$

[15] Cf. 2.17a of [Oxb].

Proof. We show that $H^0(Z, L_Z^n) \cong H^0(\mathcal{N}, S^n(T_{\mathcal{N}}))$ for every n, the rest of the theorem will follow from Theorem 6.2 of [Hit2].

By Proposition 4.5.3 we get that $H^0(Z, L_Z^n) \cong H^0(PT^*_{\mathcal{N}}, L^n_{PT^*_{\mathcal{N}}})$. Let $\mathrm{pr}_{\mathcal{N}} : PT^*_{\mathcal{N}} \to \mathcal{N}$ denote the projection. It is well known that the Leray spectral sequence for $\mathrm{pr}_{\mathcal{N}}$ degenerates at the E^2 term. Moreover, we have[16] that $R^i(\mathrm{pr}_{\mathcal{N}})_*(L^n_{PT^*_{\mathcal{N}}}) = 0$ if $0 < i < 3g-4$. Therefore $H^0(PT^*_{\mathcal{N}}, L^n_{PT^*_{\mathcal{N}}}) \cong H^0(\mathcal{N}, (\mathrm{pr}_{\mathcal{N}})_*(L^n_{PT^*_{\mathcal{N}}}))$. Finally the sheaf $(\mathrm{pr}_{\mathcal{N}})_*(L^n_{PT^*_{\mathcal{N}}})$ is $S^n(T_{\mathcal{N}})$, which proves the statement. □

We can moreover determine the first cohomology group corresponding to the infinitesimal deformations of the holomorphic contact structure on Z and can interpret it in a nice way.

Corollary 4.5.13 *There are canonical isomorphisms*

$$H^1(Z, L_Z) \cong (H^1(\mathcal{M}, \mathcal{O}_{\mathcal{M}}))_1 \cong H^1(\mathcal{N}, T_{\mathcal{N}}) \cong H^1(\Sigma, K_\Sigma^{-1}),$$

where $(H^1(\mathcal{M}, \mathcal{O}_{\mathcal{M}}))_1 \subset H^1(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ *is the vector space of elements of* $H^1(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ *homogeneous of degree* 1.

Proof. We may use the cohomological version[17] of Hartog's theorem to show that

$$H^1(Z, L_Z) \cong H^1(PT^*_{\mathcal{N}}, L_{PT^*_{\mathcal{N}}}),$$

as Z and $PT^*_{\mathcal{N}}$ are isomorphic on an analytic set of codimension ≥ 3 from Proposition 4.5.3.

The proof of the other isomorphisms can be found in [Hit3]. □

Remark. We can interpret this result as saying that the deformation of the complex structure on Σ corresponds to the deformation of complex structure on $\mathcal{N}$, to the deformation[18] of holomorphic contact structure on Z and to the deformation of the holomorphic symplectic structure of homogeneity 1 on $\mathcal{M}$.

As an easy corollary of the above we note the following

Corollary 4.5.14 *The line orbibundle L_Z is nef but neither trivial nor ample.*

Proof. The line bundle L_Z is certainly not ample since $c_1(L_Z)$ is trivial on the Kummer variety. On the other hand L_Z^2 being the pullback of an ample bundle is not trivial and is nef itself, hence the result. □

The next theorem will describe the inherited Kähler structures of Z. Considering the one-parameter family of Kähler quotients Q_t, for $t > c_{g-1}$ we get a one-parameter family of Kähler forms ω_t on Z. Theorem 1.1 from [Du,He] gives the following result for our case[19].

Theorem 4.5.15 (Duistermaat, Heckman) *The complex orbifold Z has a one-parameter family of Kähler forms ω_t, $t > c_{g-1}$ such that*

$$[\omega_{t_1}(Z)] - [\omega_{t_2}(Z)] = (t_1 - t_2)c_1(L_Z)$$

where $t_1, t_2 > c_{g-1}$ and $[\omega_t] \in H^2(Z, \mathbb{R})$ is the cohomology class of ω_t.

Many of the above results will help us to prove the following theorem.

Theorem 4.5.16 *Z is a projective algebraic variety.*

[16] Cf. [Har] Theorem 5.1b.
[17] Cf. [Sche].
[18] Cf. [Leb].
[19] Cf. Theorem 2.1.3.

Proof. By the Kodaira embedding theorem for orbifolds[20] we have only to show that Z with a suitable Kähler form is a Hodge orbifold, i.e. the Kähler form is integer. For this to see we show that the Kähler cone of Z contains a subcone, which is open in $H^2(Z,\mathbb{R})$. This is sufficient since such an open subcone must contain an integer Kähler form i.e. a Hodge form.

Since Corollary 4.5.7 shows that $\mathrm{Pic}_0(Z)$ is trivial, by Corollary 4.5.14 we see that $c_1(L_Z) \neq 0$. Therefore the previous theorem exhibited a half line in the Kähler cone of Z. Thus to find an open subcone in the 2 dimensional vector space $H^2(Z,\mathbb{R})$ (Corollary 4.5.7) it is sufficient to show that this line does not go through the origin or in other words $c_1(L)$ is not on the line. But this follows from Corollary 4.5.14, because L being not ample $c_1(L)$ cannot contain a Kähler form. Hence the result. □

Remark. We see from this proof that $c_1(L_Z)$ lies on the closure of the Kähler cone, thus L_Z is nef. This reproves a statement of Corollary 4.5.14.

Example. In the case of the toy example the lowest level Kähler quotient Z_0 is the projectivized cotangent bundle $PT^*_{\mathcal{N}_{toy}}$ of $\mathcal{N}_{toy}$, which is isomorphic to $\mathcal{N}_{toy} = \mathbb{P}^1$, and the blowups and blowdowns add the four marked points to $\mathbb{P}^1$. Therefore Z_{toy} is isomorphic to the orbifold $\mathbb{P}^1_4$, where the marked points correspond to the fixed point set of the involution σ, namely these are the projectivized bundles PE_i^2, i.e. points.

Moreover the principal $\mathbb{C}^*$-orbibundle $X_{Z_{toy}}$ on $\mathbb{P}^1_4$ has the form

$$X_{Z_{toy}} = (P \times \mathbb{C}^*)/(\sigma_P \times \tau).$$

Thus in the toy example, unlike in the ordinary Higgs case, we have $c_1(L_{Z_{toy}}) = 0$. This latter assertion can be seen using 4.5.11 and noting that the target of the reduced toy Hitchin map $\overline{\chi}_{toy} : Z_{toy} \to \mathbb{P}^0$ is a point.

There is another difference, namely the Picard group of Z_{toy} is of rank 1, because $L^2_{Z_{toy}}$ is the trivial bundle on Z_{toy}.

In the next section we show how to compactify $\mathcal{M}$ by sewing in Z at infinity.

[20] Cf. [Bai].

4.6 The compactification $\overline{\mathcal{M}}$

In this section we compactify $\mathcal{M}$ by adding to each non-relatively compact $\mathbb{C}^*$-orbit an extra point i.e. sewing in Z at infinity. Another way of saying the same is to glue together $\mathcal{M}$ and E the total space of L_Z along the principal $\mathbb{C}^*$-orbibundle $X_Z^* = E \setminus E_0 = \mathcal{M} \setminus N$. To be more precise we use the construction of Lerman, called the symplectic cut[21].

Since the complex structure on the Kähler quotients depends only on the connected component of the level, we can make the following definition.

Definition 4.6.1 *Let $\overline{\mathcal{M}}_d$ denote the underlying compact complex orbifold of the Kähler quotient of $\mathcal{M}\times\mathbb{C}$ by the product $U(1)$-action*

$$\overline{\mathcal{M}}_{\mu<t} = (\mu + \mu_{\mathbb{C}})^{-1}(t)/U(1),$$

with $c_d < t < c_{d+1}$.

Let $X_{\overline{\mathcal{M}}_d}$ denote the corresponding principal $\mathbb{C}^*$-bundle on $\overline{\mathcal{M}}_d$. For simplicity we let $\overline{\mathcal{M}}$ denote $\overline{\mathcal{M}}_{g-1}$ and $X_{\overline{\mathcal{M}}}$ denote $X_{\overline{\mathcal{M}}_{g-1}}$.

As a consequence of the construction of symplectic cutting we have the following theorem[22]:

Theorem 4.6.2 *The compact orbifold $\overline{\mathcal{M}} = \mathcal{M} \cup Z$ is a compactification of $\mathcal{M}$ such that $\mathcal{M}$ is an open complex submanifold and Z is a codimension one suborbifold, i.e. a divisor.*

Moreover, $\mathbb{C}^*$ acts on $\overline{\mathcal{M}}$ extending the action on $\mathcal{M}$ with the points of Z being fixed.

In addition to the above we see that we have another decomposition $\overline{\mathcal{M}} = N \cup E$ of $\overline{\mathcal{M}}$ into the nilpotent cone and the total space E of the contact line bundle L_Z on Z. Thus the compactification by symplectic cutting produced the same orbifold as the two constructions we started this section with.

We start to list the properties of $\overline{\mathcal{M}}$. We will mention properties analogous to properties of Z (these correspond to the fact that both spaces were constructed by a Kähler quotient procedure) and we will clarify the relation between Z and $\overline{\mathcal{M}}$.

Theorem 2.1.4 and Theorem 2.1.3 give the following result in our case.

Theorem 4.6.3 *$\overline{\mathcal{M}}_d$ is a compact orbifold. It has a decomposition $\overline{\mathcal{M}}_d = \mathcal{M}_d \cup Z_d$ into an open complex suborbifold $\mathcal{M}_d$ (which is actually a complex manifold) and a codimension one suborbifold Z_d, i.e. a divisor. The singular locus of $\overline{\mathcal{M}}_d$ coincides with that of Z_d:*

$$\mathrm{Sing}(\overline{\mathcal{M}}_d) = \mathrm{Sing}(Z_d) = \bigcup_{0<i\leq d} P(E_i^2)$$

where E_i^2 is a component of the fixed points set of the involution $\sigma(E,\Phi) = (E,-\Phi)$.

Furthermore, the $\mathbb{C}^*$-action on $\mathcal{M}_d$ extends to $\overline{\mathcal{M}}_d$ with an extra component Z_d of the fixed point set.

We have the corresponding statement of Theorem 4.5.4.

Theorem 4.6.4 *$\overline{\mathcal{M}} = \overline{\mathcal{M}}_{g-1}$ is birationally isomorphic to $\overline{\mathcal{M}}_0 = P(T^*_{\mathcal{N}} \oplus \mathcal{O}_{\mathcal{N}})$. Moreover, they are isomorphic outside an analytic subset of codimension at least* 3.

Proof. In a similar manner to the proof of Corollary 4.5.4 we can argue by noting that $X_{\overline{\mathcal{M}}_0}$ is obviously isomorphic to $T^*_{\mathcal{N}} \oplus \mathcal{O}_{\mathcal{N}}$ with the standard action of $\mathbb{C}^*$. Hence indeed $\mathcal{M}_0 = P(T^*_{\mathcal{N}} \oplus \mathcal{O}_{\mathcal{N}})$.

By Theorem 2.1.3 it is clear that $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}_0$ are related by a sequence of blowups and blowdowns. The codimensions of the submanifolds we apply the blowups are at least 3 by a calculation analogous to the one in the proof of Proposition 4.5.3. □

Notation 4.6.5 *Let*

- *$\mathcal{L}_{P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})}$ denote the pullback of $\mathcal{L}_{\mathcal{N}}$ to $P(T^*_{\mathcal{N}} \oplus \mathcal{O}_{\mathcal{N}})$,*
- *$\mathcal{L}_{\overline{\mathcal{M}}}$ be the corresponding line bundle on $\overline{\mathcal{M}}$,*

[21] Cf. Subsection 2.1.3 and [Ler].
[22] Cf. Theorem 2.1.4.

- $L_{P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})}$ *be the dual of the tautological line bundle on the projective bundle* $P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})$,
- $L_{\overline{\mathcal{M}}} = X_{\overline{\mathcal{M}}} \times_{\mathbb{C}^*} \mathbb{C}$ *be the corresponding line orbibundle on* $\overline{\mathcal{M}}$.

Corollary 4.6.6 $\operatorname{Pic}\overline{\mathcal{M}}$ *is isomorphic to* $\operatorname{Pic}(P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}}))$ *and therefore is of rank* 2 *and freely generated by* $L_{\overline{\mathcal{M}}}$ *and* $\mathcal{L}_{\overline{\mathcal{M}}}$.

Proof. The previous theorem shows that $\overline{\mathcal{M}}$ and $P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})$ are isomorphic outside an analytic subset of codimension at least 2, thus their Picard groups are naturally isomorphic.

However, $\operatorname{Pic}(P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}}))$ is freely generated by $L_{P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})}$ and $\mathcal{L}_{P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})}$. The result follows. □

Corollary 4.6.7 *The canonical line orbibundle* $K_{\overline{\mathcal{M}}}$ *of* $\overline{\mathcal{M}}$ *coincides with* $L_{\overline{\mathcal{M}}}^{-(3g-2)}$. *Moreover,* $L_{\overline{\mathcal{M}}}$ *is the line bundle of the divisor* Z, *therefore (3g-2)Z is the anticanonical divisor of* $\overline{\mathcal{M}}$. *Finally,* $L_{\overline{\mathcal{M}}}$ *restricts to* L_Z *to* Z.

Proof. $L_{\overline{\mathcal{M}}}$ by its construction clearly restricts to L_Z on Z and it is the line bundle of Z, as the corresponding statement is obviously true for $P(T^*_{\mathcal{N}}\oplus\mathcal{O}_{\mathcal{N}})$.

The restriction of $K_{\overline{\mathcal{M}}}$ to $\mathcal{M}$ has a non-zero section, namely the holomorphic Liouville form ω_h^{3g-3}, thus trivial. Hence $K_{\overline{\mathcal{M}}} = L^k_{\overline{\mathcal{M}}}$ for some $k\in\mathbb{Z}$.

By the adjunction formula $K_Z = (K_{\overline{\mathcal{M}}}\otimes[Z])\mid_Z$. The right hand side equals $L_Z^{-(3g-3)}$ as L_Z is a contact line bundle[23]. The left hand side can be written as $(L^k_{\overline{\mathcal{M}}}\otimes L_{\overline{\mathcal{M}}})\mid_Z = L_Z^{k+1}$, therefore $k=-(3g-2)$. □

Lemma 4.6.8 χ *has an extension to* $\overline{\mathcal{M}}$,

$$\overline{\chi}:\overline{\mathcal{M}}\to\mathbb{P}^{3g-3}$$

such that $\overline{\chi}$ *restricted to* Z *gives the map of Lemma 4.5.10.*

Proof. We let $\mathbb{C}^*$ act on $\mathbb{C}^{3g-3}\times\mathbb{C}$ by $\lambda(x,z)=(\lambda^2x,\lambda z)$. With respect to this action the map $(\chi, id_{\mathbb{C}}):\mathcal{M}\times\mathbb{C}\to\mathbb{C}^{3g-3}\times\mathbb{C}$ is equivariant. Therefore making the symplectic cut it reduces to a map $\overline{\chi}:\overline{\mathcal{M}}\to\mathbb{P}^{3g-3}$ since the quotient space $(\mathbb{C}^{3g-3}\setminus 0)\times\mathbb{C}/\mathbb{C}^*$ is isomorphic to $\mathbb{P}^{3g-3}$.

The result follows. □

Remark. In the higher rank case, where $\mathbb{C}^*$ acts on the target space of the Hitchin map with different weights, the target space of the compactified Hitchin map is a weighted projective space.

Corollary 4.6.9 $L^2_{\overline{\mathcal{M}}} = \overline{\chi}^*\mathcal{H}_{3g-3}$.

Proof. Obviously, $\overline{\chi}^*\mathcal{H}_{3g-3}\mid_{\mathcal{M}}$ is trivial, therefore $\overline{\chi}^*\mathcal{H}_{3g-3}$ is some power of $L_{\overline{\mathcal{M}}}$. By 4.5.11 this power is 2. □

Theorem 4.6.10 (Duistermaat, Heckman) $\overline{\mathcal{M}}$ *has a one-parameter family of Kähler forms* $\omega_t(\overline{\mathcal{M}})$, $t>c_{g-1}$ *such that*

$$[\omega_{t_1}(\overline{\mathcal{M}})]-[\omega_{t_2}(\overline{\mathcal{M}})]=(t_1-t_2)c_1(L_{\overline{\mathcal{M}}}).$$

Furthermore this one-parameter family of Kähler forms restricts to Z *as the one-parameter family of Kähler forms of Theorem 4.5.15.*

Proof. This is just the application of Theorem 2.1.3 and Theorem 2.1.4 to our situation. □

Corollary 4.6.11 $\overline{\mathcal{M}}$ *is a projective algebraic variety.*

[23] Cf. Theorem 4.5.9.

Proof. The argument is the same as for Theorem 4.5.16, noting that by Corollary 4.6.6 $H^2(\overline{\mathcal{M}}, \mathbb{R})$ is two dimensional and $L_{\overline{\mathcal{M}}}$ is neither trivial nor ample since $L_{\overline{\mathcal{M}}} \mid Z = L_Z$ (by Corollary 4.6.7) is neither trivial nor ample (by Corollary 4.5.14). □

Remark. 1. The above proof yields that the cohomology class $c_1(L_{\overline{\mathcal{M}}})$ sits in the closure of the Kähler cone of $\overline{\mathcal{M}}$, hence $L_{\overline{\mathcal{M}}}$ is nef.

2. From the previous remark and Corollary 4.6.9 we can deduce that there is a complete hyperkähler (hence Ricci flat) metric on $\mathcal{M} = \overline{\mathcal{M}} \setminus Z$, the complement of a nef anticanonical divisor of a compact orbifold.

Therefore our compactification of $\mathcal{M}$ is compatible with Yau's problem, which addresses the question: which non-compact complex manifolds possess a complete Ricci flat metric? Tian and Yau in [Ti,Ya] could show that this is the case for the complement of an ample anticanonical divisor in a compact complex manifold. (Such manifolds are called Fano manifolds.)

The similar statement with ample replaced by nef is an unsolved problem.

Theorem 4.6.12 *$\overline{\mathcal{M}}$ has Poincaré polynomial*

$$P_t(\overline{\mathcal{M}}) = P_t(\mathcal{M}) + t^2 P_t(Z).$$

Proof. We have three different ways of calculating the Poincaré polynomial of $\overline{\mathcal{M}}$. The first is through Kirwan's formula in [Kir1], the second is due to Thaddeus in [Tha3], which we used to calculate the Poincaré polynomial of Z.

For $\overline{\mathcal{M}}$ there is a third method, namely direct Morse theory. All we have to note is that the $U(1)$-action $\overline{\mathcal{M}}$ is Hamiltonian with respect to any Kähler form of Theorem 4.6.10, and the critical submanifolds and corresponding indices are the same as for $\mathcal{M}$ with one extra critical submanifold Z of index 2. Hence the result. □

Example. We can describe $\overline{\mathcal{M}}_{toy} = \mathcal{M}_{toy} \cup Z_{toy}$ as follows. As we saw above $\mathcal{M}_{toy} \setminus N_{toy} = X_{Z_{toy}}$. Thus gluing together $\mathcal{M}_{toy}$ and E_{toy}, the total space of the line orbibundle $L_{Z_{toy}}$, along $X_{Z_{toy}}$ yields

$$\overline{\mathcal{M}}_{toy} = \mathcal{M}_{toy} \cup_{X_{Z_{toy}}} E_{toy}.$$

One can construct $\overline{\mathcal{M}}_{toy}$ directly, as follows. Take $\mathbb{P}^1 = \mathbb{C} \cup \infty$ extending the involution τ from $\mathbb{C}$ to $\mathbb{P}^1$. Consider the quotient $(P \times \mathbb{P}^1)/(\sigma_P \times \tau)$. This is a compact orbifold with eight $\mathbb{Z}_2$-quotient singularities. Blow up four of them corresponding to $0 \in \mathbb{C}$. The resulting space will be isomorphic to $\overline{\mathcal{M}}_{toy}$. The remaining four isolated $\mathbb{Z}_2$ quotient singularities will just be the four marked points of $Z_{toy} \subset \overline{\mathcal{M}}_{toy}$, the singular locus of $\overline{\mathcal{M}}_{toy}$.

We finish this section with a result which gives an interesting relation between the intersections of the component of the nilpotent cone N in $\mathcal{M}$ (equivalently the intersection form on the middle compact cohomology $H^{6g-6}_{cpt}(\mathcal{M})$ from Corollary 4.4.4) and the contact structure of Z.

Theorem 4.6.13 *There is a canonical isomorphism between the cokernel of $j_{\mathcal{M}}$ and the cokernel of L, where*

$$j_{\mathcal{M}} : H^{6g-6}_{cpt}(\mathcal{M}) \to H^{6g-6}(\mathcal{M})$$

is the canonical map and

$$L : H^{6g-8}(Z) \to H^{6g-6}(Z)$$

is multiplication with $c_1(L_Z)$.

Proof. We will read off the statement from the following diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^{6g-8}(Z) \arrow[d] \arrow[dr, "L"] & & \\
0 \arrow[r] & H^{6g-6}_{cpt}(\mathcal{M}) \arrow[r] \arrow[dr, "j"] & H^{6g-6}(\overline{\mathcal{M}}) \arrow[r] \arrow[d] & H^{6g-6}(Z) \arrow[r] & 0 \\
 & & H^{6g-6}(\mathcal{M}) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

We show that both the vertical and horizontal sequences are exact and the two triangles commute.

From the Bialynicki-Birula decomposition of $\overline{\mathcal{M}}$ we get the short exact sequence of middle dimensional cohomology groups[24]:

$$0 \to H^{6g-6}_{cpt}(E) \to H^{6g-6}(\overline{\mathcal{M}}) \to H^{6g-6}(\mathcal{M}) \to 0.$$

Applying the Thom isomorphism[25] we can identify $H^{6g-6}_{cpt}(E)$ with $H^{6g-8}(Z)$, this gives the vertical short exact sequence of the diagram. The horizontal one is just its dual short exact sequence.

Finally, the left triangle clearly commutes as all the maps are natural, while the right triangle commutes because the original triangle commuted as above and the canonical map $j_E : H^{6g-6}_{cpt}(E) \to H^{6g-6}(E)$ transforms to $L : H^{6g-8}(Z) \to H^{6g-6}(Z)$ by the Thom isomorphism.

Now the theorem is the consequence of the Butterfly lemma[26], or can be proved by an easy diagram chasing.

Hence the result follows. $\square$

Remark. 1. In the next Chapter we shall prove Theorem 0.2.1, that $j_{\mathcal{M}}$ is 0. Combined with the above theorem we have that the cokernel of L is g-dimensional! We will say more about this in Section 8.1.

2. If the line bundle L_Z was ample, the map L would just be the Lefschetz isomorphism, and therefore the cokernel would be trivial. In our case we have L_Z being only nef and the map is certainly not an isomorphism, the cohomology class of the Kummer variety lies in the kernel. Therefore the cokernel measures how far L_Z is from being ample.

3. Notice that the proof did not use any particular property of $\mathcal{M}$, therefore the statement is true in the general setting of Section 2.1.

Example. We can simply calculate the dimension of the cokernels in our toy example. Namely, the dimension of $\mathrm{coker}(L_{toy})$ is clearly 1, as the map $L_{toy} : H^0(Z_{toy}) \to H^2(Z_{toy})$ is the multiplication[27] with $c_1(L_{Z_{toy}}) = 0$.

Thus, by the above theorem, we have that $\mathrm{coker}(j_{\mathcal{M}_{toy}})$ is 1-dimensional. It can be seen directly, using Zariski's lemma[28], that the kernel of the map $j_{\mathcal{M}_{toy}}$ is generated by the cohomology class of the elliptic curve P, the generic fibre of the toy Hitchin map, hence it is 1-dimensional, indeed.

Thus $j_{\mathcal{M}_{toy}}$ is not 0 unlike $j_{\mathcal{M}}$. This indicates a profound difference between $\mathcal{M}$ and $\mathcal{M}_{toy}$.

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[24] Recall that $E \subset \overline{\mathcal{M}}$ denotes the total space of the contact line bundle L_Z on Z.

[25] This also holds in the orbifold category with coefficients from $\mathbb{Q}$.

[26] Cf. [Lan] IV.4 p.102.

[27] Cf. the example at the end of Section 4.5.

[28] Lemma 8.2 in [B,P,V] p. 90.

Chapter 5

Intersection numbers

The aim of the present chapter is to prove Theorem 0.2.1. We discussed in the Introduction the physical motivation for Theorem 0.2.1.

From an algebraic geometric point of view Theorem 0.2.1 can be interpreted as follows. First of all it is really about middle dimensional cohomology, because we know that $\mathcal{M}$ does not have cohomology beyond the middle dimension, and equivalently by Poincaré duality $\mathcal{M}$ does not have compactly supported cohomology below the middle dimension. Thus the main content of Theorem 0.2.1 is the vanishing of the canonical map $j_{\mathcal{M}} : H^{6g-6}_{cpt}(\mathcal{M}) \to H^{6g-6}(\mathcal{M})$ between g-dimensional spaces[1]. This in turn is equivalent to the vanishing of the intersection form on $H^{6g-6}_{cpt}(\mathcal{M})$.

There are $g+1$ intersection numbers whose vanishing follows easily. One vanishing is obtained by recalling that the moduli space $\mathcal{N}$ of stable bundles of real dimension $6g-6$ sits inside $\mathcal{M}$ with normal bundle $T^*_{\mathcal{N}}$, thus its self-intersection number is its Euler characteristic up to sign, which is known[2] to vanish.

The other g vanishings follow from the fact that the ordinary cohomology class of the Prym variety, the generic fibre of the Hitchin map, is 0 i.e. $j_{\mathcal{M}}(\overline{\eta}^{\mathcal{M}}_P) = 0$. This can be seen by thinking of the Hitchin map as a section of the trivial rank $3g-3$ vector bundle on $\mathcal{M}$ and considering the ordinary cohomology class of the Prym variety as the Euler class of this trivial vector bundle, and as such, the ordinary cohomology class of the Prym variety is trivial indeed. Note that for the case $g=2$, the above vanishings are already enough to have $j_{\mathcal{M}} = 0$.

The vanishing of the intersection form on $\mathcal{M}$ for any genus, proved in this chapter, can be considered as a generalization of these facts. We should also mention that as we explained at the end of the previous chapter, the intersection form is not trivial in the case of the toy example $\mathcal{M}_{toy}$.

The structure of this section is as follows: In the next section we develop the theory of stable Higgs bundles analogously to the stable case, and prove an important vanishing theorem. In Section 5.2 we prove that $\widetilde{\mathcal{M}}$ is a fine moduli space, and define certain universal bundles. In Section 5.3 we construct the virtual Dirac bundle[3], as the analogue of the virtual Mumford bundle, and show that it can be considered as the degeneracy sheaf of a homomorphism of vector bundles. In Section 5.4 we determine the degeneracy locus of the above homomorphism in terms of the components of the nilpotent cone. Finally in Section 5.5 we prove our main Theorem 0.2.1 using Porteous' formula for the degeneracy locus of the virtual Dirac bundle.

[1] Cf. Corollary 4.4.4.

[2] Substitute $t=-1$ into (3.19)!

[3] For its gauge theoretic construction see Subsection 1.7.

5.1 A vanishing theorem

Definition 5.1.1 *The complex $E \stackrel{\Phi}{\to} E \otimes K$ with E a vector bundle on Σ, K the canonical bundle of Σ, and $\Phi \in H^0(\Sigma, \mathrm{Hom}(E, E \otimes K))$, is called a* Higgs bundle*, while Φ is called the* Higgs field.

A morphism $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ between two Higgs bundles $\mathcal{E}_1 = E_1 \stackrel{\Phi_1}{\to} E_1 \otimes K$ and $\mathcal{E}_2 = E_2 \stackrel{\Phi_2}{\to} E_2 \otimes K$ is defined to be a homomorphism of vector bundles $\Psi \in \mathrm{Hom}(E_1, E_2)$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi_1} & E_1 \otimes K \\ \Psi \downarrow & & \downarrow \Psi \otimes id_K \\ E_2 & \xrightarrow{\Phi_2} & E_2 \otimes K \end{array}$$

Moreover we say that $\mathcal{E}_1$ is a Higgs subbundle *of $\mathcal{E}_2$ if $\Psi \in \mathrm{Hom}(E_1, E_2)$ is injective and a morphism of Higgs bundles. We denote this by $\mathcal{E}_1 \subset \mathcal{E}_2$. In this case we can easily construct the quotient Higgs bundle $\mathcal{E}_2/\mathcal{E}_1$ together with a surjective morphism of Higgs bundles $\pi : \mathcal{E}_2 \to \mathcal{E}_2/\mathcal{E}_1$ whose kernel is exactly $\mathcal{E}_1$.*

Remark. It is a tautology that morphisms of Higgs bundles form the hypercohomology[4] vector space

$$\mathbb{H}^0(\Sigma, E_1^* \otimes E_2 \stackrel{[\Phi_1,\Phi_2]}{\longrightarrow} E_1^* \otimes E_2 \otimes K),$$

where the homomorphism $[\Phi_1, \Phi_2]$ is given by:

$$[\Phi_1, \Phi_2]\,(\Psi) := (\Psi \otimes id_K)\Phi_1 - \Phi_2 \Psi$$

for $\Psi \in \mathrm{Hom}(E_1, E_2)$.

Now recall Definition 1.2.4 where we defined the notion of stability of Higgs bundles. The main result[5] of this section is the following theorem about morphisms between stable Higgs bundles.

Theorem 5.1.2 *Let $\mathcal{E} = E \stackrel{\Phi}{\to} E \otimes K$ and $\mathcal{F} = F \stackrel{\Psi}{\to} F \otimes K$ be stable Higgs bundles with $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Then the only morphism from $\mathcal{E}$ to $\mathcal{F}$ is the trivial one. In other words*

$$\mathbb{H}^0(\Sigma, E^* \otimes F \stackrel{[\Phi,\Psi]}{\longrightarrow} E^* \otimes F \otimes K) = 0.$$

Moreover if $\mu(\mathcal{F}) = \mu(\mathcal{E})$, then there is a non-trivial morphism $f : \mathcal{E} \to \mathcal{F}$ if and only if $\mathcal{E} \cong \mathcal{F}$ in which case every non-trivial morphism $f : \mathcal{E} \to \mathcal{F}$ is an isomorphism and

$$\dim(\mathbb{H}^0(\Sigma, E^* \otimes F \stackrel{[\Phi,\Psi]}{\longrightarrow} E^* \otimes F \otimes K)) = 1. \tag{5.1}$$

Proof. A characteristic property of the ring $\mathbb{C}[x_1]$ is that it is the only principal ideal domain among the rings $\mathbb{C}[x_1, \ldots, x_n]$. It follows that $\mathcal{O}_\Sigma$ is a sheaf of principal ideal domains, and that every torsion free sheaf[6] over Σ, such as a subsheaf of a locally free sheaf, will be locally free. An easy consequence of this is the lemma of Narasimhan and Seshadri[7]:

Lemma 5.1.3 *Let E and F be two vector bundles over the Riemann surface Σ with a non-zero homomorphism $f : E \to F$, then f has the following canonical factorisation:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E & \xrightarrow{\eta} & E_2 & \longrightarrow & 0 \\ & & & & \downarrow f & & \downarrow g & & \\ 0 & \longleftarrow & F_2 & \longleftarrow & F & \xleftarrow{i} & F_1 & \longleftarrow & 0 \end{array}$$

where E_1, E_2, F_1 and F_2 are vector bundles, each row is exact, $f = ig\eta$ and g is of maximal rank, i.e. $\mathrm{rank}(E_2) = \mathrm{rank}(F_1) = n$ *and $\Lambda^n(g) : \Lambda^n(E_2) \to \Lambda^n(F_1)$ is a non-zero homomorphism. In other words g is an isomorphism on a Zariski open subset U of Σ. F_1 is called the subbundle of F generated by the image of f.* $\Box$

[4] In connection with Higgs bundles the language of hypercohomology was first used in [Sim1]. In [Bi,Ra] it was used to describe the tangent space to $\mathcal{M}$.

[5] The second part of which is Proposition (3.15) in [Hit1].

[6] Every sheaf over Σ is assumed to be a sheaf of $\mathcal{O}_\Sigma$-modules.

[7] Cf. section 4 in [Na,Se].

Let $f : \mathcal{E} \to \mathcal{F}$ be a non-zero morphism of Higgs bundles. In particular $f : E \to F$ is a homomorphism of vector bundles.

Construct the canonical factorisation of f of Lemma 5.1.3. Consider the Zariski open subset U of Σ where g is an isomorphism. Here clearly $\ker(f\mid_U) = \ker(\eta\mid_U) = E_1\mid_U$. Now $\ker(f\mid_U)$ being the kernel of a morphism of Higgs bundles, is Φ-invariant, i.e. a Higgs subbundle of $\mathcal{E}\mid_U$. Thus $E_1\mid_U$ is a Higgs subbundle of $\mathcal{E}\mid_U$. This means that $\Phi(E_1)$ is contained in $E_1 \otimes K \subset E \otimes K$ on U. Because U is Zariski open in Σ, it follows that $\mathcal{E}_1 = E_1 \xrightarrow{\Phi} E_1 \otimes K$ is a Higgs subbundle of $\mathcal{E}$. Let $\mathcal{E}_2 = E_2 \xrightarrow{\Phi} E_2 \otimes K$ denote the quotient Higgs bundle.

Similarly $\mathrm{im}(\alpha)\mid_U = F_1\mid_U$ is Ψ-invariant, thus $\mathcal{F}_1 = F_1 \xrightarrow{\Psi} F_1 \otimes K$ is a Higgs subbundle of F.

By assumption $\mu(\mathcal{F}) < \mu(\mathcal{E})$, stability of $\mathcal{E}$ gives $\mu(\mathcal{E}) \leq \mu(\mathcal{E}_2)$ (it may happen that $E = E_2$) and because g is of maximal rank we get $\mu(\mathcal{E}_2) = \mu(E_2) \leq \mu(F_1) = \mu(\mathcal{F}_1)$. Thus $\mu(\mathcal{F}) < \mu(\mathcal{F}_1)$ contradicting the stability of $\mathcal{F}$.

If $\mu(\mathcal{E}) = \mu(\mathcal{F})$ then the above argument leaves the only possibility that η, g and i are isomorphisms, showing that f must be an isomorphism. Suppose that we have such an isomorphism f of Higgs bundles. Then consider $h : \mathcal{E} \to \mathcal{F}$ another non-zero morphism of Higgs bundles. In particular $h \in \mathrm{Hom}(E, F)$. Let λ be an eigenvalue of the homomorphism $f_p^{-1}h_p \in \mathrm{Hom}(E_p, E_p)$. Then the homomorphism $h - \lambda f$ is not an isomorphism, though clearly a morphism of Higgs bundles. From the above argument $h - \lambda f = 0$.

The result follows. □

Corollary 5.1.4 *For any stable Higgs bundle $\mathcal{E}$ with $\mu(\mathcal{E}) < 0$:*

$$\mathbb{H}^0(\Sigma, \mathcal{E}) = 0, \tag{5.2}$$

for any stable Higgs bundle $\mathcal{E}$ with $\mu(\mathcal{E}) > 0$:

$$\mathbb{H}^2(\Sigma, \mathcal{E}) = 0. \tag{5.3}$$

If $\mathcal{E}$ is a stable Higgs bundle with $\mu(\mathcal{E}) = 0$ and $\mathcal{E} \ncong \mathcal{E}_0 = \mathcal{O}_\Sigma \xrightarrow{0} \mathcal{O}_\Sigma \otimes K$ then both (5.2) and (5.3) hold.

Proof. For the first part consider the Higgs bundle $\mathcal{E}_0 = \mathcal{O}_\Sigma \xrightarrow{0} \mathcal{O}_\Sigma \otimes K$. Being of rank 1 it is obviously stable, with $\mu(\mathcal{E}_0) = 0$. Now the previous theorem yields that there are no nontrivial morphisms from $\mathcal{E}_0$ to $\mathcal{E}$, which in the language of hypercohomology is exactly $\mathbb{H}^0(\Sigma, \mathcal{E}) = 0$, which we had to prove.

For the second part Serre duality gives that $\mathbb{H}^2(\Sigma, \mathcal{E}) \cong (\mathbb{H}^0(\Sigma, \mathcal{E}^* \otimes K))^*$. Now clearly $\mathcal{E}^* \otimes K$ is stable and $\mu(\mathcal{E}^* \otimes K) = -\mu(\mathcal{E}) < 0$. Thus the first part gives the second.

Likewise, the third statement follows by referring to the last part of Theorem 5.1.2. □

It is convenient to include here the analogue of the Harder-Narasimhan filtration for Higgs bundles, which will be used in Chapter 6:

Corollary 5.1.5 *Every Higgs bundle $\mathcal{E}$ has a* canonical filtration*:*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_r = \mathcal{E}, \tag{5.4}$$

with $\mathcal{D}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ semi-stable and

$$\mu(\mathcal{D}_1) > \mu(\mathcal{D}_2) > \ldots > \mu(\mathcal{D}_r).$$

Proof. It follows from Theorem 5.1.2 just as in the case of vector bundles. □

5.2 Universal bundles

Nitsure showed that $\widetilde{\mathcal{M}}$ is a coarse moduli space. Here we show that $\widetilde{\mathcal{M}}$ is in fact a *fine* moduli space. We closely follow the proof of Theorem 5.12 in [New1] and (1.19) of [Tha3]. All the ingredients have already appeared in the unpublished [Tha1].

Definition 5.2.1 *Two families $\mathcal{E}_T$ and $\mathcal{E}'_T$ of stable Higgs bundles over $T \times \Sigma$ are said to be* equivalent*, (in symbols $\mathcal{E}_T \sim \mathcal{E}'_T$) if there exists a line bundle L on T such that $\mathcal{E}'_T \cong \mathcal{E}_T \otimes \mathrm{pr}_T^*(L)$.*

The next lemma, which is taken from [Tha1], shows that two families are equivalent iff they give rise to the same map to the coarse moduli space $\widetilde{\mathcal{M}}$.

Lemma 5.2.2 *If $\mathcal{E}_T = \mathbb{E}_T \xrightarrow{\mathbf{\Phi}} \mathbb{E}_T \otimes K_\Sigma$ and $\mathcal{E}'_T = \mathbb{E}'_T \xrightarrow{\mathbf{\Phi}'} \mathbb{E}'_T \otimes K_\Sigma$ are families of stable Higgs bundles over $T \times \Sigma$ such that*

$$\mathcal{E}_T \mid_{\{t\}\times\Sigma} \cong \mathcal{E}'_T \mid_{\{t\}\times\Sigma} \tag{5.5}$$

for each $t \in T$, then $\mathcal{E}_T \sim \mathcal{E}'_T$.

Proof. Let

$$\mathcal{F} := \mathbb{E}_T^* \otimes \mathbb{E}'_T \stackrel{[\mathbf{\Phi}_T, \mathbf{\Phi}'_T]}{\longrightarrow} \mathbb{E}_T^* \otimes \mathbb{E}'_T \otimes K_\Sigma.$$

We define $L = \mathbb{R}^0 \mathrm{pr}_{T*}(\mathcal{F})$. By (5.5) and (5.1) this is a line bundle over T. It follows from the projection formula[8] that the sheaf $\mathbb{R}^0\mathrm{pr}_{T*}(\mathcal{F} \otimes \mathrm{pr}_T^*(L^*))$ is just $\mathcal{O}_T$, the structure sheaf. A non-zero section

$$\mathbf{\Psi} \in H^0(T, \mathbb{R}^0\mathrm{pr}_{T*}(\mathcal{F} \otimes \mathrm{pr}_T^*(L^*)))$$

for every $t \in T$ gives

$$\mathbf{\Psi} \mid_{\{t\}\times\Sigma} : (\mathcal{E}_T \otimes \mathrm{pr}_T^*(L)) \mid_{\{t\}\times\Sigma} \to \mathcal{E}'_T \mid_{\{t\}\times\Sigma}$$

a non-zero morphism of Higgs bundles, which is by Theorem 5.1.2 an isomorphism.

The result follows. □

Now we prove the existence of universal Higgs bundles[9]:

Proposition 5.2.3 *Universal Higgs bundles $\mathcal{E}_{\widetilde{\mathcal{M}}} = \mathbb{E}_{\widetilde{\mathcal{M}}} \xrightarrow{\mathbf{\Phi}} \mathbb{E}_{\widetilde{\mathcal{M}}} \otimes K_\Sigma$ over $\widetilde{\mathcal{M}} \times \Sigma$ do exist.*

Proof. We construct a *holomorphic* universal Higgs bundle over $\widetilde{\mathcal{M}} \times \Sigma$ by using the gauge theoretic construction of $\widetilde{\mathcal{M}}$ from Subsection 1.2.3. An analogous construction in the GIT framework of [Nit] however gives an *algebraic* universal Higgs bundle. We preferred here the gauge theoretic proof, because it fits better into this thesis.

To construct $\mathbb{E}_{\widetilde{\mathcal{M}}}$ we proceed similarly to p.579-580 of [At,Bo]. First we note that there is an obvious tautological rank 2 holomorphic bundle $\mathbb{E}_\mathcal{C}$ over $\mathcal{C} \times \Sigma$ with the constant scalars $\mathbb{C}^* \subset \mathcal{G}^c$ acting trivially on the base and as scalars in the fibre of $\mathbb{E}_\mathcal{C}$.

We also need a $\mathcal{G}^c$-invariant holomorphic line bundle $\mathbb{L}$ over $\mathcal{C}_{<g}$ such that $\mathbb{C}^* \subset \mathcal{G}^c$ acts via scalar multiplication. To construct such a line bundle we choose k large enough such that $H^1(M, E \otimes L_p^k) = 0$ for each $E \in \mathcal{C}_{<g}$. Then $(\mathrm{pr}_\mathcal{C})_*(\mathbb{E}_\mathcal{C} \otimes L_p^k)$ is a $\mathcal{G}^c$-equivariant holomorphic vector bundle over $\mathcal{C}_{<g}$ of degree $2k+1-2(g-1)$. Taking determinants gives a $\mathcal{G}^c$-equivariant line bundle A_k on $\mathcal{C}_{<g}$, with the group $\mathbb{C}^*$ of scalar automorphisms of E acting on this with weight $m = 2k+1-2(g-1)$. Now taking the determinant of $(\mathrm{pr}_\mathcal{C})_*(\mathbb{E}_\mathcal{C} \otimes L_p^{k+1})$ gives a line bundle A_{k+1} over $\mathcal{C}_{<g}$ such that the weight of the $\mathbb{C}^*$-action is $2k+3-2(g-1)$. It follows that we can find a and b such that $\mathbb{L} = A_k^a \otimes A_{k+1}^b$ is a holomorphic $\mathcal{G}^c$-equivariant line bundle over $\mathcal{C}_{<g}$, with $\mathbb{C}^* \subset \mathcal{G}^c$ acting on it with weight 1.

From Subsection 1.2.3 we have $(\mathcal{B})^s \subset \mathcal{B}$ the subspace of stable Higgs bundles and a map $\mathrm{pr} : \mathcal{B} \to \mathcal{C}$. The restricted map $\mathrm{pr}^s : (\mathcal{B})^s \to \mathcal{C}$ has image in $\mathcal{C}_{<g}$. Thus we can pullback the bundle $\mathbb{E}_\mathcal{C} \otimes \mathbb{L}^{-1}$ from $\mathcal{C}_{<g} \times \Sigma$ to $(\mathcal{B})^s \times \Sigma$ gaining $\mathbb{E}_\mathcal{B}$. This bundle is a priori a $\mathcal{G}^c$-equivariant bundle. However the subgroup

[8] Cf. p.124 of [Har].
[9] Cf. [Tha1].

of constant scalar automorphisms $\mathbb{C}^* \subset \mathcal{G}^c$ acts trivially on $\mathbb{E}_{\mathcal{B}}$, therefore it reduces to a $\overline{\mathcal{G}}^c = \mathcal{G}^c/\mathbb{C}^*$-equivariant bundle. Since $\overline{\mathcal{G}}^c$ acts freely on $(\mathcal{B})^s$ it follows that $\mathbb{E}_{\mathcal{C}} \otimes \mathbb{L}^{-1}$ reduces to a rank 2 holomorphic vector bundle $\mathbb{E}_{\widetilde{\mathcal{M}}}$ over $\widetilde{\mathcal{M}} \otimes \Sigma$, with the property that $\mathbb{E}_{\widetilde{\mathcal{M}}} |_{(E,\Phi)} \cong E$.

Now the fibre of $\mathrm{pr} : \mathcal{B} \to \mathcal{C}$ over a point E, can be identified canonically with $H^0(\Sigma, \mathrm{End}(E) \otimes K)$, thus there is an obvious tautological section

$$\mathbf{\Phi}_{\mathcal{B}} \in H^0(\mathcal{B}; \mathrm{End}(\mathrm{pr}^*(\mathbb{E}_{\mathcal{C}})) \otimes K).$$

However $\mathrm{End}(\mathrm{pr}^*(\mathbb{E}_{\mathcal{C}}) \cong \mathrm{End}(\mathbb{E}_{\mathcal{B}})$ canonically. It follows that we have a tautological Higgs bundle $\mathbb{E}_{\mathcal{B}} \xrightarrow{\mathbf{\Phi}_{\mathcal{B}}} \mathbb{E}_{\mathcal{B}} \otimes K$ over $(\mathcal{B})^s \times \Sigma$, which is a priori a $\mathcal{G}^c$-equivariant complex but reduces to a $\overline{\mathcal{G}}^c$-equivariant complex as proved above. It follows that it reduces to $\widetilde{\mathcal{M}} \times \Sigma$ to give a universal Higgs bundle $\mathbb{E}_{\widetilde{\mathcal{M}}} \xrightarrow{\mathbf{\Phi}_{\widetilde{\mathcal{M}}}} \mathbb{E}_{\widetilde{\mathcal{M}}} \otimes K$. $\square$

As in Theorem 5.12 of [New1] and (1.19) of [Tha3] our Lemma 5.2.2 and Proposition 5.2.3 gives:

Corollary 5.2.4 *The space $\widetilde{\mathcal{M}}$ is a fine moduli space for rank 2 stable Higgs bundles of degree 1 with respect to the equivalence $\sim$ of families of stable Higgs bundles.*

As another consequence of Proposition 5.2.3 and Lemma 5.2.2, we see that although $\mathbb{E}_{\widetilde{\mathcal{M}}}$ is not unique $\mathrm{End}(\mathbb{E}_{\widetilde{\mathcal{M}}})$ is. Moreover it is clear that by setting $\mathbb{E}_{\mathcal{M}} = \mathbb{E}_{\widetilde{\mathcal{M}}} |_{\mathcal{M}\times\Sigma}$ we have

$$c(\mathrm{End}(\mathbb{E}_{\widetilde{\mathcal{M}}})) = c(\mathrm{End}(\mathbb{E}_{\mathcal{M}})) \otimes 1 \tag{5.6}$$

in the decomposition (3.25).

Thus from the Künneth decomposition of $\mathrm{End}(\mathbb{E}_{\mathcal{M}})$ we get universal classes

$$c_2(\mathrm{End}(\mathbb{E}_{\mathcal{M}})) = 2\alpha_{\mathcal{M}} \otimes \sigma^{\Sigma} + \sum_{i=1}^{2g} 4\psi^i_{\mathcal{M}} \otimes \xi^{\Sigma}_i - \beta_{\mathcal{M}} \otimes 1 \tag{5.7}$$

in $H^4(\mathcal{M} \times \Sigma) \cong \sum_{r=0}^4 H^r(\mathcal{M}) \otimes H^{4-r}(\Sigma)$ for some $\alpha_{\mathcal{M}} \in H^2(\mathcal{M})$, $\psi^i_{\mathcal{M}} \in H^3(\mathcal{M})$ and $\beta_{\mathcal{M}} \in H^4(\mathcal{M})$.

Though $\mathbb{E}_{\mathcal{M}}$ is not unique we can still write its Chern classes in the Künneth decomposition[10], getting[11]

$$c_1(\mathbb{E}_{\mathcal{M}}) = 1 \otimes \sigma^{\Sigma} + \beta_1 \otimes 1,$$

where $\beta_1 \in H^2(\mathcal{M})$ and

$$c_2(\mathbb{E}_{\mathcal{M}}) = \alpha_2 \otimes \sigma^{\Sigma} + \sum_{i=1}^{2g} a_i \otimes \xi^{\Sigma}_i + \beta_2 \otimes 1,$$

where $\alpha_2 \in H^2(\mathcal{M})$, $a_i \in H^3(\mathcal{M})$ and $\beta_2 \in H^4(\mathcal{M})$. Since

$$4c_2(\mathbb{E}_{\mathcal{M}}) - c_1^2(\mathbb{E}_{\mathcal{M}}) = c_2(\mathrm{End}(\mathbb{E}_{\mathcal{M}})),$$

we get $\alpha_{\mathcal{M}} = 2\alpha_2 - \beta_1$ and $\beta = \beta_1^2 - 4\beta_2$. Because $\mathrm{Pic}(\mathcal{M}) \cong H^2(\mathcal{M}, \mathbb{Z})$ we can normalize $\mathbb{E}_{\mathcal{M}}$ uniquely such that $c_1((\mathbb{E}_{\mathcal{M}})_p) = \alpha_{\mathcal{M}}$, where

$$(\mathbb{E}_{\mathcal{M}})_p := \mathbb{E}_{\mathcal{M}} |_{\mathcal{M}\times\{p\}} .$$

Definition 5.2.5 *The universal Higgs bundle $\mathcal{E}_{\mathcal{M}}$ is normalized if $c_1((\mathbb{E}_{\mathcal{M}})_p) = \alpha_{\mathcal{M}}$.*

We also need to work out the Chern classes of $\mathbb{E}_{\widetilde{\mathcal{M}}}$. It is easy to see that $c(\mathbb{E}_{\widetilde{\mathcal{M}}})$ in the decomposition (3.25) is the product of $c(\mathbb{E}_{\widetilde{\mathcal{M}}}) |_{\mathcal{M}\times\Sigma}$ and $c(\mathbb{L}_1)$, where $\mathbb{L}_1$ is some universal line bundle over $\mathcal{J} \times \Sigma$.

Definition 5.2.6 *We call the universal Higgs bundle $\mathcal{E}_{\widetilde{\mathcal{M}}}$* normalized *if in the decomposition (3.25)*

$$c_1((\mathbb{E}_{\widetilde{\mathcal{M}}})_p) = \alpha_{\mathcal{M}}, \tag{5.8}$$

where $(\mathbb{E}_{\widetilde{\mathcal{M}}})_p = \mathbb{E}_{\widetilde{\mathcal{M}}} |_{\widetilde{\mathcal{M}}\times\{p\}}$.

[10] Cf. proof of Newstead's theorem in [Tha2]).
[11] Note that $\mathcal{M}$ being simply connected by [Hit1] $H^1(\mathcal{M}) = 0$.

Remark. Since

$$4c_2((\mathbb{E}_{\widetilde{\mathcal{M}}})_p) - c_1((\mathbb{E}_{\widetilde{\mathcal{M}}})_p)^2 = c_2\left(\mathrm{End}((\mathbb{E}_{\widetilde{\mathcal{M}}})_p)\right),$$

for a normalized universal Higgs bundle over $\widetilde{\mathcal{M}} \times \Sigma$ (5.6) and (5.8) yield:

$$c_2((\mathbb{E}_{\widetilde{\mathcal{M}}})_p) = \frac{(\alpha_{\mathcal{M}}^2 - \beta_{\mathcal{M}})}{4} \tag{5.9}$$

Finally, given a universal Higgs bundle $\mathcal{E}_{\widetilde{\mathcal{M}}}$ over $\widetilde{\mathcal{M}} \times \Sigma$, we introduce a *universal Higgs bundle of degree* $2k-1$ by setting $\mathcal{E}^k_{\widetilde{\mathcal{M}}} := \mathcal{E}_{\widetilde{\mathcal{M}}} \otimes \mathrm{pr}^*_\Sigma(L_p^{k-1})$, where L_p is the line bundle of the divisor of the point $p \in \Sigma$. It is called normalized if $\mathcal{E}_{\widetilde{\mathcal{M}}}$ is normalized.

5.3 The virtual Dirac bundle $\mathbf{D}_k$

The strategy of the proof of Theorem 0.2.1 will be to examine the virtual Dirac bundle $\mathbf{D}_k$ which is defined in the following:

Definition 5.3.1 *The* virtual Dirac bundle *is*[12]

$$\mathbf{D}_k := -\mathrm{pr}_{\widetilde{\mathcal{M}}!}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \in K(\widetilde{\mathcal{M}}),$$

where $\mathcal{E}^k_{\widetilde{\mathcal{M}}}$ is a normalized universal Higgs bundle of degree $2k-1$ and $\mathrm{pr}_{\widetilde{\mathcal{M}}} : \widetilde{\mathcal{M}} \times \Sigma \to \widetilde{\mathcal{M}}$ is the projection to $\widetilde{\mathcal{M}}$.

The name is justified by Hitchin's construction[13] of $\mathbf{D}_k$ related to the space of solutions of an equation on Σ, which is locally the dimensional reduction of the Dirac equation in $\mathbb{R}^4$ coupled to a self-dual Yang-Mills field.

The virtual Dirac bundle is a priori

$$-\mathrm{pr}_{\mathcal{M}!}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) = -\mathbb{R}^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) + \mathbb{R}^1\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) - \mathbb{R}^2\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \in K(\widetilde{\mathcal{M}})$$

a formal sum of three coherent sheaves. Corollary 5.1.4 shows that one of these sheaves always vanishes: if $k > 0$, then $\mathbb{R}^2 = 0$, if $k \leq 0$ then $\mathbb{R}^0 = 0$. From now on k is assumed to be positive.

We would like to use Porteous' Theorem 2.2.6 for $\mathbf{D}_k$. As we explained in Subsection 2.2.2 for this we need to show that we can think of the virtual Dirac bundle as the virtual degeneracy sheaf of a homomorphism of vector bundles. More precisely we prove:

Theorem 5.3.2 *There exist two vector bundles V and W over $\widetilde{\mathcal{M}}$ together with a homomorphism $f : V \to W$ of vector bundles, whose kernel and cokernel are respectively $\mathbb{R}^0\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}})$ and $\mathbb{R}^1\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}})$. In other words there is an exact sequence of sheaves:*

$$0 \to \mathbb{R}^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \to V \xrightarrow{f} W \to \mathbb{R}^1\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \to 0.$$

Proof[14]. First we need a lemma.

Lemma 5.3.3 *Let X be a smooth quasi-projective variety and Σ a smooth projective curve. If E is a locally free sheaf over $X \times \Sigma$ then there exists a vector bundle F over $X \times \Sigma$ with a surjective vector bundle homomorphism $g_E : F \to E$ such that $R^0\mathrm{pr}_{X*}(F) = 0$. We will call F a* sectionless resolution *of E.*

Proof. The lemma is a special case of Proposition 2.1.10 of [Hu,Le]. We only have to note that $\mathrm{pr}_{X*} : X \times \Sigma \to X$ is a smooth projective morphism of relative dimension 1 and E being locally free is flat over X. The proof is rather simple so we sketch it here.

Let us denote by E_x the vector bundle $E\mid_{\{x\}\times\Sigma}$ over Σ. Fix an ample line bundle L on Σ. Then it is well known that for big enough k the vector bundle $E_x \otimes L^k$ is generated by its sections and $H^1(\Sigma; E_x \otimes L^k) = 0$. Let us denote by $X_k \subset X$ those points x for which $E_x \otimes L^k$ is generated by its sections and $H^1(\Sigma; E_x \otimes L^k) = 0$. It is standard that X_k is a Zariski open subset of X. Thus we have a covering $X = \bigcup X_k$ of X by Zariski open subsets. It is again standard that the Zariski topology of an algebraic variety is noetherian[15], which yields that we have some k such that $X_k = X$. It is now immediate that

$$F = \mathrm{pr}^*_\Sigma(L^{-k}) \otimes \mathrm{pr}^*_X \left((\mathrm{pr}_X)_*(E \otimes \mathrm{pr}^*_\Sigma(L^k))\right)$$

has the required properties.

The result follows. $\square$

[12] Recall the definition of the pushforward of a complex from Subsection 2.2.3.
[13] Cf. Subsection 1.1.5.
[14] The idea of the proof was suggested by Manfred Lehn.
[15] Cf. Example 3.2.1 on p. 84 of [Har].

Proposition 5.3.4 *Let Σ be a smooth projective curve and X be a smooth quasi-projective variety. Let $\mathcal{E} = E \xrightarrow{f} F$ be a complex of vector bundles on $X \times \Sigma$. Let $g_F : A \to F$ be a sectionless resolution of F. Let M be the fibred product of f and g_F. This comes with projection maps $p_F : M \to F$ and $p_A : M \to A$. Let $g_M : A_2 \to M$ be a sectionless resolution of M, and denote $j = g_M \circ p_{A_2}$. Finally, let $A_1 = \ker g_M$ and $i : A_1 \to A_2$ the embedding. The situation is shown in the following diagram:*

$$\begin{tikzcd}
 & & E \arrow[rr, "f"] & & F \\
 & & & M \arrow[ul] \arrow[dr] & \\
0 \arrow[r] & A_1 \arrow[r, "i"] & A_2 \arrow[ur] \arrow[rr, "j"] & & A \arrow[uu]
\end{tikzcd}$$

In this case the cohomology of the complex

$$R^1\mathrm{pr}_{X*}(A_1) \xrightarrow{i_*} R^1\mathrm{pr}_{X*}(A_2) \xrightarrow{j_*} R^1\mathrm{pr}_{X*}(A)$$

calculates the sheaves $\mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{E})$, $\mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{E})$ and $\mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{E})$ respectively. In other words

$$\mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{E}) \cong \ker(i_*) \tag{5.10}$$
$$\mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{E}) \cong \ker(j_*)/\operatorname{im}(i_*) \tag{5.11}$$
$$\mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{E}) \cong \operatorname{coker}(j_*). \tag{5.12}$$

Proof. Let us recall the definition of the fibred product:

$$M = \ker(f - g_F : E \oplus A \to F).$$

This comes equipped with two obvious projections $p_E : M \to E$ and $p_A : M \to A$. Because g_F is surjective, $f - g_F$ is also surjective. Thus M is a vector bundle. By construction the kernel of p_E is isomorphic to the kernel of g_F. Denote it by B. This says that the following diagram is commutative and has two exact columns:

$$\begin{tikzcd}
0 \arrow[r] & 0 \\
E \arrow[u] \arrow[r, "f"] & F \arrow[u] \\
M \arrow[u, "p_E"] \arrow[r, "p_A"] & A \arrow[u, "g_F"'] \\
B \arrow[u] \arrow[r, "\cong"] & B \arrow[u] \\
0 \arrow[u] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

If $\mathcal{A}$ denotes the complex $\mathcal{A} = M \xrightarrow{p_A} A$ and $\mathcal{B}$ the complex $\mathcal{B} = B \xrightarrow{\cong} B$, then the above diagram is just a short exact sequence of complexes

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Clearly $\mathbb{R}^i\mathrm{pr}_{X*}(\mathcal{B})$ vanishes for all i. (Any hypercohomology of an isomorphism is 0.) Thus the long exact sequence of the above short exact sequence gives the isomorphisms

$$\mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{E}) \cong \mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{A}) \tag{5.13}$$
$$\mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{E}) \cong \mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{A}) \tag{5.14}$$
$$\mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{E}) \cong \mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{A}) \tag{5.15}$$

Because A is a sectionless resolution of M, we have $R^0\mathrm{pr}_{X*}(A) = 0$ thus the long exact sequence of the push forward of the complex $\mathcal{A}$ breaks up into two exact sequences:

$$0 \to \mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{A}) \to R^0\mathrm{pr}_{X*}(M) \to 0,$$

and

$$0 \longrightarrow \mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{A}) \longrightarrow R^1\mathrm{pr}_{\mathcal{W}*}(M) \xrightarrow{p_{A*}} R^1\mathrm{pr}_{X*}(A) \longrightarrow \mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{A}) \longrightarrow 0.$$

Thus

$$\begin{aligned} \mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{A}) &\cong R^0\mathrm{pr}_{X*}(M) && (5.16)\\ \mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{A}) &\cong \ker(p_{A*}) && (5.17)\\ \mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{A}) &\cong \mathrm{coker}(p_{A*}). && (5.18) \end{aligned}$$

Now consider the short exact sequence:

$$0 \longrightarrow A_1 \xrightarrow{i} A_2 \xrightarrow{g_M} M \longrightarrow 0. \tag{5.19}$$

$R^0\mathrm{pr}_{X*}(A_2) = 0$ because A_2 is a sectionless resolution of M and hence we get the exact sequence of sheaves:

$$0 \longrightarrow R^0\mathrm{pr}_{X*}(M) \longrightarrow R^1\mathrm{pr}_{X*}(A_1) \xrightarrow{i_*} R^1\mathrm{pr}_{X*}(A_2) \xrightarrow{g_{M*}} R^1\mathrm{pr}_{\mathcal{W}*}(M) \longrightarrow 0. \tag{5.20}$$

Thus $\ker(i_*) \cong R^0\mathrm{pr}_{X*}(M)$ which by (5.16) and (5.13) proves (5.10). Since g_{M*} is a surjection $\mathrm{coker}(j_*) \cong \mathrm{coker}(p_{A*})$. This together with (5.18) and (5.15) give (5.12).

Finally, consider the commutative diagram:

$$\begin{array}{ccc} R^1\mathrm{pr}_{\mathcal{W}*}(M) & \xrightarrow{\cong} & R^1\mathrm{pr}_{X*}(M) \\ {\scriptstyle g_{M*}}\uparrow & & \downarrow{\scriptstyle p_{A*}} \\ R^1\mathrm{pr}_{X*}(A_2) & \xrightarrow{j_*} & R^1\mathrm{pr}_{X*}(A) \end{array}$$

Since g_{M*} surjective by (5.20) we get that $\ker(j_*)/\ker(g_{M*}) \cong \ker(p_{A*})$. From (5.20) clearly $\ker(g_{M*}) \cong \mathrm{im}(i_*)$, thus $\ker(j_*)/\mathrm{im}(i_*) \cong \ker(p_{A*})$. This together with (5.17) and (5.14) prove (5.11). □

Corollary 5.3.5 *If $\mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{E}) = 0$, in the situation of Proposition 5.3.4, then there exist two vector bundles V and W over X together with a homomorphism $f : V \to W$, whose kernel and cokernel are $\mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{E})$ and $\mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{E})$ respectively. I.e. the following sequence is exact:*

$$0 \to \mathbb{R}^0\mathrm{pr}_{X*}(\mathcal{E}) \to V \xrightarrow{f} W \to \mathbb{R}^1\mathrm{pr}_{X*}(\mathcal{E}) \to 0.$$

Proof. From the long exact sequence corresponding to (5.19), we have $R^0\mathrm{pr}_{X*}(A_1) = 0$. Let V be the vector bundle $R^1\mathrm{pr}_{X*}(A_1)$.

Moreover $R^1\mathrm{pr}_{X*}(A_2)$ and $R^1\mathrm{pr}_{X*}(A)$ are also vector bundles because A_2 and A are sectionless resolutions. Furthermore the assumption $\mathbb{R}^2\mathrm{pr}_{X*}(\mathcal{E}) = 0$ shows that j_* is surjective. Let W be the vector bundle $\ker(j_*)$, and f be the map $i_* : V \to W$.

The result follows from Proposition 5.3.4. □

The proof of Theorem 5.3.2 is completed by Corollary 5.3.5 noting that by Corollary 5.1.4 we have $\mathbb{R}^2\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) = 0$. □

5.4 The downward degeneracy locus DD_k

Definition 5.4.1 *The* downward degeneracy locus

$$DD_k := \{\mathcal{E} \in \widetilde{\mathcal{M}} : \mathbb{H}^0(\Sigma, \mathcal{E}^k_{\widetilde{\mathcal{M}}}) \neq 0)\}$$

is the locus where $\mathbf{D}_k$ fails to be a vector bundle, i.e. where f of Theorem 5.3.2 fails to be an injection.

The aim of this section is to give a description of the degeneracy locus DD_k. For this we need a refinement of Theorem 4.4.5, which still follows from the proof of Proposition (19) of [Tha1].

Proposition 5.4.2 *The nilpotent cone is a compact union of $3g-3$ dimensional manifolds:*

$$N = \mathcal{N} \cup \bigcup_{k=1}^{g-1} D_k,$$

where each D_k is biholomorphic to the total space of the negative vector bundle E_k^- over F_k, the k-th component of the fixed point set of the $\mathbb{C}^*$-action. The component D_k can also be characterised as the locus of those stable Higgs bundles $\mathcal{E} = E \xrightarrow{\Phi} E \otimes K$ which have a unique subbundle $L_\mathcal{E}$ of degree $(1-k)$ killed by the non-zero Higgs field Φ.

Proof. The first part is proved in Theorem 4.4.5.

For the second part consider a universal Higgs bundle $\mathcal{E}_\mathcal{M}$ over $\mathcal{M} \times \Sigma$ restricted to $D_k \times \Sigma$. Let us denote it by $\mathcal{E}_k = \mathbb{E}_k \xrightarrow{\mathbf{\Phi}_k} \mathbb{E}_k \times K_\Sigma$. Consider the kernel of $\mathbf{\Phi}_k$. Because D_k parametrizes nilpotent stable Higgs bundles with non-zero Higgs field $\ker(\mathbf{\Phi}_k)$ is a line bundle over $D_k \times \Sigma$. Recall from Proposition 7.1 of [Hit1] that for $E \xrightarrow{\Phi} E \otimes K \in F_k \subset D_k$ we have $\deg(\ker(\Phi)) = 1-k$. Since D_k is smooth we have that $\deg(\ker(\Phi)) = 1-k$ for every $E \xrightarrow{\Phi} E \otimes K \in D_k$.

The result follows. $\square$

Remark. A completely analogous result holds for $\widetilde{N}$ with $\widetilde{\mathcal{N}}$, $\widetilde{D}_k$ and $\widetilde{F}_k$ instead of $\mathcal{N}$, D_k and F_k.

Theorem 5.4.3 *Let $k = 1, .., g-1$. The degeneracy locus DD_k has the following decomposition:*

$$DD_k = \widetilde{\mathcal{N}}^k \cup \bigcup_{i=1}^{k} \widetilde{D}_i^k,$$

where $\widetilde{\mathcal{N}}^k = DD_k \cap \widetilde{\mathcal{N}}$, and $\widetilde{D}_i^k \subset \widetilde{D}_i$ are those nilpotent stable Higgs bundles whose unique line bundle $L_\mathcal{E}$ of Proposition 5.4.2 has the property that $H^0(\Sigma, L_\mathcal{E} \otimes L_p^{k-1}) \neq 0$.

Furthermore

$$\widetilde{D}_k^k := \{\mathcal{E} \in \widetilde{D}_k : L_\mathcal{E} = L_p^{1-k}\}$$

and hence[16]

$$\eta^{\widetilde{\mathcal{M}}}_{\widetilde{D}_k^k} \setminus [\mathcal{J}] = \eta^{\mathcal{M}}_{D_k} \in H^{6g-6}(\mathcal{M}), \tag{5.21}$$

where $\eta^{\widetilde{\mathcal{M}}}_{\widetilde{D}_k^k} \setminus [\mathcal{J}]$ means the coefficient of $\eta^{\mathcal{J}}_{pt}$ in the decomposition of (3.25).

Proof. Let $\mathcal{E} = E \xrightarrow{\Phi} E \otimes K$ be a stable Higgs bundle with $\Phi \neq 0$ and $\mathbb{H}^0(\Sigma, \mathcal{E} \otimes L_p^{k-1}) \neq 0$. It is easy to see that this hypercohomology is the vector space of all morphisms from $\mathcal{E}_0 \otimes L_p^{1-k} = L_p^{1-k} \xrightarrow{0} L_p^{1-k} \otimes K$ to $\mathcal{E}$. Consider a nonzero such morphism f. Consider L the line subbundle of E generated by the image of f of Lemma 5.1.3. Clearly L is killed by the Higgs field Φ. This shows that $\mathcal{E} \in \widetilde{N}$ and $L = L_\mathcal{E}$. We also see that $\mathbb{H}^0(\Sigma, \mathcal{E} \otimes L_p^{k-1}) \cong H^0(L_\mathcal{E} \otimes L_p^{k-1})$. The first part of the statement follows.

By the above argument it follows that $\widetilde{D}_k^k = \{\mathcal{E} \in \widetilde{D}_k : H^0(\Sigma, L_\mathcal{E} \otimes L_p^{k-1}) \neq 0\}$, however $L_\mathcal{E}$ is of degree $1-k$, thus $\widetilde{D}_k^k = \{\mathcal{E} \in \widetilde{D}_k : L_\mathcal{E} = L_p^{1-k}\}$, as claimed. This means that for every $\mathcal{E} \in D_k$ there is a unique line bundle $L = L_p^{1-k} \otimes L_\mathcal{E}^*$ such that $\mathcal{E} \otimes L \in \widetilde{D}_k^k$. This shows (5.21). $\square$

[16] Recall the definition of η^Y_X from Notation 2.2.5.

Remark. By definition $\widetilde{\mathcal{N}}^k = W^0_{2,2k-1}$ are non-Abelian Brill-Noether loci as defined in [Sun][17].

[17]Cf. [Tei].

5.5 Proof of Theorem 0.2.1

In this final section we prove Theorem 0.2.1.

Proof of Theorem 0.2.1. The proof proceeds by first showing that $\mathrm{ch}_0(\mathbf{D}_k) = 4g-4$, then $c_{4g-3}(\mathbf{D}_k) = 0$ and concludes using Porteous' Theorem 2.2.6 for $\mathbf{D}_k$.

First we make some calculations.

Lemma 5.5.1 *The virtual bundle $\mathbf{D}_k$ has rank $4g-4$, i.e. $ch_0(\mathbf{D}_k) = 4g-4$. Moreover*

$$c(\mathbf{D}_k) = \left(1 + \alpha_{\mathcal{M}} + \frac{\alpha_{\mathcal{M}}^2 - \beta_{\mathcal{M}}}{4}\right)^{2g-2} \tag{5.22}$$

in the decomposition (3.25).

Proof. It follows from the hypercohomology long exact sequence that

$$\mathbf{D}_k = -\mathrm{pr}_{\widetilde{\mathcal{M}}!}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) = \mathrm{pr}_{\widetilde{\mathcal{M}}!}(\mathbb{E}^k_{\widetilde{\mathcal{M}}} \otimes K_\Sigma) - \mathrm{pr}_{\widetilde{\mathcal{M}}!}(\mathbb{E}^k_{\widetilde{\mathcal{M}}}).$$

We can calculate the Chern character of the right hand side by the Grothendieck-Riemann-Roch theorem. This gives

$$\mathrm{ch}(\mathbf{D}_k) = \mathrm{pr}_{\widetilde{\mathcal{M}}*}\left(\mathrm{ch}(\mathbb{E}^k_{\widetilde{\mathcal{M}}})(\mathrm{ch}(K_\Sigma) - 1)\,\mathrm{td}(\Sigma)\right).$$

Now $\mathrm{td}(\Sigma) = 1 - (g-1)\sigma$ and $\mathrm{ch}(K_\Sigma) = 1 + (2g-2)\sigma$. Moreover $\mathrm{pr}_{\widetilde{\mathcal{M}}*}$ maps a cohomology class $a \in H^*(\widetilde{\mathcal{M}}) \otimes H^*(\Sigma)$ of the form

$$a = a_0 \otimes 1 + \sum_{i=1}^{2g} a_1^i \otimes e_i + a_2 \otimes \sigma$$

to the class $a_2 \in H^*(\widetilde{\mathcal{M}})$. We will use the notation $a \setminus \sigma = a_2$ and $a \setminus 1 = a_0$. From this it follows that

$$\mathrm{ch}(\mathbf{D}_k) = \left(\mathrm{ch}(\mathbb{E}^k_{\widetilde{\mathcal{M}}})((2g-2)\sigma)(1-(g-1)\sigma)\right) \setminus \sigma = (2g-2)(\mathrm{ch}(\mathbb{E}^k_{\widetilde{\mathcal{M}}}) \setminus 1).$$

Observe that

$$\mathrm{ch}(\mathbb{E}^k_{\widetilde{\mathcal{M}}}) \setminus 1 = \mathrm{ch}((\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p) \in H^*(\widetilde{\mathcal{M}}),$$

where $(\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p = \mathbb{E}^k_{\widetilde{\mathcal{M}}}|_{\widetilde{\mathcal{M}}\times\{p\}}$. It follows from (5.8) and (5.9) that $c_1((\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p) = \alpha_{\mathcal{M}}$ and $c_2((\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p) = (\alpha_{\mathcal{M}}^2 - \beta_{\mathcal{M}})/4$. Hence the formal Chern roots of $(\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p$ are $(\alpha_{\mathcal{M}} + \sqrt{\beta_{\mathcal{M}}})/2$ and $(\alpha_{\mathcal{M}} - \sqrt{\beta_{\mathcal{M}}})/2$. Thus

$$\mathrm{ch}((\mathbb{E}^k_{\widetilde{\mathcal{M}}})_p) = \exp\left(\frac{\alpha_{\mathcal{M}} + \sqrt{\beta_{\mathcal{M}}}}{2}\right) + \exp\left(\frac{\alpha_{\mathcal{M}} - \sqrt{\beta_{\mathcal{M}}}}{2}\right) = 2e^{\alpha_{\mathcal{M}}/2}\cosh\left(\sqrt{\beta_{\mathcal{M}}}/2\right),$$

and hence

$$\mathrm{ch}(\mathbf{D}_k) = (4g-4)e^{\alpha_{\mathcal{M}}/2}\cosh\left(\sqrt{\beta_{\mathcal{M}}}/2\right).$$

This shows that $\mathrm{rank}(\mathbf{D}_k) = \mathrm{ch}_0(\mathbf{D}_k) = 4g-4$ and formal calculation gives (5.22). □

(5.22) has the following immediate corollary:

Corollary 5.5.2 $c_{4g-3}(\mathbf{D}_k) = 0$.□

To prove Theorem 0.2.1 we exhibit g linearly independent elements $r_0, r_1, .., r_{g-1} \in H^{6g-6}_{cpt}(\mathcal{M})$ for which $j_{\mathcal{M}}(r_i) = 0$.

To construct r_k for $0 < k < g$ consider the Zariski open subvarieties

$$\widetilde{D}_{\geq k} = \widetilde{\mathcal{M}} \setminus (\widetilde{\mathcal{N}} \bigcup_{i=1}^{k-1} \widetilde{D}_i)$$

and

$$D_{\geq k} = \mathcal{M} \setminus (\mathcal{N} \bigcup_{i=1}^{k-1} D_i)$$

of $\widetilde{\mathcal{M}}$ and $\mathcal{M}$ respectively. Restricting the sequence of Theorem 5.3.2 to $\widetilde{D}_{\geq k}$ yields:

$$0 \longrightarrow \mathbb{R}^0\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \mid_{\widetilde{D}_{\geq k}} \longrightarrow V \mid_{\widetilde{D}_{\geq k}} \stackrel{f|_{\widetilde{D}_{\geq k}}}{\longrightarrow} W \mid_{\widetilde{D}_{\geq k}} \longrightarrow \mathbb{R}^1\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}}) \mid_{\widetilde{D}_{\geq k}} \longrightarrow 0. \tag{5.23}$$

The degeneracy locus of $f \mid_{\widetilde{D}_{\geq k}}$ (where $f \mid_{\widetilde{D}_{\geq k}}$ fails to be an injection) is $DD_k \cap \widetilde{D}_{\geq k}$ which is $\widetilde{D}^k_k$ from Theorem 5.4.3. This has codimension $4g-3$. Furthermore

$$\mathrm{rank}(W) - \mathrm{rank}(V) = \mathrm{rank}\left(\mathbb{R}^1\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}})\right) - \mathrm{rank}\left(\mathbb{R}^0\mathrm{pr}_{\mathcal{M}*}(\mathcal{E}^k_{\widetilde{\mathcal{M}}})\right) = \mathrm{rank}(\mathbf{D}_k) = 4g-4$$

by Lemma 5.5.1. Thus the degeneracy locus has the expected codimension hence we are in the situation of Porteous' Theorem 2.2.6, which gives:

$$\eta^{\widetilde{D}_{\geq k}}_{\widetilde{D}^k_k} = c_{4g-3}(W \mid_{\widetilde{D}_{\geq k}} - V \mid_{\widetilde{D}_{\geq k}}) \in H^{8g-6}(\widetilde{D}_{\geq k}).$$

The right hand side equals $c_{4g-3}(\mathbf{D}_k \mid_{\widetilde{D}_{\geq k}})$ by (5.23), which vanishes by Corollary 5.5.2. Also

$$\eta^{\widetilde{D}_{\geq k}}_{\widetilde{D}^k_k} \frown [\mathcal{J}] = \eta^{D_{\geq k}}_{D_k}$$

by (5.21). It follows that

$$\eta^{D_{\geq k}}_{D_k} = 0 \in H^{6g-6}(D_{\geq k}). \tag{5.24}$$

From now on we work over $\mathcal{M}$. We show by induction on i that there is a formal linear combination

$$r^i_k = \sum_{j=k-i}^{k} \lambda_j \cdot \left[\eta^{D_{\geq k-i}}_{D_j}\right]$$

of cohomology classes in $H^{6g-6}(D_{\geq k-i})$, such that $\lambda_k = 1$ and the corresponding cohomology class $\sum_{j=k-i}^{k} \lambda_i \cdot \eta^{D_{k-i}}_{D_j}$ is 0 in $H^{6g-6}(D_{\geq k-i})$.

For $i = 0$ the statement is just (5.24). Suppose that there is such formal linear combination r^i_k. Consider the following bit of the long exact sequence of the pair $D_{\geq k-i} \subset D_{\geq k-i-1}$:

$$H^{6g-6}(D_{\geq k-i}, D_{\geq k-i-1}) \longrightarrow H^{6g-6}(D_{\geq k-i-1}) \longrightarrow H^{6g-6}(D_{\geq k-i}).$$

Because $D_{\geq k-i-1} \setminus D_{\geq k-i} = D_{k-i-1}$ is of real codimension $6g-6$, the Thom isomorphism transforms this sequence to:

$$H^0(D_{k-i-1}) \stackrel{\tau}{\longrightarrow} H^{6g-6}(D_{\geq k-i-1}) \stackrel{\rho}{\longrightarrow} H^{6g-6}(D_{\geq k-i}), \tag{5.25}$$

where τ is the Thom map and ρ is restriction. Clearly $\rho\left(\eta^{D_{\geq k-i-1}}_{D_j}\right) = \eta^{D_{\geq k-i}}_{D_j}$. Thus

$$\rho\left(\sum_{j=k-i}^{k} \lambda_j \cdot \eta^{D_{\geq k-i-1}}_{D_j}\right) = \sum_{j=k-i}^{k} \lambda_j \cdot \eta^{D_{\geq k-i}}_{D_j} = 0.$$

The exactness of (5.25) yields that the cohomology class

$$\sum_{j=k-i}^{k} \lambda_j \cdot \eta^{D_{\geq k-i-1}}_{D_j}$$

is in the image of τ. Because $H^0(D_k) \cong \mathbb{Q}$ there is a rational number $-\lambda_{k-i-1} \in \mathbb{Q}$ such that

$$\tau(-\lambda_{k-i-1}) = \sum_{j=k-i}^{k} \lambda_j \cdot \eta_{D_j}^{D_{\geq k-i-1}} \in H^{6g-6}(D_{\geq k-i-1}). \tag{5.26}$$

However a well known property of the Thom map gives $\tau(1) = \eta_{D_{k-i-1}}^{D_{\geq k-i-1}}$, thus from (5.26) the formal linear combination

$$r_k^{i+1} = \sum_{j=k-i-1}^{k} \lambda_j \cdot \left[\eta_{D_j}^{D_{\geq k-i-1}}\right]$$

is 0, when considered as a class in $H^{6g-6}(D_{\geq k-i-1})$. This proves the existence of formal linear combinations r_k^i for all $0 \leq i \leq k-1$.

Using r_k^{k-1} an identical argument gives the formal linear combination

$$r_k' = \lambda \cdot \left[\eta_{\mathcal{N}}^{\mathcal{M}}\right] + \sum_{j=1}^{k} \lambda_j \cdot \left[\eta_{D_j}^{\mathcal{M}}\right]$$

with the property that $\lambda_k = 1$ and r_k' when considered as an element of $H^{6g-6}(\mathcal{M})$ is 0. Now the compactly supported cohomology class

$$r_k = \lambda \cdot \overline{\eta}_{\mathcal{N}}^{\mathcal{M}} + \sum_{j=1}^{k} \lambda_j \cdot \overline{\eta}_{D_j}^{\mathcal{M}} \in H_{cpt}^{6g-6}(\mathcal{M})$$

has the property that $j_{\mathcal{M}}(r_k) = r_k' = 0$, where by abuse of notation r_k' denotes the cohomology class in $H^{6g-6}(\mathcal{M})$ corresponding to the formal linear combination r_k'.

We have found $g-1$ linearly independent compactly supported cohomology classes $r_1, .., r_{g-1} \in H_{cpt}^{6g-6}(\mathcal{M})$. Clearly $\overline{\eta}_{\mathcal{N}}^{\mathcal{M}}$ is not in the span of $r_1, .., r_{g-1}$. Moreover for each $0 < i < g$ we have $\int_{\mathcal{M}} \overline{\eta}_{\mathcal{N}}^{\mathcal{M}} \wedge r_i = 0$ since $j_{\mathcal{M}}(r_i) = 0$. Furthermore

$$\int_{\mathcal{M}} \overline{\eta}_{\mathcal{N}}^{\mathcal{M}} \wedge \overline{\eta}_{\mathcal{N}}^{\mathcal{M}} = \int_{\mathcal{N}} c_{3g-3}(T_{\mathcal{N}}^*) = 0.$$

Thus $\overline{\eta}_{\mathcal{N}}^{\mathcal{M}}$ is perpendicular to $r_1, .., r_{g-1}$ and $\overline{\eta}_{\mathcal{N}}^{\mathcal{M}}$, which constitutes a basis for $H_{cpt}^{6g-6}(\mathcal{M})$, and so $j_{\mathcal{M}}(\overline{\eta}_{\mathcal{N}}^{\mathcal{M}}) = 0$.

Putting all this together: we have g linearly independent middle dimensional compactly supported classes $r_0 = \overline{\eta}_{\mathcal{N}}^{\mathcal{M}}$ and $r_1, .., r_{g-1}$ in the kernel of the forgetful map $j_{\mathcal{M}} : H_{cpt}^{6g-6}(\mathcal{M}) \to H^{6g-6}(\mathcal{M})$.

Theorem 0.2.1 is finally proved. $\square$

Chapter 6

Cohomology

As we already noted in the Introduction, to understand the physical model of [BJSV] one needs to have a good understanding of the cohomology ring of $\widetilde{\mathcal{M}}$. The present chapter[1] attempts to fill the gap in the literature by providing at least a half-proved complete description of $H^*(\mathcal{M})^\Gamma$, and in turn of $H^*(\widetilde{\mathcal{M}})$, which agrees with computer calculations of genus up to 7.

We start with constructing equivariant structures on the universal bundles of Section 5.2. Then we construct the equivariant virtual Mumford bundle in Section 6.2, and investigate its degeneracy locus in Section 6.3. Applying the equivariant Porteous' theorem we are able to deduce a fundamental proposition, which implies, through the general arguments in Subsection 2.2.1, that $H^*_\circ(\widetilde{\mathcal{M}})$ and in turn $H^*(\widetilde{\mathcal{M}})$ are generated by some universal classes.

We finish by providing a conjectured complete description of the subring $H^*_I(\mathcal{M}) \subset H^*(\mathcal{M})^\Gamma$, generated by the classes $\alpha_\mathcal{M}$, $\beta_\mathcal{M}$ and $\gamma_\mathcal{M}$. We support it by a few results. We prove that $H^*_I(\mathcal{M})$ has the same Poincaré polynomial as the conjectured ring. We also find the first two relations and prove here that the second is Newstead's relation $\beta^g = 0$, which easily yields that the Chern classes of $\mathcal{M}$ are zero in degrees at least $2g$.

[1] This chapter describes a joint work with Michael Thaddeus.

6.1 Equivariant universal bundles

We would like to incorporate into the universal bundle the $\mathbb{C}^*$-action on $\widetilde{\mathcal{M}}$. Recall that $\mathbb{C}^*$ acts on $\widetilde{\mathcal{M}}$ by scalar multiplication of the Higgs field. Moreover let $\mathbb{C}^*$ act trivially on Σ so that we get the diagonal $\mathbb{C}^*$-action on $\widetilde{\mathcal{M}} \times \Sigma$. We will need equivariant universal bundles with respect to this action:

Proposition 6.1.1 *Let $\mathcal{E}_{\widetilde{\mathcal{M}}} = \mathbb{E}_{\widetilde{\mathcal{M}}} \xrightarrow{\mathbf{\Phi}} \mathbb{E}_{\widetilde{\mathcal{M}}} \otimes \mathrm{pr}_\Sigma^*(K)$ be a universal Higgs bundle over $\widetilde{\mathcal{M}} \times \Sigma$. Then there is a $\mathbb{C}^*$-equivariant structure on $\mathcal{E}_{\widetilde{\mathcal{M}}}$, i.e. an equivariant bundle structure on $\mathbb{E}_{\widetilde{\mathcal{M}}}$ and $\mathbb{E}_{\widetilde{\mathcal{M}}} \otimes K_\Sigma$ such that the universal Higgs field $\mathbf{\Phi} : \mathbb{E}^\circ_{\widetilde{\mathcal{M}}} \to \mathbb{E}^\circ_{\widetilde{\mathcal{M}}} \otimes K^\circ_\Sigma$ is equivariant.*

Proof. Recall the construction of the universal Higgs bundle over $\widetilde{\mathcal{M}} \times \Sigma$ from Proposition 5.2.3. In the gauge theory picture $\mathbb{C}^*$ acts on $\mathcal{B}$ by scalar multiplication of the Higgs field[2]. If we let $\mathbb{C}^*$ act trivially on $\mathcal{C}$, then we have that $\mathrm{pr} : \mathcal{B} \to \mathcal{C}$ is $\mathbb{C}^*$-equivariant. Also let $\mathbb{E}_\mathcal{C} \otimes \mathbb{L}^{-1}$ be a $\mathbb{C}^*$-equivariant holomorphic bundle over $\mathcal{C} \times \Sigma$ with the trivial $\mathbb{C}^*$-action. Thus the pullback of $\mathbb{E}_\mathcal{C} \otimes \mathbb{L}^{-1}$ by the $\mathbb{C}^*$-equivariant map pr gives a $\mathbb{C}^*$-equivariant structure on $\mathbb{E}_\mathcal{B}$, we denote the resulting $\mathbb{C}^*$-equivariant bundle by $\mathbb{E}^\circ_\mathcal{B}$. This descends in the quotient to an equivariant universal bundle $\mathbb{E}^\circ_{\widetilde{\mathcal{M}}}$.

We need a $\mathbb{C}^*$-equivariant structure on $\mathcal{O}_{\widetilde{\mathcal{M}}}$ with homogeneity 1, in other words a $\mathbb{C}^*$-equivariant line bundle with equivariant first Chern class u, where u is the integer generator of the $\mathbb{C}^*$-equivariant cohomology of a point. Such an equivariant line bundle exists by Theorem 1 of [EdGr2]. In our case we can think of this line bundle as the invertible subsheaf of the sheaf $\Omega^2(\widetilde{\mathcal{M}})$ of holomorphic two-forms on $\widetilde{\mathcal{M}}$ generated by the holomorphic symplectic form ω_h on $\widetilde{\mathcal{M}}$, which is homogeneous of degree 1. In any case let us denote it by $\mathcal{O}^\circ_{\widetilde{\mathcal{M}}}$. Moreover we denote by K°_Σ the equivariant line bundle

$$\mathrm{pr}^*_{\widetilde{\mathcal{M}}}(\mathcal{O}^\circ_{\widetilde{\mathcal{M}}}) \otimes \mathrm{pr}^*_\Sigma(K) \tag{6.1}$$

on $\widetilde{\mathcal{M}} \times \Sigma$, where by abuse of notation K stands for the trivial equivariant structure on the canonical bundle K on Σ.

Now it is easy to check that the universal Higgs field $\mathbf{\Phi}_{\widetilde{\mathcal{M}}} : \mathbb{E}^\circ_{\widetilde{\mathcal{M}}} \to \mathbb{E}^\circ_{\widetilde{\mathcal{M}}} \otimes K^\circ_\Sigma$ is $\mathbb{C}^*$-equivariant. □

Having proved the existence of the equivariant universal bundle $\mathbb{E}^\circ_{\widetilde{\mathcal{M}}}$, we can restrict it to $\mathbb{E}_\mathcal{M}$ to get a universal bundle over $\mathcal{M} \times \Sigma$ and consider, analogously to (5.7), the Künneth decomposition of $\mathrm{End}(\mathbb{E}^\circ_\mathcal{M})$ to get equivariant universal classes:

$$c_2(\mathrm{End}(\mathbb{E}^\circ_\mathcal{M})) = 2\alpha^\circ \otimes \sigma^\Sigma + \sum_{i=1}^{2g} 4\psi^\circ_i \otimes \xi^\Sigma_i - \beta^\circ_\mathcal{M} \otimes 1 \tag{6.2}$$

in

$$H^4_\circ(\mathcal{M} \times \Sigma) \cong \sum_{r=0}^{4} H^r_\circ(\mathcal{M}) \otimes_{\mathbb{Q}[u]} H^{4-r}_\circ(\Sigma) \cong \sum_{r=0}^{4} H^r_\circ(\mathcal{M}) \otimes H^{4-r}(\Sigma)$$

for some *equivariant universal classes* $\alpha^\circ \in H^2_\circ(\mathcal{M})$, $\psi^\circ_i \in H^3_\circ(\mathcal{M})$ and $\beta^\circ \in H^4_\circ(\mathcal{M})$.

We will need to know the restriction of the equivariant universal classes to the fixed point set of the $\mathbb{C}^*$-action on $\mathcal{M}$. First consider $\mathcal{N} = F_0$. The equivariant universal bundle $\mathbb{E}^\circ_\mathcal{M}$ as constructed in the proof of Proposition 6.1 clearly restricts to $\mathcal{N} \times \Sigma$ to $\mathbb{E}_\mathcal{N}$ with the trivial $\mathbb{C}^*$-action on it. Consequently

$$\alpha^\circ \mid_\mathcal{N} = \alpha_\mathcal{N} \in H^2(\mathcal{N}) \subset H^2_\circ(\mathcal{N}),$$

$$\psi^\circ_i \mid_\mathcal{N} = \psi^i_\mathcal{N} \in H^3(\mathcal{N}) \subset H^3_\circ(\mathcal{N})$$

and

$$\beta^\circ \mid_\mathcal{N} = \beta_\mathcal{N} \in H^4(\mathcal{N}) \subset H^4_\circ(\mathcal{N}).$$

Consider now the restriction of the universal classes to F_d for $d > 0$. Because the $\mathbb{C}^*$-action is trivial on F_d we have $H^*_\circ(F_d) \cong H^*(F_d) \otimes H^*_\circ(pt) \cong H^*(F_d) \otimes \mathbb{Q}[u]$. Note that since all the universal classes are invariant under Γ, their restrictions to F_d live[3] in $(H^*(F_d))^\Gamma \cong H^*(\Sigma_{\bar{d}})$. Recall the ring $H^*(\Sigma_{\bar{d}})$ from Subsection 3.3.

[2]This $\mathbb{C}^*$ does not have anything to do so far with the constant scalar gauge transformations $\mathbb{C}^* \subset \mathcal{G}^c$. However see Section 8.1, where we suggest that they are closely related.

[3]Recall that $\bar{d} = 2g - 2d - 1$.

Lemma 6.1.2 *The equivariant universal classes restrict to F_d as follows:*

$$\begin{aligned} \alpha^\circ\,|_{F_d} &= (2d-1)(\eta-u)+\sigma; \\ \psi_i^\circ\,|_{F_d} &= \begin{cases} \frac{(\eta-u)}{2}\xi_{i+g} & \text{if} \quad i\le g; \\ -\frac{(\eta-u)}{2}\xi_{i-g} & \text{if} \quad i>g; \end{cases} \\ \beta^\circ\,|_{F_d} &= (\eta-u)^2. \end{aligned}$$

Proof. First we construct an equivariant universal Higgs bundle over $F_d\times\Sigma$. Let $\mathbb{P}_d$ a normalized Poincaré bundle over $\mathcal{J}_d\times\Sigma$ as in Subsection 3.2. By abuse of notation we also denote by $\mathbb{P}_d$ the pullback[4] of $\mathrm{pr}^*_{\mathcal{J}_d}(\mathbb{P}_d)$ to $F_d\times\Sigma$. We also denote by $\Delta_{\bar d}$ the pullback $\mathrm{pr}^*_{\Sigma_{\bar d}}(\Delta_{\bar d})$ of the universal divisor[5] on $\Sigma_{\bar d}\times\Sigma$. Then $\mathbb{P}_d^2K_\Sigma^{-1}\Lambda_\Sigma^{-1}(\Delta_{\bar d})$ defines a line bundle over $F_d\times\Sigma$ which is trivial over $x\times\Sigma$ for all $x\in F_d$. By the push-pull formula this is the pullback of some line bundle L_{F_d} over F_d. So

$$\mathcal{O}(\Delta_{\bar d})\cong L_{F_d}K_\Sigma\Lambda_\Sigma\mathbb{P}_d^{-2};$$

hence the latter has a section, which we denote by ϕ_d, vanishing on $\Delta_{\bar d}$. Now we let

$$\mathbb{E}_{F_d}=\mathbb{P}_d\oplus\mathbb{P}_d^{-1}\Lambda_\Sigma L_{F_d}$$

and

$$\mathbf{\Phi}_d=\begin{pmatrix} 0 & 0 \\ \phi_d & 0 \end{pmatrix},$$

so $\mathbf{\Phi}_d\in H^0(F_d\times\Sigma;\mathrm{Hom}(\mathbb{E}_{F_d},\mathbb{E}_{F_d}\otimes K_\Sigma))$. By construction $\mathbb{E}_{F_d}\stackrel{\mathbf{\Phi}_d}{\to}\mathbb{E}_{F_d}\otimes K_\Sigma$ is a universal Higgs bundle over $F_d\times\Sigma$.

Then we let $\mathcal{O}^\circ$ be the weight 1 equivariant structure on the trivial bundle $\mathcal{O}_{F_d\times\Sigma}$. Moreover we let $\mathbb{P}_d^\circ=\mathcal{O}^\circ\mathbb{P}_d$, $K_\Sigma^\circ=\mathcal{O}^\circ K_\Sigma$ and $\Lambda_\Sigma^\circ=\mathcal{O}^\circ\Lambda_\Sigma$. Then it follows that as equivariant bundles:

$$\mathcal{O}(\Delta_{\bar d})\cong L_{F_d}K_\Sigma^\circ\Lambda_\Sigma^\circ(\mathbb{P}_d^\circ)^{-2}. \tag{6.3}$$

Now if we let $\mathbb{E}^\circ_{F_d}=\mathbb{P}^\circ_d\oplus(\mathbb{P}^\circ_d)^{-1}\Lambda^\circ_\Sigma L_{F_d}$, then $\mathbb{E}^\circ_{F_d}\stackrel{\mathbf{\Phi}_d}{\to}\mathbb{E}^\circ_{F_d}\otimes K^\circ_\Sigma$ is an equivariant universal Higgs bundle over $F_d\times\Sigma$. It follows from Lemma 5.2.2 that the equivariant universal classes restricted to F_d will appear as the Künneth components of $c_2^\circ(\mathrm{End}(\mathbb{E}^\circ_{F_d}))$. We can calculate them as follows:

$$c_2^\circ(\mathrm{End}(\mathbb{E}^\circ_{F_d}))=-\left(c_1^\circ((\mathbb{P}^\circ_d)^2(\Lambda^\circ_\Sigma)^{-1}L^{-1}_{F_d})\right)^2=-\left(c_1^\circ\left(\mathcal{O}(-\Delta_{\bar d})(K^\circ_\Sigma)\right)\right)^2$$

from (6.3). We can calculate this since from (3.1) we have

$$c_1(\Delta_{\bar d})=\eta\otimes 1+\sum_{i=1}^g(\xi_i\otimes\xi^\Sigma_{i+g}-\xi_{i+g}\otimes\xi^\Sigma_i)+\bar d\otimes\sigma^\Sigma\in H^2(F_d\times\Sigma)\cong\sum_{r=0}^2 H^r(F_d)\otimes H^{2-r}(\Sigma)$$

and from (6.1)

$$c_1^\circ(K^\circ_\Sigma)=u\otimes 1+(2g-2)\otimes\sigma^\Sigma\in H^2(F_d\times\Sigma)\cong\sum_{r=0}^2 H^r(F_d)\otimes H^{2-r}(\Sigma).$$

Thus we have

$$\begin{aligned} c_2^\circ(\mathrm{End}(\mathbb{E}^\circ_{F_d})) &= -\left(c_1^\circ\left(\mathcal{O}(-\Delta_{\bar d})(K^\circ_\Sigma)\right)\right)^2 \\ &= -\left((\bar d-(2g-2))\otimes\sigma^\Sigma+\sum_{i=1}^g(\xi_i\otimes\xi^\Sigma_{i+g}-\xi_{i+g}\otimes\xi^\Sigma_i)+(\eta-u)\otimes 1\right)^2 \end{aligned}$$

[4]Recall the projection $\mathrm{pr}_{\mathcal{J}_d}$ from (3.31).
[5]Cf. Subsection 3.3.

$$
\begin{aligned}
&= -\left(\sum_{i=1}^{g}(\xi_i \otimes \xi_{i+g}^{\Sigma} - \xi_{i+g} \otimes \xi_i^{\Sigma})\right)^2 - 2(\eta - u)\sum_{i=1}^{g}(\xi_i \otimes \xi_{i+g}^{\Sigma} - \xi_{i+g} \otimes \xi_i^{\Sigma}) \\
&\quad -(\eta - u)^2 - 2(\eta - u)(1 - 2d) \otimes \sigma^{\Sigma} \\
&= 2\left((2d-1)(\eta - u) + \sigma^{\Sigma}\right) \otimes \sigma_{\Sigma}^{\Sigma} - 2(\eta - u)\sum_{i=1}^{g}(\xi_i \otimes \xi_{i+g}^{\Sigma} - \xi_{i+g} \otimes \xi_i^{\Sigma}) \\
&\quad -(\eta - u)^2 \otimes 1.
\end{aligned}
$$

Comparing this with (6.2) proves the result. □

6.2 The equivariant virtual Mumford bundle $\mathbf{M}_d^\circ$

In Section 6.4 we will prove the generation theorem by considering the degeneration locus of the equivariant virtual Mumford bundle. To do this, first we define the equivariant virtual Mumford bundle as:

$$\mathbf{M}_d^\circ = -\mathrm{pr}_{\widetilde{\mathcal{M}}!}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) = -R^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) + R^1\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) \in K(\widetilde{\mathcal{N}}),$$

where $\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}$ denotes $\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ} \otimes \mathrm{pr}_\Sigma^*(L_p^{-d})$.

The next theorem says that the equivariant virtual Mumford bundle can be thought of as the equivariant degeneracy sheaf of a homomorphism of equivariant vector bundles.

Theorem 6.2.1 *There exist two equivariant vector bundles V° and W°, together with an equivariant homomorphism $f^\circ : V^\circ \to W^\circ$ such that the following sequence of coherent sheaves is exact:*

$$0 \to R^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) \to V^\circ \xrightarrow{f^\circ} W^\circ \to R^1\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) \to 0.$$

Proof. Choose an effective divisor D on Σ such that $H^1(\Sigma, E \otimes \mathcal{O}(D)) = 0$ for all stable Higgs bundle $E \xrightarrow{\Phi} E \otimes K \in \widetilde{\mathcal{M}}$. Such D exists because the Harder-Narasimhan type of vector bundles occurring in stable Higgs pairs is bounded[6]. Then tensoring

$$0 \to \mathcal{O}_\Sigma \to \mathcal{O}(D) \to \mathcal{O}_D \to 0$$

with $\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}$ and pushing down by $\mathrm{pr}_{\widetilde{\mathcal{M}}}$ yields

$$\begin{aligned} 0 &\to R^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) \to R^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d} \otimes \mathcal{O}(D)) \to \\ &\to R^0\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d} \otimes \mathcal{O}_D) \to R^1\mathrm{pr}_{\widetilde{\mathcal{M}}*}(\mathbb{E}_{\widetilde{\mathcal{M}}}^{\circ-d}) \to 0, \end{aligned}$$

because of the condition on D. However our hypothesis on D also implies that the second and third terms of the exact sequence are vector bundles, proving the desired result. $\square$

[6] Cf. Corollary 3.3 of [Nit].

6.3 The upward degeneration locus UD_d

Here we consider the degeneracy locus UD_d of f° of the above Theorem 6.2.1. It can be defined as

$$UD_d = \left\{(E,\Phi) \in \widetilde{\mathcal{M}} : H^0(\Sigma; E \otimes L_p^{-d}) \neq 0\right\},$$

and called the *upward degeneracy locus*, in contrast with the downward degeneracy locus of Subsection 5.4. If E is stable then we see that $H^0(\Sigma; E \otimes L_p^{-d}) = 0$, thus $T^*_{\widetilde{\mathcal{N}}} \cap UD_d = \emptyset$. Furthermore if the destabilizing bundle of E is of degree less than d, then $(E,\Phi) \notin UD_d$. More specifically we have the following description of the degeneration locus UD_d:

Theorem 6.3.1 *For $1 \leq d \leq g-1$ the degeneration locus UD_d has the following decomposition:*

$$UD_d = \bigcup_{k=d}^{g-1} UD_d^k,$$

where $UD_d^k \subset \widetilde{U}_k$ are those stable Higgs bundles (E,ϕ) for which L, the destabilizing line bundle of E, has the property that $H^0(\Sigma, L \otimes L_p^{-d}) \neq 0$.

Finally the real codimension of UD_d^d in $\widetilde{\mathcal{M}}$ is $2(2g+2d-2)$.

Proof. By definition $E \xrightarrow{\Phi} E \otimes K \in UD_d$ if and only if $H^0(\Sigma, E \otimes L_p^{-d}) \neq 0$. A non-zero section however generates a line bundle of $E \otimes L_p^{-d}$ of non-negative degree, which in turn gives a line subbundle L of E of degree d (which, by its uniqueness, should be the destabilizing line bundle of E), such that $H^0(\Sigma; L \otimes L_p^{-d}) \neq 0$. The first statement follows.

Now consider the map $f_k : U_k \to \mathcal{J}_{k-d}$ sending $E \xrightarrow{\Phi} E \otimes K$ to $L \otimes L_p^{-d}$ and consider also the Abel-Jacobi map $u_k : \Sigma_{k-d} \to \mathcal{J}_{k-d}$. We have $(E,\Phi) \in U_d^k$ if and only if $f_k(E,\Phi) \in u_k(\Sigma_{k-d})$. The theorem now easily follows. $\square$

6.4 Generation theorem

The aim of this section is to prove that $H^*_\circ(\widetilde{\mathcal{M}})$ is generated by the equivariant universal classes. To prove this we apply Corollary 2.2.7 to UD^d_d. First we need some notation.

Consider the equivariant Chern class

$$c^\circ_{2g+2d-2}(\mathbf{M}^\circ_d) \in H^{2(2g+2d-2)}_\circ(\widetilde{\mathcal{M}}).$$

Write it down in the Künneth decomposition

$$H^*_\circ(\widetilde{\mathcal{M}}) \cong H^*_\circ(\mathcal{M}) \otimes_{\mathbb{Q}[u]} H^*_\circ(T^*_{\mathcal{J}}) \cong H^*_\circ(\mathcal{M}) \otimes H^*(\mathcal{J}),$$

to get[7]

$$c^\circ_{2g+2d-2}(\mathbf{M}^\circ_d) = \sum_{S\subset\{1\ldots 2g\}} \zeta^\circ_{d,S} \otimes \tau_S. \tag{6.4}$$

In this way, for any $S \subset \{1\ldots 2g\}$ we get $\zeta^\circ_{d,S} \in \mathbb{Q}[\alpha^\circ, \beta^\circ, \psi^\circ_i]$ of degree $2g+2d-2-\deg(\tau_S)$. When $d=1$ we also express

$$c^\circ_{2g+r}(\mathbf{M}^\circ_1) = \sum_{S\subset\{1\ldots 2g\}} \zeta^{\circ r}_S \otimes \tau_S,$$

to get

$$\zeta^{\circ r}_S \in \mathbb{Q}[\alpha^\circ, \beta^\circ, \psi^\circ] \tag{6.5}$$

for each $S \subset \{1\ldots 2g\}$, which we call the *equivariant Mumford relations*[8].

Now we prove the following fundamental proposition.

Proposition 6.4.1 *For any $S \subset \{1\ldots 2g\}$ and its complement $S' = \{1\ldots 2g\}\setminus S$ we have*

$$\zeta^\circ_{d,S'}\mid_{\widetilde{U}_d} = \widetilde{e}^\circ_d \cdot \xi_S, \tag{6.6}$$

where $\widetilde{e}^\circ_d \in H^*_\circ(\widetilde{U}_d)$ is the equivariant Euler class of the normal bundle of $\widetilde{U}_d$ in $\widetilde{\mathcal{M}}$ and

$$\xi_S = \prod_{i\in S} \xi_i \in H^*(\Sigma_{\bar{d}}) \subset H^*_\circ(\widetilde{F}_d) \cong H^*_\circ(\widetilde{U}_d).$$

Equivalently

$$(i_d)_*(\xi_S) = \zeta^\circ_{d,S'}\mid_{\widetilde{U}_{\leq d}}, \tag{6.7}$$

where $i_d : \widetilde{U}_d \to \widetilde{U}_{\leq d}$ is the embedding.

Proof. Recall the equivariant virtual Mumford bundle $\mathbf{M}^\circ_d$ from the previous section, and restrict it to $\widetilde{U}_{\leq d}$. A simple calculation gives that $\mathrm{ch}_0(\mathbf{M}^\circ_d) = 2g+2d-3$ and in turn that

$$\mathrm{rank}(W^\circ) - \mathrm{rank}(V^\circ) + 1 = 2g+2d-2.$$

According to Theorem 6.3.1 the codimension of the degeneracy locus of f° of Theorem 6.2.1 is $2g+2d-2$, thus the degeneracy locus has the expected dimension. Corollary 2.2.7 then yields:

$$\eta^{\circ\widetilde{\mathcal{M}}}_{UD^d_d} = c^\circ_{2g+2d-2}(W^\circ - V^\circ) = c^\circ_{2g+2d-2}(\mathbf{M}^\circ_d).$$

Let us denote $Gr_d = UD^d_d \cap \widetilde{F}_d$. Then it is immediate that

$$c^\circ_{2g+2d-2}(\mathbf{M}^\circ_d)\mid_{\widetilde{F}_d} = \eta^{\circ\widetilde{\mathcal{M}}}_{UD^d_d}\mid_{\widetilde{F}_d} = \widetilde{e}^\circ_d \cdot \eta^{\circ\widetilde{F}_d}_{Gr_d} \in H^*_\circ(\widetilde{F}_d) \cong H^*_\circ(\widetilde{U}_d). \tag{6.8}$$

[7] Recall that $\tau_S = \prod_{i\in S}\tau_i \in H^*(\mathcal{J})$.

[8] The name is justified by noting that forgetting the $U(1)$-equivariant structure $\zeta^{\circ r}_S$ goes to the Mumford relation ζ^r_S, defined in (3.20).

On the other hand recall that $F_d \times \mathcal{J}$ is a 2^{2g}-fold cover of $\widetilde{F}_d$, where F_d is the moduli space of complexes $L \xrightarrow{s} L\Lambda K$ with $\deg L = d$ and fixed line bundle Λ with $\deg(\Lambda) = 1$. We can define a map $f : F_d \to \mathcal{J}$ by sending $f(L \xrightarrow{s} L\Lambda K) = L \otimes L^{1-d}$.

Let us denote by abuse of notation $Gr_d \subset F_d \times \mathcal{J}$ the pullback of Gr_d to $F_d \times \mathcal{J}$. Then Theorem 6.3.1 shows that Gr_d is nothing else than the graph of the map f.

Now let $S \subset \{1 \ldots 2g\}$. Then clearly $f^*(\tau_S) = \xi_S$ so denoting by $e_d^\circ \in H_\circ^*(F_d) \cong H_\circ^*(U_d)$ the equivariant Euler class of the normal bundle of U_d in $\mathcal{M}$ we get

$$\begin{array}{rcll} e_d^\circ \cdot \xi_S & = & e_d^\circ \cdot f^*(\tau_S) & \\ & = & e_d^\circ \cdot (\mathrm{pr}_{F_d})_* \left(\eta_{Gr_d}^{F_d \times \mathcal{J}} \cdot \mathrm{pr}_{\mathcal{J}}^*(\tau_S)\right) & \text{from (6.11)} \\ & = & (\mathrm{pr}_{F_d})_* \left(\widetilde{e}_d^\circ \cdot \eta_{Gr_d}^{F_d \times \mathcal{J}} \cdot \mathrm{pr}_{\mathcal{J}}^*(\tau_S)\right) & \text{since } \mathrm{pr}_{F_d}^*(e_d^\circ) = \widetilde{e}_d^\circ \\ & = & (\mathrm{pr}_{F_d})_* \left(c_{2g+2d-2}^\circ(\mathbf{M}_d^\circ) \cdot \mathrm{pr}_{\mathcal{J}}^*(\tau_S)\right) & \text{from (6.8)} \\ & = & (\mathrm{pr}_{F_d})_* \left((\sum_{R \subset \{1 \ldots 2g\}} \zeta_{d,R}^\circ \otimes \tau_R) \cdot \mathrm{pr}_{\mathcal{J}}^*(\tau_S)\right) & \text{from (6.4)} \\ & = & \zeta_{d,S'}^\circ |_{F_d} & \end{array}$$

which proves (6.6). $\square$

Remark. For $S = \emptyset$, (6.7) says that the cohomology class

$$\eta_{\widetilde{U}_d}^{\circ \widetilde{\mathcal{M}}} = (i_d)_*(1) = (i_d)_*(\xi_\emptyset) = \zeta_{d,\{1 \ldots 2g\}}^\circ \tag{6.9}$$

in

$$H_\circ^{2(g+2d-2)}(\widetilde{U}_{\leq d}) \cong H_\circ^{2(g+2d-2)}(\widetilde{\mathcal{M}}),$$

since for $k > d$ the stratum $\widetilde{U}_k$ has codimension at least $2(g+2d)$. In particular for $d = 1$ and forgetting the $U(1)$-equivariant structure, we have that

$$\eta_{\widetilde{U}_1}^{\widetilde{\mathcal{M}}} = \zeta_{\{1 \ldots 2g\}}^0 \in H^{2g}(\widetilde{\mathcal{M}}),$$

i.e. that the cohomology class of the first stratum $\widetilde{U}_1$ in $\widetilde{\mathcal{M}}$ agrees with the first Mumford relation of degree $2g$! This is not so surprising if we believe the generation theorem –to be proved at the end of the section– i.e. that the $\eta_{\widetilde{U}_1}^{\widetilde{\mathcal{M}}}$ cohomology class is expressed as a (complex) degree g polynomial of the universal classes, since then this polynomial –$\widetilde{\mathcal{N}}$ and $\widetilde{U}_1$ being disjoint in $\widetilde{\mathcal{M}}$ – should restrict to 0 on $\widetilde{\mathcal{N}}$, which therefore should be some multiple of the first Mumford relation. A similar argument shows that the cohomology classes (6.9) of the higher strata are also generated by the Mumford relations. The next formula gives the exact statement:

$$\sum_{i=0}^{\infty} c_i(\mathbf{M}_d^\circ) = \left(1 + \alpha^\circ + \frac{\alpha^\circ - \beta^\circ}{4}\right)^{d-1} \sum_{i=0}^{\infty} c_i^\circ(\mathbf{M}_1^\circ), \tag{6.10}$$

which can be obtained similarly to (5.22). From this it follows that the equivariant classes $\zeta_{d,S}^\circ$, and in particular the equivariant cohomology classes of the higher strata, are in the ideal generated by the equivariant Mumford relations of (6.5). This fact will be used in the proof of Theorem 7.3.2.

To complete the proof of Proposition 6.4.1 we need only to prove the following lemma:

Lemma 6.4.2 *Let $f : X \to Y$ be a map of compact manifolds. Then if we denote by $Gr \subset X \times Y$ the graph of f and let $a \in H^*(Y)$ then we have the formula:*

$$f^*(a) = (\mathrm{pr}_X)_* \left(\eta_{Gr}^{X \times Y} \cdot \mathrm{pr}_Y^*(a)\right). \tag{6.11}$$

Proof. Let $b \in H^*(X)$. Note that by definition of the graph Gr

$$\mathrm{pr}_Y \circ i_{Gr} = f \circ \mathrm{pr}_X \circ i_{Gr} : Gr \to Gr. \tag{6.12}$$

and

$$\mathrm{pr}_X \circ i_{Gr} : Gr \to X \text{ is a homeomorphism,} \tag{6.13}$$

where $i_{Gr} : Gr \to X \times Y$ is the embedding. Since $(\mathrm{pr}_X)_*$ is an $H^*(X)$-module homomorphism we have

$$(\mathrm{pr}_X)_*(\eta_{Gr}^{X\times Y} \cdot \mathrm{pr}_Y^*(a)) \cdot b = (\mathrm{pr}_X)_*(\eta_{Gr}^{X\times Y} \cdot \mathrm{pr}_Y^*(a) \cdot \mathrm{pr}_X^*(b)).$$

Therefore

$$\begin{aligned}
\int_X (\mathrm{pr}_X)_*(\eta_{Gr}^{X\times Y} \cdot \mathrm{pr}_Y^*(a)) \cdot b &= \int_{X\times Y} \eta_{Gr}^{X\times Y} \cdot \mathrm{pr}_Y^*(a) \cdot \mathrm{pr}_X^*(b) \\
&= \int_{Gr} i_{Gr}^*(\mathrm{pr}_Y^*(a)) \cdot i_{Gr}^*(\mathrm{pr}_X^*(b)) \\
&= \int_{Gr} i_{Gr}^*(\mathrm{pr}_X^*(f^*(a))) \cdot i_{Gr}^*(\mathrm{pr}_X^*(b)) \\
&= \int_X f^*(a) \cdot b,
\end{aligned}$$

where we used (6.12) and (6.13). Now Poincaré duality yields the result. □

Now we have the following important corollary of Proposition 6.4.1: It will yield the generation theorem at the end of the present section, and an obvious generalization of it will provide a purely geometric proof for the Mumford conjecture in Section 7.3.

Corollary 6.4.3 *Set $\mathcal{R}_0$ to be the $\mathbb{Q}[u]$-submodule of $H^*_\circ(\widetilde{\mathcal{M}})$ generated by the universal classes of Theorem 6.4.4. Furthermore for $d = 1 \ldots g-1$ set*

$$\mathcal{R}_d = \langle \{\zeta_{d,S}^\circ : S \subset \{1 \ldots 2g\}\} \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]},$$

where $\langle , \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]}$ stands for the generated $\mathbb{Q}[u,\alpha^\circ,\tau_i]$-module. Then

$$i_d^*(\mathcal{R}_{d'}) = 0 \text{ for } d < d' \text{ and } i_d^*(\mathcal{R}_d) = \langle \widetilde{e}_d^\circ \rangle \subset H^*_\circ(\widetilde{U}_d), \tag{6.14}$$

where $\widetilde{e}_d^\circ$ is the equivariant normal bundle to the stratum $\widetilde{U}_d$.

Proof. When $d = 0$ the statement (6.14) is equivalent[9] to the corresponding generation theorem for $\widetilde{\mathcal{N}} = \widetilde{F}_0$, which we already know.

For $d > 0$ recall that $H^*_\circ(\widetilde{F}_d) = H^*(\widetilde{F}_d) \otimes \mathbb{Q}[u]$ is generated by classes $u \in H^2_\circ(\widetilde{F}_d)$, $\eta \in H^2_\circ(\widetilde{F}_d)$ and $\tau_i \in H^1_\circ(\widetilde{F}_d)$, $\xi_i \in H^1_\circ(\widetilde{F}_d)$ for $i = 1 \ldots 2g$. Since from Lemma 6.1.2 we have $\alpha^\circ |_{\widetilde{F}_d} = (2d-1)(\eta - u) + \sigma$ it follows easily that

$$\langle \xi_S : S \subset \{1 \ldots 2g\} \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]} = H^*_\circ(\widetilde{F}_d) \cong H^*_\circ(\widetilde{U}_d).$$

Therefore the statement (6.14) follows from Proposition 6.4.1. □

The main result of this section is the following corollary:

Theorem 6.4.4 *The equivariant cohomology ring $H^*_\circ(\widetilde{\mathcal{M}})$ is generated as a $\mathbb{Q}[u]$-module by the equivariant universal classes $\alpha^\circ \in H^2_\circ(\widetilde{\mathcal{M}})$, $\beta^\circ \in H^4_\circ(\widetilde{\mathcal{M}})$ and $\tau_i \in H^1_\circ(\widetilde{\mathcal{M}})$, $\psi_i^\circ \in H^3(\widetilde{\mathcal{M}})$ for $i = 1 \ldots 2g$.*

[9] Recall that we set $\widetilde{e}_0^\circ = 1$.

Proof. The proof rests on Proposition 2.2.1 as explained in Remark 2 after it. Namely we have a strongly $U(1)$-perfect stratification $\widetilde{\mathcal{M}} = \bigcup_{d=0}^{g-1} \widetilde{U}_d$, and the sets $\mathcal{R}_d$ of Corollary 6.4.3, which contain elements generated by the universal classes, satisfy the conditions of Proposition 2.2.1. The result follows. □

Since the forgetful map $H^*_\circ(\widetilde{\mathcal{M}}) \to H^*(\widetilde{\mathcal{M}})$ is surjective, we have the following immediate corollary.

Corollary 6.4.5 *The ordinary cohomology ring $H^*(\widetilde{\mathcal{M}})$ is generated by the ordinary universal classes $\alpha \in H^2(\widetilde{\mathcal{M}})$, $\beta \in H^4(\widetilde{\mathcal{M}})$ and $\tau_i \in H^1(\widetilde{\mathcal{M}})$, $\psi_i \in H^3(\widetilde{\mathcal{M}})$ for $i = 1 \dots 2g$. Consequently the ring $H^*(\mathcal{M})^\Gamma$ is generated by universal classes: $\alpha \in H^2(\mathcal{M})$, $\beta \in H^4(\mathcal{M})$ and $\psi_i \in H^3(\mathcal{M})$ for $i = 1 \dots 2g$.*

6.5 A conjectured complete set of relations for $H_I^*(\mathcal{M})$

In the previous chapter we showed that the ring $H^*(\mathcal{M})^\Gamma$ is generated by universal classes. To have a complete description of the ring it is sufficient to give a complete set of relations in these universal classes. The idea is to first determine the relations in $H_\circ^*(\mathcal{M})^\Gamma$ based on the injectiveness of (2.7). Namely we consider R_d the kernel of the composition of the projection $\mathbb{Q}[\alpha^\circ, \beta^\circ, \psi_i^\circ]$ to $H_\circ^*(\mathcal{M})^\Gamma$ with the restriction of $H_\circ^*(\mathcal{M})^\Gamma$ to $H_\circ^*(F_d)^\Gamma$ for $0 \le d \le g-1$. Then (2.7) gives that $\bigcap_{d=0}^{g-1} R_d$ is the ideal of relations of $H_\circ^*(\mathcal{M})^\Gamma$, in other words the ring

$$H_\circ^*(\mathcal{M})^\Gamma \cong \mathbb{Q}[\alpha^\circ, \beta^\circ, \psi_i^\circ] / \bigcap_{d=0}^{g-1} R_d$$

is given by generators and relations.

Though we could not yet proceed this way, computer calculations with the software package[10] Macaulay 2 gave us enough numerical evidence to be able to formulate a conjecture about a complete description of the subring of $H^*(\widetilde{\mathcal{M}})^\Gamma$ generated by $\alpha_\mathcal{M}$,$\beta_\mathcal{M}$ and $\gamma_\mathcal{M}$.

To explain this conjecture we have to define certain polynomials:

Definition 6.5.1 *Define polynomials in $\mathbb{Q}[\alpha, \beta, \gamma]$ as follows*

$$\rho_{r,s,t} = \sum_{i=0}^{\min(r,s)} \binom{r}{i} \binom{g-t-i}{g-t-s} \alpha^{r-i} \beta^{s-i} (2\gamma)^{t+i},$$

for $r, s, t \ge 0$.

Let R denote the graded ring given by generators α, β, γ of degree 2, 4 and 6 respectively, and relations:

$$\rho_{r,s,t} \quad \text{for} \quad r, s, t \ge 0 \text{ a } \ r + 3s + 3t > 3g - 3.$$

Conjecture 2 *The subring $H_I^*(\mathcal{M}) \subset H^*(\mathcal{M})^\Gamma$, generated by $\alpha_\mathcal{M}, \beta_\mathcal{M}$ and $\gamma_\mathcal{M}$ is isomorphic to R.*

Remark. 1. Computer calculations with Macaulay 2 show that the statement is true for $2 \le g \le 7$.

2. We used the notation $H_I^*(\mathcal{M})$ for the subring generated by $\alpha_\mathcal{M}, \beta_\mathcal{M}$ and $\gamma_\mathcal{M}$ because it can be thought of as the invariant subring of the action of $Sp(2g, \mathbb{Z})$ on $H^*(\mathcal{M})^\Gamma$, which is given abstractly by letting $Sp(2g, \mathbb{Z})$ act on $H^3(\mathcal{M})^\Gamma$ in the usual symplectic manner, which induces an action[11] on the whole of $H^*(\mathcal{M})^\Gamma$, because of Corollary 6.4.5.

We give some further evidence supporting Conjecture 2 in the rest of the section. First we show that $H_I^*(\mathcal{M})$ and the conjectured ring have the same Poincaré polynomial.

Lemma 6.5.2 *An additive basis for the ring R is given by $\alpha^r \beta^s \gamma^t$ for $r, s, t \ge 0$ and $r + 3s + 3t \le 3g - 3$. Consequently its Poincaré polynomial is of the form*

$$\sum_{\substack{r,s,t \ge 0 \\ r+3s+3t \le 3g-3}} T^{r+2s+3t}. \tag{6.15}$$

Proof. The result easily follows by noting that the highest order term of $\rho_{r,s,t}$ in the lexicographical ordering is clearly $\alpha^r \beta^s \gamma^t$. □

Theorem 6.5.3 *The Poincaré polynomial $P_T^I(\mathcal{M})$ of $H_I^*(\mathcal{M})$ equals (6.15).*

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[10]Cf. http://www.math.uiuc.edu/Macaulay2/.

[11]As a matter of fact this action is induced from the action of the mapping class group of Σ on the cohomology of the representation space of $\pi(\Sigma)$ to $SL(2, \mathbb{C})$, but we do not need this fact, so we simply think of this action as given abstractly.

Proof. First we define an $Sp(2g,\mathbb{Z})$-action on $H^*(\Sigma_n)$ induced by the usual symplectic action on $H^1(\Sigma_n)$. This gives $H^*(F_d)^\Gamma$ an $Sp(2g,\mathbb{Z})$-module structure. Observe that $H^*(\mathcal{M}_{\le d})^\Gamma$ being generated[12] by tautological classes admits a natural $Sp(2g,\mathbb{Z})$-module structure such that the surjective map

$$i^*_{\le d} : H^*(\mathcal{M})^\Gamma \to H^*(\mathcal{M}_{\le d})^\Gamma$$

is an $Sp(2g,\mathbb{Z})$-module homomorphism. Moreover from (6.7) it easily follows that

$$(i_d)_* : H^*(U_d)^\Gamma \to H^*(\mathcal{M}_{\le d})^\Gamma$$

is an $Sp(2g,\mathbb{Z})$-module homomorphism. It follows that the short exact sequence (3.29) is a sequence of $Sp(2g,\mathbb{Z})$-modules yielding the short exact sequence:

$$0 \to H^*_I(U_d) \to H^*_I(\mathcal{M}_{\le d}) \to H^*_I(\mathcal{M}_{<d}) \to 0,$$

where H^*_I denotes the $Sp(2g,\mathbb{Z})$-invariant part of $H^*()^\Gamma$. Recall $P^I_T(U_d) = P^I_T(\Sigma_{\bar d})$ from (3.6) and $P^I_T(\mathcal{N})$ from (3.24). We have

$$\begin{aligned}
P^I_T(\mathcal{M}) &= P^I_T(\mathcal{N}) + \sum_{d=1}^{g-1} T^{g+2d-2} P^I_T(U_d) \\
&= \sum_{\substack{r,s,t\ge 0\\ r+s+t\le g-1}} T^{r+2s+3t} + \sum_{d=1}^{g-1} T^{g+2d-2} \sum_{\substack{q,s\ge 0\\ q+2s\le \bar d}} T^{q+s} \\
&= \sum_{\substack{r,s,t\ge 0\\ r+s+t\le g-1}} T^{r+2s+3t} + \sum_{t=0}^{g-2} T^{3t} \sum_{\substack{q,s\ge 0\\ q+2s\le \bar d}} T^{g-t+q+s} \\
&= \sum_{\substack{r,s,t\ge 0\\ r+s+t\le g-1}} T^{r+2s+3t} + \sum_{t=0}^{g-2} T^{3t} \sum_{\substack{r,s\ge 0, r+s\ge g-t\\ r+3s\le 3g-3-3t}} T^{r+2s} \\
&= \sum_{\substack{r,s,t\ge 0\\ r+3s+3t\le 3g-3}} T^{r+2s+3t}.
\end{aligned}$$

We first introduced $t = d-1$ and then $r = q - s + g - t$. The result follows. □

Theorem 6.5.4 *The polynomial $\rho_{0,g,0} = \beta^g$ of complex degree $2g$ is zero in $H^*(\mathcal{M})^\Gamma$.*

Proof. In order to prove such a statement we want to extend it to an equivariant relation. Namely we prove that

$$\sum_{r=0}^{g} \zeta^\circ_{r,g-r} u^r = 0$$

on $\mathcal{M}$. By (2.7) it is sufficient to check this relation on F_d for every $0 \le d \le g-1$. For $d = 0$ the vanishing is automatic since Zagier's relations (3.22) hold on $\mathcal{N} = F_0$. For $d > 0$ we use Lemma 6.1.2 and substitute $x = u$ and $y = 1$ in Zagier's generating function (3.23) to get:

$$\sum_{r=0}^{g} \zeta^\circ_{r,g-r} u^r \,|_{F_d} = \left(e^{\sigma u} \frac{(1-(\eta-u)(\eta-2u))^{d-1}}{(1-\eta(\eta-u))^d} \right)_{2g}.$$

Recall from Subsection 3.3 that $(\ldots)_m$ means the parts of degree $2m$. To prove that it vanishes in $H^*(F_d)^\Gamma \cong H^*(\Sigma_{\bar d})$, express it as

$$e^{\sigma u} \frac{(1-(\eta-u)(\eta-2u))^{d-1}}{(1+\eta u)^d \left(1 - \frac{\eta^2}{1+\eta u}\right)^d}$$

[12] Just like Corollary 6.4.5 it follows from Corollary 6.4.3.

which equals

$$\sum_{i=1}^{\infty} \binom{d+i}{i} \frac{\eta^{2i} e^{\sigma u}}{(1+\eta u)^{d+i}} (1-(\eta-u)(\eta-2u))^{d-1}. \tag{6.16}$$

It immediately follows from Lemma 3.3.2 that

$$\left(\frac{e^{\sigma u}(\eta u)^{2k}}{(1+(\eta u))^{d+k}} \right)_{2(g-d+k+1)+n} = 0$$

for $n \geq 0$ and hence that

$$\left(\frac{e^{\sigma u}\eta^{2k}}{(1+(\eta u))^{d+k}} \right)_{2g-2d+2+n} = 0$$

for $n \geq 0$. Consequently in (6.16) each term vanishes at total degree $2g$.

The result follows. $\square$

Remark. 1. A similar argument shows the vanishing of the first relation

$$\rho_{1,g-1,0} = g\alpha^\circ(\beta^\circ)^{g-1} + (g-1)(\beta^\circ)^{g-2}(2\gamma^\circ) = \zeta^\circ_{1,g-1}$$

of complex degree $2g-1$, by showing the vanishing of the equivariant class

$$\sum_{r=1}^{g} r\zeta^\circ_{r,g-r} u^{r-1} = 0$$

on $\mathcal{M}$. This goes similarly to the argument above, though the calculation is more tedious.

2. To settle Conjecture 2 'all' we have to do is to extend these polynomials to equivariant relations as we did above in a special case. However it seems that our polynomials are simpler than the ones which may occur naturally from equivariant relations. The reason for this may be that $\rho_{r,s,t}$ can be defined in a much simpler fashion, than Zagier's polynomials $\zeta_{r,s,t}$, which do have some geometric origin.

To picture this difference consider a conjectured relation $\rho_{r,s,t}$ of Conjecture 2. Since it should also be a relation on $\mathcal{N}$ it has to be written as a linear combination of Zagier's relations (3.22). Computer calculations show that this is indeed the case, though the linear combinations tend to be fairly complicated. We do not even have a general formula for this linear combination!

Chapter 7

Resolution tower

By considering the spaces $\widetilde{\mathcal{M}}_k$ as defined in Definition 1.2.5, we complete here[1] the picture of the cohomology ring $H^*(\widetilde{\mathcal{M}})$ in the framework described at the beginning of Section 3. Namely we show that the tower

$$\widetilde{\mathcal{M}} \cong \widetilde{\mathcal{M}}_0 \subset \widetilde{\mathcal{M}}_1 \subset \ldots \subset \widetilde{\mathcal{M}}_k \subset \ldots$$

gives a resolution of the cohomology ring $H^*(\widetilde{\mathcal{M}})$, in the sense that

$$i_0^* : H^*(\widetilde{\mathcal{M}}_\infty) \to H^*(\widetilde{\mathcal{M}})$$

is a free graded commutative resolution of $H^*(\widetilde{\mathcal{M}})$, where

$$\widetilde{\mathcal{M}}_\infty = \lim_{\rightarrow} \widetilde{\mathcal{M}}_k$$

is defined as the direct limit of the above tower.

We also show that the space $\widetilde{\mathcal{M}}_\infty$ is important on its own right. To prove this, we use it to give a simple geometric proof of the Mumford conjecture in Section 7.3. In the last section we explain why $\widetilde{\mathcal{M}}_\infty$ is so useful by showing that it is a model for the classifying space of $\overline{\mathcal{G}}$, the gauge group modulo constant scalars. Also we prove that this homotopy equivalence is even preserved on the level of the strata. We conclude by showing that the above tower has the property that its homotopy groups are stabilizing, thus getting a picture similar to the Atiyah-Jones theorem for the moduli space of instantons on S^4.

Finally we mention that $\widetilde{\mathcal{M}}_k$ and even $\widetilde{\mathcal{M}}_\infty$ appeared already in the work of Donagi and Markman [Do,Ma], where they showed that $\widetilde{\mathcal{M}}_k$ is a complex Poisson manifold, and its Hitchin map is an algebraic completely integrable Hamiltonian system.

[1] This chapter is based on a joint work with Michael Thaddeus.

7.1 The moduli space of Higgs k-bundles $\widetilde{\mathcal{M}}_k$

In this section we list some basic properties of the spaces $\widetilde{\mathcal{M}}_k$. They are completely analogous to the properties of $\widetilde{\mathcal{M}}$. As the proofs are also following the same lines we do not spell out the details here, but hope, that in case of any doubt, the reader can complete the arguments.

The $\mathbb{C}^*$-action on $\mathcal{M}_k$

Recall from Subsection 1.2.2 that $\mathcal{M}_k$ is a smooth quasi-projective varieties of dimension $6g-6+3k$. Moreover $\mathbb{C}^*$ acts on $\mathcal{M}_k$ by multiplication of the Higgs k-field. Completely analogously[2] to $\mathcal{M}$, $\mathcal{M}_k$ is a Kähler manifold, and $U(1) \subset \mathbb{C}^*$ acts on it in a Hamiltonian way, with proper moment map μ_k of absolute minimum 0. Thus as in Subsection 2.1.1 we have a stratification $\mathcal{M}_k = \bigcup_{d=0}^n U_d^k$ with upward flows, where n is the number of components of the fixed point set, to be determined later. We call this the *Hitchin stratification.* Just as in Subsection 4.3 we have the Shatz stratification $\mathcal{M}_k = \bigcup_{d=0}^n U_d^{\prime k}$, defined by $U_d^{\prime k} = \{(E, \Phi_k) : E \in \mathcal{C}_d\}$, which coincides with the Hitchin stratification.

From Subsection 2.1.1 we know that U_d^k retracts to F_d^k, the d-th component of the fixed point set of the $U(1)$-action. A stable Higgs k-pair (E, Φ_k) is fixed by the circle action either if E is stable and $\Phi_k = 0$ or if

$$E = L \oplus L^{-1}\Lambda$$

and

$$\Phi_k = \begin{pmatrix} 0 & 0 \\ \phi_k & 0 \end{pmatrix},$$

where $0 \neq \phi_k \in H^*(\Sigma; L^{-2}\Lambda K L_p^k)$. From the stability of the pair we have that $\deg(L) > 0$, and from the assumption $\phi_k \neq 0$ that $\deg(L) \leq g-1+k$. It follows that $n = g-1+k$ and the components of the fixed point set of the $U(1)$-action are $F_0^k \cong \mathcal{N}$ and F_d^k for $0 < d \leq g-1+k$ are 2^{2g}-fold covers of the symmetric product $\Sigma_{\bar{d}+k}$, with covering group Γ. Now the tangent space of $\mathcal{M}_k$ at a point $(E, \Phi_k) \in F_d^k$ is naturally

$$\mathbb{H}^1(\Sigma; \mathrm{End}_0(E) \xrightarrow{[\Phi_k, \cdot]} \mathrm{End}_0(E) \otimes K \otimes L_p^k).$$

Tracing back the action of $\mathbb{C}^*$ on it one gets that the only negative weight appearing is -1 and the corresponding weight space is $\mathbb{H}^1(\Sigma; L^{-2}\Lambda \to 0) \cong H^1(\Sigma; L^{-2}\Lambda)$. By Riemann-Roch it has dimension $g + 2d - 2$. Thus the real codimension of U_d^k in $\mathcal{M}_k$ which is the same as the index of the critical submanifold F_d^k is $2(g+2d-2)$.

Poincaré polynomial of $\widetilde{\mathcal{M}}_k$

The fact that the indices are even implies that the stratification is perfect, thus we have the following formula for the Γ-invariant Poincaré polynomial of $\mathcal{M}_k$:

$$P_t(\mathcal{M}_k)^\Gamma = P_t(\mathcal{N}) + \sum_{d=1}^{g-1+k} t^{2(g+2d-2)} P_t(\Sigma_{\bar{d}+k}),$$

from which we get the following formula for the Poincaré polynomial of $\widetilde{\mathcal{M}}_k$:

$$P_t(\widetilde{\mathcal{M}}_k) = P_t(\mathcal{J}) P_t(\mathcal{M}_k)^\Gamma = P_t(\widetilde{\mathcal{N}}) + \sum_{d=1}^{g-1+k} t^{2(g+2d-2)} P_t(\Sigma_{\bar{d}+k}) P_t(\mathcal{J}). \tag{7.1}$$

Generators for $H^*(\widetilde{\mathcal{M}}_k)$

We have an equivariant universal bundle $\mathbb{E}^\circ_{\mathcal{M}_k}$, which gives equivariant universal classes α°, β° and ψ_i° in $H_\circ^*(\mathcal{M}_k)^\Gamma$. We have now the analogue of Corollary 6.4.3:

[2] This follows from the gauge theory construction of $\mathcal{M}_k$ of Section 1.2.3.

Proposition 7.1.1 *Set $\mathcal{R}_0$ to be the $\mathbb{Q}[u,\tau_i]$-submodule of $H^*_\circ(\widetilde{\mathcal{M}}_k)$ generated by the above equivariant universal classes. Furthermore for $d = 1 \dots g-1+k$ set*

$$\mathcal{R}_d = \langle \zeta^\circ_{d,S} : S \subset \{1 \dots 2g\} \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]},$$

where $\langle , \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]}$ stands for the generated $\mathbb{Q}[u,\alpha^\circ,\tau_i]$-module. Then $i^*_d(\mathcal{R}_{d'}) = 0$ for $d < d'$ and

$$i^*_d(\mathcal{R}_d) = \langle e_d \rangle \subset H^*_\circ(\widetilde{U}^k_d). \tag{7.2}$$

It follows from Proposition 2.2.1 that the equivariant cohomology ring of $H^*(\widetilde{\mathcal{M}}_k)$ is generated as an algebra by u and universal classes α°,β°, ψ°_i and τ_i.

7.2 The moduli space of Higgs ∞-bundles $\widetilde{\mathcal{M}}_\infty$

Let us fix s_p, a non-zero section of L_p. This induces embeddings $i_k : \widetilde{\mathcal{M}}_k \to \widetilde{\mathcal{M}}_{k+1}$ given by $i_k(E, \Phi_k) = (E, \Phi_k \otimes s_p)$. It clearly respects the $\mathbb{C}^*$-action and $i_k(\widetilde{N}_d^k) \subset \widetilde{N}_d^{k+1}$ which for $d > 0$ is induced from the map $\Sigma_{\bar{d}+k} \to \Sigma_{\bar{d}+k+1}$ given by $D \mapsto D + p$. It follows from (3.5) and Corollary 2.2.2 that

$$i_k^* : H^*(\widetilde{\mathcal{M}}_k) \to H^*(\widetilde{\mathcal{M}}_{k+1}) \text{ is a surjection.} \tag{7.3}$$

Now consider the direct limit of the embeddings i_k, and denote it by

$$\widetilde{\mathcal{M}}_\infty = \varinjlim \widetilde{\mathcal{M}}_k.$$

Then we have the inverse limit

$$H^*(\widetilde{\mathcal{M}}_\infty) = \varprojlim H^*(\widetilde{\mathcal{M}}_k),$$

since H^* is a contravariant functor. Recall $\overline{\mathcal{G}}$ and $P_t(B\overline{\mathcal{G}})$ from (3.9). From (7.3) we have that

$$\begin{aligned}
P_t(\widetilde{\mathcal{M}}_\infty) &= \lim_{k\to\infty} P_t(\widetilde{\mathcal{M}}_k) = \lim_{k\to\infty}\left(P_t(\widetilde{\mathcal{N}}) + \sum_{d=1}^{g-1+k} t^{2(g+2d-2)} P_t(\Sigma_{\bar{d}+k}) P_t(\mathcal{J})\right) \\
&= P_t(\widetilde{\mathcal{N}}) + \sum_{d=1}^{\infty} t^{2(g+2d-2)} P_t(\mathcal{J}) \lim_{k\to\infty}\left(P_t(\Sigma_{\bar{d}+k})\right) \\
&= P_t(\widetilde{\mathcal{N}}) + \sum_{d=1}^{\infty} t^{2(g+2d-2)} P_t(\mathcal{J}) P_t(\Sigma_\infty) \\
&= (1+t)^{2g}\left(\frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}\right) + \sum_{d=1}^{\infty} t^{2(g+2d-2)} (1+t)^{2g} \frac{(1+t)^{2g}}{(1-t^2)} \\
&= \frac{\left\{(1+t)(1+t^3)\right\}^{2g}}{(1-t^2)(1-t^4)} = P_t(B\overline{\mathcal{G}})
\end{aligned} \tag{7.4}$$

On the other hand $H^*(\widetilde{\mathcal{M}}_\infty)$ is generated by universal classes, because the same is true for $H^*(\widetilde{\mathcal{M}}_k)$. It follows that $H^*(\widetilde{\mathcal{M}}_\infty)$ is a free graded commutative algebra, and thus

$$H^*(\widetilde{\mathcal{M}}_\infty) \to H^*(\widetilde{\mathcal{M}})$$

is a resolution of the cohomology ring $H^*(\widetilde{\mathcal{M}})$. It shows that $H^*(\widetilde{\mathcal{M}})$ can be understood in the framework described at the beginning of Section 3.

In the rest of the chapter we give some applications of $\widetilde{\mathcal{M}}_\infty$ to emphasize its significance.

7.3 Geometric proof of the Mumford conjecture

As the embeddings $i_k : \widetilde{\mathcal{M}}_k \to \widetilde{\mathcal{M}}_{k+1}$ are respecting the $U(1)$-action, we have a $U(1)$-action on $\widetilde{\mathcal{M}}_\infty$. Just like above $H^*_\circ(\widetilde{\mathcal{M}}_\infty)$ is also generated over $\mathbb{Q}[u]$ by the universal equivariant classes. Furthermore a calculation, completely analogous to (7.4), shows that

$$P^\circ_t(\widetilde{\mathcal{M}}_\infty) = \frac{P_t(B\overline{\mathcal{G}})}{1-t^2}$$

from which we see that $H^*_\circ(\widetilde{\mathcal{M}}_\infty)$ is a free graded commutative algebra on the equivariant universal classes and u. Observe also that the stratification

$$\widetilde{\mathcal{M}}_\infty = \bigcup_{d=0}^{\infty} \widetilde{U}^\infty_d$$

is $U(1)$-perfect, so we are in a position to apply Proposition 2.2.1 as explained in Remark 1 after it. Namely Proposition 7.1.1 in the direct limit yields:

Proposition 7.3.1 *Set $\mathcal{R}_0$ to be the subring of $H^*_\circ(\widetilde{\mathcal{M}}_\infty)$ generated by the universal classes. Furthermore for $d \geq 1$ set*

$$\mathcal{R}_d = \langle \{ \zeta^\circ_{d,S} : S \subset \{1 \dots 2g\} \} \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]},$$

where $\langle , \rangle_{\mathbb{Q}[u,\alpha^\circ,\tau_i]}$ stands for the generated $\mathbb{Q}[u,\alpha^\circ,\tau_i]$-module. Then $i^*_d(\mathcal{R}_{d'}) = 0$ for $d < d'$ and

$$i^*_d(\mathcal{R}_d) = \langle e_d \rangle \subset H^*_\circ(\widetilde{U}^\infty_d) \tag{7.5}$$

Now as explained in Remark 1 after Proposition 2.2.1, the above proposition yields that $\bigcup_{d=1}^\infty \mathcal{R}_d$ additively generates the kernel of

$$H^*_\circ(\widetilde{\mathcal{M}}_\infty) \to H^*_\circ(\widetilde{\mathcal{N}}).$$

However $H^*(\widetilde{\mathcal{M}}_\infty)$ is a free graded commutative algebra, thus $\bigcup_{d=1}^\infty \mathcal{R}_d$ is a complete set of relations for $\widetilde{\mathcal{N}}$. Moreover combining with (6.10), we have the following theorem:

Theorem 7.3.2 *The Mumford relations ζ^r_S for each $S \subset \{1 \dots 2g\}$ and $r \geq 0$ generate the relation ideal of $H^*(\widetilde{\mathcal{N}})$.*

7.4 Gauge theoretic construction of $\widetilde{\mathcal{M}}_\infty$

To construct $\widetilde{\mathcal{M}}_\infty$ gauge theoretically, first recall the gauge theoretic construction of $\widetilde{\mathcal{M}}_k$ from Subsection 1.2.3. Recall also that $s_p \in H^0(\Sigma; L_p)$ is a fixed holomorphic section of L_p. It follows that there are embeddings $\Omega_k^{1,0} \subset \Omega_{k+1}^{1,0}$ and $\Omega_k^{1,1} \subset \Omega_{k+1}^{1,1}$, given by tensoring with s_p. Since s_p is holomorphic $\overline{\partial}_{k+1} \mid_{\mathcal{C}\times\Omega_k} = \overline{\partial}_k$, consequently

$$\mathcal{B}_k \subset \mathcal{B}_{k+1}. \tag{7.6}$$

Thus if we define the direct limit

$$\Omega_\infty^{1,i} = \lim_{k\to\infty} \Omega_k^{1,i},$$

then the direct limit of the maps $\overline{\partial}_k$ will be

$$\overline{\partial}_\infty : \mathcal{C} \times \Omega_\infty^{1,0} \to \Omega_\infty^{1,1},$$

and the direct limit

$$\mathcal{B}_\infty = \lim_{k\to\infty} \mathcal{B}_k$$

coincides with $\overline{\partial}_\infty^{-1}(0)$, which is the space of pairs (E,Φ) where Φ is a holomorphic Higgs ∞-field. We denote by $\mathrm{pr}_\infty : \mathcal{B}_\infty \to \mathcal{C}$, the direct limit of pr_k. We also have $i_k((\mathcal{B}_k)^s) \subset (\mathcal{B}_{k+1})^s$, and we let

$$(\mathcal{B}_\infty)^s = \varinjlim (\mathcal{B}_k)^s$$

denote the space of stable Higgs ∞-bundles.

It follows from the foregoing that we can think of $\widetilde{\mathcal{M}}_\infty$ as the quotient $(\mathcal{B}_\infty)^s/\mathcal{G}^c$. In order to apply this construction to obtain topological results about $\widetilde{\mathcal{M}}_\infty$, we make a detailed study of the spaces occurring:

A condition for $\overline{\partial}_k$ to be a submersion. The map $\overline{\partial}_k$ of (1.9) is a smooth map of Banach manifolds and the following theorem gives a sufficient condition for the derivative $T_{\overline{\partial}_k}$ to be surjective:

Theorem 7.4.1 *The derivative $T_{\overline{\partial}_k}$ is surjective at the point $(E,\Phi) \in \mathcal{C} \times \Omega_k^{1,0}$ if and only if*

$$\mathbb{H}^0 \left(\Sigma; \mathrm{End}(E) \otimes L_p^{-k} \xrightarrow{[\Phi,\cdot]} \mathrm{End}(E) \otimes L_p^{-k} \otimes K \right) \tag{7.7}$$

is trivial.

Proof. At the point (E,Φ) the derivative of $\overline{\partial}_k$

$$T_{\overline{\partial}_k} : \Omega_k^{0,1} \times \Omega_k^{1,0} \to \Omega_k^{1,1}$$

is given by

$$T_{\overline{\partial}_k}(\alpha,\beta) = \overline{\partial}_k^E \beta + [\alpha, \Phi],$$

where

$$\alpha \in \Omega_k^{0,1} \text{ and } \beta \in \Omega_k^{1,0}.$$

There is a natural non-degenerate pairing between $\Omega_k^{1,1}$ and $\Omega_{-k}^{0,0}$ given by integrating over Σ the trace of the tensor product. Suppose now that $\psi \in \Omega_{-k}^{0,0}$ is perpendicular to the image of $T_{\overline{\partial}_k}$, i.e.

$$\int_\Sigma \mathrm{tr} \left(T_{\overline{\partial}_k}(\alpha,\beta) \otimes \psi \right) = 0, \tag{7.8}$$

for all α and β. Then for all $\beta \in \Omega_k^{1,0}$

$$\begin{aligned} \int_\Sigma \mathrm{tr}\left(\beta \otimes \overline{\partial}_{-k}^E \psi\right) &= \int_\Sigma \mathrm{tr}\left(\overline{\partial}_0^E(\beta\otimes\psi) - \overline{\partial}_k^E(\beta)\otimes\psi\right) \\ &= \int_\Sigma d_0^E\, \mathrm{tr}(\beta\otimes\psi) - \int_\Sigma \mathrm{tr}\left(\overline{\partial}_k^E(\beta)\otimes\psi\right) \\ &= 0, \end{aligned}$$

the first term vanishes because of Stokes' theorem, the second because of (7.8) for the choice of $\alpha = 0$. However the pairing between $\Omega^{1,0}_k$ and $\Omega^{1,0}_{-k}$ is non-degenerate, which gives that $\overline{\partial}^E_{-k}(\psi) = 0$. On the other hand we have for all $\alpha \in \Omega^{1,0}_k$ that

$$\int_\Sigma [\alpha, \Phi] \otimes \psi = 0$$

from (7.8) for the choice of $\beta = 0$. It follows that $[\psi, \Phi] = 0$.

Putting everything together we have that $T_{\overline{\partial}_k}$ is surjective at (E, Φ) if and only if $\overline{\partial}^E_{-k}(\psi) = 0$ and $[\psi, \Phi] = 0$ imply $\psi = 0$. However this is exactly the Dolbeault description of the hypercohomology vector space (7.7). The result follows. □

The following lemma will be useful later:

Lemma 7.4.2 *If $k > 0$ and (E, Φ) is a stable Higgs k-bundle, then the hypercohomology (7.7) vanishes and thus $T_{\overline{\partial}_k}$ is surjective.*

If $k > 0$, $0 \le 2d \le k$ and $E \in \mathcal{C}_d$, then $H^0\left(\Sigma; \mathrm{End}(E) \otimes L_p^{-k}\right) = 0$. Consequently the hypercohomology (7.7) vanishes, and thus $T_{\overline{\partial}_k}$ is surjective at (E, Φ) for any Φ.

Proof. The first statement follows since $(E \otimes L_p^{-k}, \Phi)$ is also stable, thus a result analogous to Theorem 5.1.2 for k-Higgs bundles for $k > 0$ gives the vanishing of the hypercohomology in question.

For the second part consider

$$0 \to L \to E \to V \to 0$$

the Harder-Narasimhan filtration of E. Recall from p. 566 of [At,Bo] that $\mathrm{End}'(E)$ denotes the bundle of those endomorphisms which preserve this filtration. Any filtration-preserving endomorphism of E gives an element in $\mathrm{Hom}(L, L) \cong \mathcal{O}_\Sigma$, thus we have a bundle homomorphism $\mathrm{End}'(E) \to \mathrm{Hom}(L, L)$, whose kernel consists of endomorphisms which kill L i.e. $V^* \otimes E \subset E^* \otimes E = \mathrm{End}(E)$. Thus we have the short exact sequence

$$0 \to V^* \otimes E \to \mathrm{End}'(E) \to \mathcal{O}_\Sigma \to 0. \tag{7.9}$$

Therefore we have the following filtration of $\mathrm{End}(E)$:

$$0 \subset V^* \otimes L \subset V^* \otimes E \subset \mathrm{End}'(E) \subset \mathrm{End}(E).$$

This is not yet the Harder-Narasimhan filtration of $\mathrm{End}(E)$ since

$$\deg\left((V^* \otimes E)/(V^* \otimes L)\right) = \deg\left(\left(\mathrm{End}'(E)\right)/(V^* \otimes E)\right) = 0.$$

However this means that $\left(\mathrm{End}'(E)\right)/(V^* \otimes L)$ is semistable, thus the Harder-Narasimhan filtration of $\mathrm{End}(E)$ is:

$$0 \subset V^* \otimes L \subset \mathrm{End}'(E) \subset \mathrm{End}(E),$$

consequently the highest degree line subbundle of $\mathrm{End}(E)$ is $V^* \otimes L$ of degree $2d-1$. Therefore $\mathrm{End}(E) \otimes L_p^{-k}$ has highest degree line subbundle $V^* \otimes L \otimes L_p^{-k}$ of degree $2d - 1 - k < 0$. Such a bundle cannot have a section. The result follows. □

Stratifications on $\mathcal{B}_k$. We define two stratifications on the spaces $\mathcal{B}_k$. The first is the preimage of the Shatz stratification: $(\mathcal{B}_k)_d = \mathrm{pr}_k^{-1}(\mathcal{C}_d)$, i.e. $(\mathcal{B}_k)_d$ contains pairs (E, Φ) with $E \in \mathcal{C}_d$. The other is given by the Harder-Narasimhan filtration of Corollary 5.1.5 for Higgs k-bundles, namely we define $(\mathcal{B}_k)^0 \subset \mathcal{B}_k$ to be the subspace of stable pairs (E, Φ), and $(\mathcal{B}_k)^d$ to be the subspace of pairs (E, Φ) with destabilizing Higgs k-subbundle of degree $d > 0$.

We also let $(\mathcal{B}_k)^d_l = (\mathcal{B}_k)^d \cap (\mathcal{B}_k)_l$. It is clear that $(\mathcal{B}_k)^d \subset (\mathcal{B}_k)_d$ for $d > 0$, because the line bundle of the destabilizing Higgs k-subbundle of a Higgs k-bundle (E, Φ) will be the destabilizing line bundle of E. Thus for $d > 0$ we have

$$\text{either } (\mathcal{B}_k)^d_l = \emptyset \text{ for } d \ne l \text{ or } (\mathcal{B}_k)^d_d = (\mathcal{B}_k)^d. \tag{7.10}$$

Since these stratifications are compatible with the embeddings (7.6) we get stratifications $(\mathcal{B}_\infty)_d$ and $(\mathcal{B}_\infty)^d$ in the direct limit.

Now we have, analogously to (7.8) of [At,Bo]:

Theorem 7.4.3 *The decomposition $\mathcal{B}_\infty = \bigcup_{d=0}^{\infty} (\mathcal{B}_\infty)^d$ has the property:*

$$\overline{(\mathcal{B}_\infty)^d} \subset \bigcup_{i=d}^{\infty} (\mathcal{B}_\infty)^i.$$

Proof. First we show that $(\mathcal{B}_\infty)^0 \subset \mathcal{B}_\infty$ is open. For $k > 0$ Lemma 7.4.2 shows that $(\mathcal{B}_k)^0$ is a Banach submanifold of $\mathcal{B}_k$, moreover the tangent space of $(\mathcal{B}_k)^0$ is naturally isomorphic to the tangent space of $\mathcal{B}_k$, which proves that $(\mathcal{B}_k)^0 \subset \mathcal{B}_k$ is open indeed.

Now if $x \in \overline{(\mathcal{B}_\infty)^d}$ for $d > 0$, then $x \notin (\mathcal{B}_\infty)^0$, since $(\mathcal{B}_\infty)^0$ is open. However

$$\mathrm{pr}_\infty(x) \in \overline{\mathcal{C}_d} \subset \bigcup_{i \geq d} \mathcal{C}_i$$

from (7.8) of [At,Bo], thus (7.10) proves the result. □

Theorem 7.4.4 *For $k > 0$ and $0 \leq 2l \leq k$ the space $(\mathcal{B}_k)_{\leq l}$ is naturally a Banach manifold: it is the total space of a rank $4g - 4 + 4k$ smooth, complex vector bundle over the Banach manifold $\mathcal{C}_{\leq l}$.*

Proof. For $2l \leq k$ the derivative of $\overline{\partial}_k : \mathcal{C}_{\leq l} \times \Omega_k^{1,0} \to \Omega_k^{1,1}$ is surjective by Theorem 7.4.1 and Lemma 7.4.2. Thus the inverse function theorem gives that $(\mathcal{B}_k)_{\leq l} = \overline{\partial}_k^{-1}(0)$ is a Banach manifold indeed. Moreover the fibre of the map

$$(\mathrm{pr}_k)_{\leq l} : (\mathcal{B}_k)_{\leq l} \to \mathcal{C}_{\leq l}$$

over the point $E \in \mathcal{C}_{\leq l}$ is

$$H^0(\Sigma; \mathrm{End}(E) \otimes K \otimes L_p^k) \subset \Omega_k^{1,0}$$

of dimension $4g - 4 + 4k$ since

$$H^1(\Sigma; \mathrm{End}(E) \otimes K \otimes L_p^k) \cong (H^0(\Sigma; \mathrm{End}(E) \otimes L_p^{-k}))^* = 0,$$

by Lemma 7.4.2. Consequently the map $(\mathrm{pr}_k)_{\leq l}$ is a locally trivial fibration with fibres $\mathbb{C}^{4g-4+4k}$. □

Corollary 7.4.5 *The projection*

$$\mathrm{pr}_\infty : \mathcal{B}_\infty \to \mathcal{C}$$

is a locally trivial fibration with fibres homeomorphic to $\mathbb{C}^\infty$.

Proof. Since $4g - 4 + 4k \to \infty$ as $k \to \infty$ Theorem 7.4.4 gives that

$$(\mathrm{pr}_\infty)_{\leq l} : (\mathcal{B}_\infty)_{\leq l} \to (\mathcal{C})_{\leq l}$$

is a locally trivial fibration with fibres $\mathbb{C}^\infty$. But clearly

$$\lim_{l \to \infty} (\mathrm{pr}_\infty)_{\leq l} = \mathrm{pr}_\infty,$$

which gives the desired result. □

Theorem 7.4.6 *If $k > 0$, $0 \leq 2l \leq k$ and $d \leq l$ the stratum $(\mathcal{B}_k)^d \subset (\mathcal{B}_k)_{\leq l}$ is a Banach submanifold of $(\mathcal{B}_k)_{\leq l}$ of complex codimension $2g - 2 + k$.*

Proof. We proceed similarly to the discussion on p. 566 of [At,Bo]. We have that the $\mathcal{G}^c$-orbit of a Higgs k-bundle $\mathcal{E} = E \xrightarrow{\Phi} E \otimes K \otimes L_p^k$ in $(\mathcal{B}_k)_{\leq l}$ is, locally, a manifold of finite codimension and its normal bundle can be identified with the hypercohomology vector space $\mathbb{H}^1(\Sigma; \mathrm{End}(\mathcal{E}))$, where

$$\mathrm{End}(\mathcal{E}) = \mathrm{End}(E) \xrightarrow{[\Phi, \cdot]} \mathrm{End}(E) \otimes K \otimes L_p^k$$

is the complex of Higgs k-endomorphisms of $\mathbb{E}$.

In the same way we can identify the normal bundle to $(\mathcal{B}_k)^d$. Let $\mathrm{End}'(\mathcal{E})$ denote the complex of Higgs k-endomorphisms which respects the Harder-Narasimhan[3] filtration of $\mathcal{E}$ and define the complex $\mathrm{End}''(\mathcal{E})$ by the exact sequence

$$0 \to \mathrm{End}'(\mathcal{E}) \to \mathrm{End}(\mathcal{E}) \to \mathrm{End}''(\mathcal{E}) \to 0. \tag{7.11}$$

Alternatively, one defines $\mathrm{End}'(\mathcal{E})$ to be the complex

$$\mathrm{End}'(\mathcal{E}) = \mathrm{End}'(E) \xrightarrow{[\Phi,\cdot]} \mathrm{End}'(E) \otimes K \otimes L_p^k$$

and

$$\mathrm{End}''(\mathcal{E}) = \mathrm{End}''(E) \xrightarrow{[\Phi,\cdot]} \mathrm{End}''(E) \otimes K \otimes L_p^k,$$

using the notation of p. 566 of [At,Bo]. From this definition and 7.4 of [At,Bo] it follows that

$$\mathbb{H}^0\left(\Sigma; \mathrm{End}''(\mathcal{E})\right) = 0.$$

Because of this vanishing, the hypercohomology long exact sequence of the short exact sequence (7.11) gives the exact sequence:

$$0 \to \mathbb{H}^1\left(\Sigma; \mathrm{End}'(\mathcal{E})\right) \to \mathbb{H}^1\left(\Sigma; \mathrm{End}(\mathcal{E})\right) \to \mathbb{H}^1\left(\Sigma; (\mathrm{End}''(\mathcal{E})\right) \xrightarrow{\delta} \mathbb{H}^2\left(\Sigma; \mathrm{End}'(\mathcal{E})\right) \to \dots$$

Clearly the conormal to $(\mathcal{B}_k)^d$ is the factor of $\mathbb{H}^1\left(\Sigma; \mathrm{End}(\mathcal{E})\right)$ by $\mathbb{H}^1\left(\Sigma; \mathrm{End}'(\mathcal{E})\right)$, which by the above exact sequence is isomorphic to $\ker(\delta)$.

Now we need the following lemma:

Lemma 7.4.7 *For $k > 0$ the vector space*

$$\mathbb{H}^2\left(\Sigma; \mathrm{End}'(\mathcal{E})\right) = 0$$

is trivial.

Proof. It is sufficient to show that $H^1\left(\Sigma; \mathrm{End}'(E) \otimes K \otimes L_p^k\right) = 0$. For this we need by Serre duality that $H^0\left(\Sigma; \left(\mathrm{End}'(E)\right)^* \otimes L_p^{-k}\right) = 0$. Taking the dual of (7.9) and tensoring by L_p^{-k} we get the short exact sequence

$$0 \to L_p^{-k} \to \left(\mathrm{End}'(\mathcal{E})\right)^* \otimes L_p^{-k} \to V \otimes E^* \otimes L_p^{-k} \to 0.$$

Since $V \otimes E^* \otimes L_p^{-k}$ has Harder-Narasimhan filtration:

$$0 \to L_p^{-k} \to V \otimes E^* \otimes L_p^{-k} \to V \otimes L^* \otimes L_p^{-k} \to 0,$$

it follows that $H^0\left(\Sigma; V \otimes E^* \otimes L_p^{-k}\right) = 0$, and in turn that $H^0\left(\Sigma; \left(\mathrm{End}'(E)\right)^* \otimes L_p^{-k}\right) = 0$. The result follows. $\square$

The above lemma yields that the conormal to $(\mathcal{B}_k)^d$ is isomorphic to $\mathbb{H}^1\left(\Sigma; \mathrm{End}''(\mathcal{E})\right)$. Finally we need the following result:

Lemma 7.4.8 *For $k > 0$ the dimension of $\mathbb{H}^1\left(\Sigma; \mathrm{End}''(\mathcal{E})\right)$ depends only on k; it is given by:*

$$\dim\left(\mathbb{H}^1(\Sigma; \mathrm{End}''(\mathcal{E})\right) = 2g - 2 + k.$$

[3] Cf. Corollary 5.1.5.

Proof. First we show that $\mathbb{H}^2\left(\Sigma;\mathrm{End}''(\mathcal{E})\right)=0$. This follows from

$$H^1\left(\Sigma;\mathrm{End}''(E)\otimes K\otimes L_p^k\right)=0.$$

Observe that $\mathrm{End}''(E)\cong L^*\otimes V$ thus $\deg\left(\mathrm{End}''(E)\right)=1-2d$, consequently

$$H^0\left(\Sigma;\left(\mathrm{End}''(E)\right)^*\otimes L_p^{-k}\right)=0.$$

Now Riemann-Roch proves the lemma. □

Theorem 7.4.6 follows. □

Corollary 7.4.9 *We have*

$$H_q((\mathcal{B}_k)^{\leq d+1},(\mathcal{B}_k)^{\leq d};\mathbb{Z})=0$$

for $2d+2\leq k$ *and* $q<2(2g-2+k)$.

Proof. This is a consequence of Theorem 7.4.3, Theorem 7.4.6 and the Thom isomorphism in homology:

$$H_q((\mathcal{B}_k)^{\leq d+1},(\mathcal{B}_k)^{\leq d};\mathbb{Z})\cong H_{q-2(2g-2+k)}((\mathcal{B}_k)^{d+1};\mathbb{Z}).$$

□

7.5 Homotopy types

We learned in Section 7.2 that the cohomology rings of $\widetilde{\mathcal{M}}_\infty$ and $B\overline{\mathcal{G}}$ are isomorphic. We show in the next subsection that this is because they are in fact *homotopy equivalent*.

7.5.1 Homotopy type of $\widetilde{\mathcal{M}}_\infty$

We start with a result in the gauge theory setting of the previous section.

Proposition 7.5.1 *The space $(\mathcal{B}_\infty)^0$ is contractible.*

Proof. First we prove that

$$\pi_i\left((\mathcal{B}_\infty)^0\right) = \begin{cases} 0 & \text{for} \quad i > 0 \\ \mathbb{Z} & \text{for} \quad i = 0 \end{cases} \tag{7.12}$$

For this we show that

$$H_i((\mathcal{B}_\infty)^0;\mathbb{Z}) = \begin{cases} 0 & \text{for} \quad i > 0 \\ \mathbb{Z} & \text{for} \quad i = 0 \end{cases} . \tag{7.13}$$

To see that $(\mathcal{B}_\infty)^0$ is connected note that the map

$$(\mathrm{pr}_\infty)^0 : (\mathcal{B}_\infty)^0 \to \mathcal{C}$$

has connected image and fibres. Note also that $\mathcal{B}_\infty$ is contractible from Corollary 7.4.5.

Thus (7.13) follows from

$$H_*(\mathcal{B}_\infty, (\mathcal{B}_\infty)^0;\mathbb{Z}) = 0.$$

Taking direct limits it follows from

$$H_*((\mathcal{B}_\infty)^{\leq d}, (\mathcal{B}_\infty)^0;\mathbb{Z}) = 0 \text{ for each } d.$$

We prove this by induction on d. For $d = 0$ it is trivial. Suppose we proved it for d and consider the homology long exact sequence of the triple $(\mathcal{B}_\infty)^0 \subset (\mathcal{B}_\infty)^{\leq d} \subset (\mathcal{B}_\infty)^{\leq d+1}$:

$$\to H_q\left((\mathcal{B}_\infty)^{\leq d}, (\mathcal{B}_\infty)^0;\mathbb{Z}\right) \to H_q\left((\mathcal{B}_\infty)^{\leq d+1}, (\mathcal{B}_\infty)^0;\mathbb{Z}\right) \to H_q\left((\mathcal{B}_\infty)^{\leq d+1}, (\mathcal{B}_\infty)^{\leq d};\mathbb{Z}\right) \to .$$

By induction $H_*((\mathcal{B}_\infty)^{\leq d}, (\mathcal{B}_\infty)^0;\mathbb{Z}) = 0$ thus we need only to prove

$$H_*\left((\mathcal{B}_\infty)^{\leq d+1}, (\mathcal{B}_\infty)^{\leq d};\mathbb{Z}\right) = 0.$$

Taking direct limits it follows from Corollary 7.4.9 since $2g - 2 + k \to \infty$ as $k \to \infty$. Thus

$$H_*\left(\mathcal{B}_\infty, (\mathcal{B}_\infty)^0;\mathbb{Z}\right) = 0$$

indeed, proving (7.13).

We also show that

$$\pi_1((\mathcal{B}_\infty)^0) \text{ is Abelian.} \tag{7.14}$$

Consider the homotopy long exact sequence of the fibration $(\mathcal{B}_\infty)^0 \to \left((\mathcal{B}_\infty)^0\right)_{\overline{\mathcal{G}}} \to B\overline{\mathcal{G}}$:

$$\pi_2(B\overline{\mathcal{G}}) \to \pi_1\left((\mathcal{B}_\infty)^0\right) \to \pi_1\left(\left((\mathcal{B}_\infty)^0\right)_{\overline{\mathcal{G}}}\right) \overset{\mathrm{pr}_*}{\to} \pi_1(B\overline{\mathcal{G}}),$$

Because the indices of the Bott-Morse function μ_k –the moment map of the $U(1)$-action on $\widetilde{\mathcal{M}}_k$– are all even it follows from Bott-Morse theory that

$$\pi_1\left(\left((\mathcal{B}_k)^0\right)_{\overline{\mathcal{G}}}\right) \cong \pi_1(\widetilde{\mathcal{M}}_k) \cong \pi_1(\widetilde{\mathcal{N}}).$$

According to p. 581 of [At,Bo]

$$\pi_1(\widetilde{\mathcal{N}}) \cong \pi_1(B\mathcal{G}) \cong \pi_1(B\overline{\mathcal{G}}).$$

Thus pr_* is an isomorphism. Thus $\pi_1\left((\mathcal{B}_\infty)^0\right)$ is a factor group of the Abelian group $\pi_2(B\overline{\mathcal{G}})$, proving (7.14).

Now (7.13) together with the Hurewitz theorem[4] imply that the abelianization of $\pi_1\left((\mathcal{B}_\infty)^0\right)$ is 0 thus from (7.14) it is 0, and in turn we get (7.12).

The next step is to show that $(\mathcal{B}_\infty)^0$ is a CW-space[5]. Consider the fibration

$$(\mathcal{B}_\infty)^0 \to ((\mathcal{B}_\infty)^0)_{\overline{\mathcal{G}}^c} \to B\overline{\mathcal{G}} \tag{7.15}$$

from (2.1). We show that its total space and base space are CW-spaces. It will then follow from Corollary (13) of [Sta], that the fibre $(\mathcal{B}_\infty)^0$ is a CW-space.

Note that $\overline{\mathcal{G}}^c$ acts freely on $(\mathcal{B}_\infty)^0$ and the quotient is $\widetilde{\mathcal{M}}_\infty$. Thus we have the fibration

$$E\overline{\mathcal{G}}^c \to ((\mathcal{B}_\infty)^0)_{\overline{\mathcal{G}}^c} \to \widetilde{\mathcal{M}}_\infty.$$

In this fibration the base space, being the direct limit of finite dimensional manifolds, is a CW-complex, and the fibre, being contractible, is a CW-space. Then Proposition (0) of [Sta] yields that the total space $((\mathcal{B}_\infty)^0)_{\overline{\mathcal{G}}^c}$ is a CW-space.

Furthermore $B\mathcal{G}$ is a CW-space because according to Proposition 2.4 of [At,Bo] it is a component of a mapping space from a compact Hausdorff space Σ to a CW-complex $BU(2)$, and a theorem of Milnor[6] says that such a space is a CW-space. Recall now (3.11), i.e. that $B\mathcal{G} \sim BU(1) \times B\overline{\mathcal{G}}$. Thus we have a fibration $B\mathcal{G} \to BU(1)$ from a CW-space to a CW-complex, according to Corollary (13) of [Sta] it follows that the fibre $B\overline{\mathcal{G}}$ is itself a CW-space.

Putting everything together we have a connected CW-space $(\mathcal{B}_\infty)^0$ with trivial homotopy groups. Whitehead's theorem[7] concludes the proof. □

Thus we have $\overline{\mathcal{G}}^c$ acting freely on the contractible space $(\mathcal{B}_\infty)^0$, with quotient $\widetilde{\mathcal{M}}_\infty$, which gives the following immediate

Corollary 7.5.2 *The space $\widetilde{\mathcal{M}}_\infty$ is homotopy equivalent to $B\overline{\mathcal{G}}$.*

7.5.2 Homotopy type of the strata

In this subsection we prove that not only the whole spaces $\widetilde{\mathcal{M}}_\infty$ and $B\overline{\mathcal{G}}$ are homotopy equivalent, but even as stratified spaces. This explains why the calculation (7.4) was the same as the Atiyah-Bott calculation (3.15) of $P_t(\widetilde{\mathcal{N}})$.

Proposition 7.5.3 *The map*

$$(\mathrm{pr}_\infty)^0_d : (\mathcal{B}_\infty)^0_d \to \mathcal{C}_d$$

is a homotopy equivalence.

Proof. First we note that $(\mathcal{B}_\infty)^0_d = (\mathcal{B}_\infty)_d \setminus (\mathcal{B}_\infty)^d$. Now both are locally trivial fibrations with fibre $\mathbb{C}^\infty$ over $\mathcal{C}_d$ and the codimension of $(\mathcal{B}_\infty)^d$ in $(\mathcal{B}_\infty)_d$ is infinite. It follows that $(\mathcal{B}_\infty)^0_d$ is a locally trivial fibration over $\mathcal{C}_d$ with fibre retracting to S^∞, which is contractible. Now $\mathcal{C}_d$ is paracompact (and consequently numerable), since it is a metric subspace of the metric space $\mathcal{C}$, and also locally contractible since it is a Banach manifold. Thus Theorem 6.3 of [Dold][8] yields that $(\mathrm{pr}_\infty)^0_d$ is a homotopy equivalence. □

Corollary 7.5.4 *We have the homotopy equivalence*

$$\widetilde{U}^\infty_d \sim (\mathcal{C}_d)_{\overline{\mathcal{G}}^c}. \tag{7.16}$$

Consequently $(\mathcal{C}_d)_{\overline{\mathcal{G}}^c}$ is homotopy equivalent to $\mathcal{J}_d \times \Sigma_\infty$.

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[4] As in Theorem 2.1.1 of Section 2 of Chapter 13 of [Jam].
[5] Which means that it has the homotopy type of a CW-complex
[6] Cf. [Miln]
[7] Cf. e.g. Theorem 2.1.3 of Section 2 of Chapter 13 of [Jam].
[8] This reference was suggested by Ioan James.

Proof. The map $(\mathrm{pr}_\infty)^0_d$ induces a map of $B\overline{\mathcal{G}}$-spaces:

$$\begin{array}{ccc} (\mathcal{B}_\infty)^0_d & \to & \mathcal{C}_d \\ \downarrow & & \downarrow \\ \big((\mathcal{B}_\infty)^0_d\big)_{\overline{\mathcal{G}}} & \to & (\mathcal{C}_d)_{\overline{\mathcal{G}}} \\ \downarrow & & \downarrow \\ B\overline{\mathcal{G}} & \cong & B\overline{\mathcal{G}} \end{array}.$$

As one can choose a CW-complex model for $B\overline{\mathcal{G}}$, e.g. from Corollary 7.5.2, the previous theorem and Theorem 6.3 of [Dold] gives (7.16).

For the last statement note that $\widetilde{U}^k_d$ is the moduli space of complexes $L \xrightarrow{\phi} VKL^k_p$, which is uniquely determined by L and ϕ, thus $\widetilde{U}^k_d \cong \mathcal{J}_d \times \Sigma_{\bar{d}+k}$. It follows that

$$\widetilde{U}^\infty_d \cong \mathcal{J}_d \times \Sigma_\infty. \tag{7.17}$$

The result follows. □

Corollary 7.5.5 *We have*

$$(\mathcal{C}_d)_{\mathcal{G}} \sim \Sigma_\infty \times \Sigma_\infty.$$

Proof. Since (3.11) is a product we get

$$(\mathcal{C}_d)_{\mathcal{G}} \sim BU(1) \times (\mathcal{C}_d)_{\overline{\mathcal{G}}} \cong BU(1) \times \mathcal{J}_d \times \Sigma_\infty.$$

The result follows from (7.19). □

7.5.3 Stabilization of homotopy groups

Finally we have two results about homotopy groups stabilizing in the resolution tower. The second of which is reminiscent of the the Atiyah-Jones conjecture about the stabilization of the homotopy groups of the moduli space of instantons on S^4.

Theorem 7.5.6 *For $k \geq 0$ we have*

$$(\mathrm{pr}^k_d)_* : \pi_i(\widetilde{U}^k_d) \to \pi_i(\widetilde{U}^\infty_d) \;\text{ for }\; 0 \leq i \leq \bar{d} + k - 1.$$

Proof. Because of (7.17), it is sufficient to show the stabilization of homotopy groups for the resolution tower (3.3) of Σ. By (12.2) of [Macd] we have an isomorphism

$$(i_n)_* : H_i(\Sigma_n; \mathbb{Z}) \to H_i(\Sigma_\infty; \mathbb{Z}) \;\text{ for }\; 0 \leq i \leq n - 1. \tag{7.18}$$

The isomorphism for the fundamental groups is clear. Thus $\pi_2(\Sigma_\infty, \Sigma_n)$, being the factor group of the Abelian group $\pi_2(\Sigma_\infty)$, is also Abelian. Now the relative Hurewitz theorem gives the result. □

Remark. 1. By a theorem of Dold and Thom we have a complete description of the homotopy type of Σ_∞, namely $\pi_k(\Sigma_\infty) \cong H_k(\Sigma)$, and

$$\Sigma_\infty \sim \prod_{i>0} K(H_i(\Sigma; \mathbb{Z}), i), \tag{7.19}$$

which combined with (7.17) gives an explicit description of the homotopy type of $\widetilde{U}^\infty_d$.

2. It is also interesting to note that (7.19) together with Proposition 2.4 and (2.6) of [At,Bo] show that $B\mathcal{G}_1 \sim \Sigma_\infty$, where $\mathcal{G}_1$ is the group of gauge transformations on a principal $U(1)$-bundle on Σ.

Our final result is the following

Theorem 7.5.7 *For $k \geq 0$ we have*

$$\pi_i(\widetilde{\mathcal{M}}_k) \overset{i^*_k}{\cong} \pi_i(\widetilde{\mathcal{M}}_\infty) \cong \pi_i(B\overline{\mathcal{G}}) \;\text{ for }\; 0 \leq i \leq 4g - 8 + k.$$

Proof. First we show that

$$H_i(\widetilde{\mathcal{M}}_\infty, \widetilde{\mathcal{M}}_k; \mathbb{Z}) = 0 \quad \text{for} \quad 0 \leq i \leq 4g-7+k.$$

This follows from

$$(i_{\widetilde{\mathcal{M}}_k})_* : H_i(\widetilde{\mathcal{M}}_k; \mathbb{Z}) \to H_i(\widetilde{\mathcal{M}}_\infty; \mathbb{Z}) \text{ is an isomorphism for } 0 \leq i \leq 4g-7+k,$$

which is a consequence of the five lemma applied to the diagram

$$\begin{array}{ccccccccc}
\dots & \to & H_q(\widetilde{U}^\infty_{<d}; \mathbb{Z}) & \to & H_q(\widetilde{U}^\infty_{\leq d}; \mathbb{Z}) & \to & H_{q-2(g+2d-2)}(\widetilde{U}^\infty_d; \mathbb{Z}) & \to & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \to & H_q(\widetilde{U}^k_{<d}; \mathbb{Z}) & \to & H_q(\widetilde{U}^k_{\leq d}; \mathbb{Z}) & \to & H_{q-2(g+2d-2)}(\widetilde{U}^k_d; \mathbb{Z}) & \to & \dots
\end{array}$$

and (7.18). We also need that $\pi_1(\widetilde{\mathcal{M}}_\infty, \widetilde{\mathcal{M}}_k) = 0$, this follows from the fact that $(i_0)_* : \pi_1(\widetilde{\mathcal{N}}) \to \pi_1(\widetilde{\mathcal{M}}_k)$ is an isomorphism for each k from standard Bott-Morse theory, since each index is even. Finally we have that $\pi_2(\widetilde{\mathcal{M}}_\infty, \widetilde{\mathcal{M}}_k)$ is Abelian, because it is a factor of the Abelian group $\pi_2(\widetilde{\mathcal{M}}_\infty)$. Now the relative Hurewitz theorem[9] gives that

$$\pi_i(\widetilde{\mathcal{M}}_\infty, \widetilde{\mathcal{M}}_k; \mathbb{Z}) = 0 \quad \text{for} \quad 0 \leq i \leq 4g-7+k,$$

which in turn proves the result. $\square$

[9]E.g. Theorem 2.1.2 of Section 2 of Chapter 13 of [Jam].

Conclusion

In this thesis we have attempted to give a general picture of the geometrical and topological properties of $\mathcal{M}$, the moduli space of rank 2 Higgs bundles with fixed determinant of degree 1 over Σ. Examining the symplectic geometry of $\mathcal{M}$ we found two Morse stratifications on it and a natural compactification of it. The downward flows were found to be responsible for the intersection numbers, and the upward flows for the cohomology ring. Investigating the latter we constructed a resolution tower for $\widetilde{\mathcal{M}}$ and found that its direct limit was a model for the classifying space of the gauge group modulo constant scalars.

However we have not yet explained the relation between the compactification and the rest of the thesis. In the next and final section we intend to fill this gap by providing a heuristic and at some places conjectural summary of the thesis from the point of view of the compactification.

8.1 Compactification of the thesis

Let us go back to the end of Chapter 4 and recall Theorem 4.6.13, where we transformed the problem of intersection numbers from $\mathcal{M}$ to a problem concerning the cohomology ring of Z. Without using this correspondence we were able to calculate the intersection numbers in Chapter 5. Here however we are focusing on Z and explain the cohomological calculations of the later chapters from this angle.

Since Z is a symplectic quotient of $\mathcal{M}$ we have the Kirwan map

$$r : H^*_\circ(\mathcal{M}) \to H^*(Z),$$

which has the fundamental property that it is surjective. Consequently generators for $H^*_\circ(\mathcal{M})$ give generators for $H^*(Z)$. We denote by α_Z, β_Z and ψ^i_Z the images of the corresponding equivariant universal classes by the Kirwan map. We have also that $r(u) = c_1(L_Z)$ the first Chern class of the contact line bundle on Z. Moreover these generators can be obtained from the universal bundle $\mathbb{E}_Z$, which is the restriction of $\mathbb{E}^\circ_{\mathcal{M}}$ in the quotient.

In order to be able to go on we have to consider $H^*_{\circ cpt}(\mathcal{M})$ the compactly supported equivariant cohomology of $\mathcal{M}$. It is trivial below the middle dimension and is g-dimensional at the middle dimension. Indeed $H^{3g-3}_{\circ cpt}(\mathcal{M})$ is generated by the equivariant compactly supported cohomology classes of the components of the nilpotent cone. An analogue of Theorem 6.4.4 for compactly supported cohomology says that $H^*_{\circ cpt}(\mathcal{M})^\Gamma$ is generated[10] as a $\mathbb{Q}[u, \alpha, \beta, \psi_i]$-module from $H^{3g-3}_{\circ cpt}(\mathcal{M})$.

Going back to Z we have, as the $\mathbb{C}^*$-equivariant analogue of Theorem 4.6.13, the isomorphism[11] of rings:

$$H^*(Z) \cong H^*_\circ(\mathcal{M})/H^*_{\circ cpt}(\mathcal{M}),$$

where on the right hand side we have the quotient ring of the ordinary equivariant cohomology by the image of the compactly supported equivariant cohomology, which is an ideal. Considering the obvious equivariant structure on the virtual Dirac bundle $\mathbf{D}_k$, we can work out the equivariant cohomology classes of the components of the nilpotent cone. The corresponding $\mathbf{D}_k$ over Z however, will be an honest vector bundle, thus its Chern classes vanish in degrees beyond the rank, giving relations in $H^*(Z)^\Gamma$. We know that each component of the nilpotent cone has trivial cohomology class, which over Z says, that all the above relations will be some multiple of $c_1(L_Z)$.

The picture which emerges is that the relations for $H^*(Z)^\Gamma$ are of two types: those which are multiples of $c_1(L_Z)$, these correspond to the intersection numbers on $H^*(\mathcal{M})$, and the rest, which correspond to

[10] Recall that H^*_{cpt} is an H^*-module.

[11] This statement seems to be true in the general setting of Subsection 2.1.2. As such, it may be interesting in its own right, especially in relation with the recent paper [To,We].

relations in $H^*(\mathcal{M})^\Gamma$. Thus from this perspective Chapter 5 and Chapter 6 attempt to give a complete description of the cohomology ring of the projective variety Z and in turn for the compactification $\overline{\mathcal{M}}$!

We can also look at Chapter 7 from the point of view of the compactification. Namely we can form $\widetilde{Z}_k$ the highest level Kähler quotients of each $\widetilde{\mathcal{M}}_k$, since their moment maps μ_k are proper, by taking the quotient of $\widetilde{\mathcal{M}}_k \setminus \tilde{N}^k$ by the $\mathbb{C}^*$-action, where $\tilde{N}^k$ denotes the downward Morse flow, or equivalently the nilpotent cone in $\widetilde{\mathcal{M}}_k$. It can be seen that the inclusions of the spaces $\widetilde{\mathcal{M}}_k$ induces inclusions for $\widetilde{Z}_k$. Thus we can form the direct limit $\widetilde{Z}_\infty$. Similarly we have $\overline{\mathcal{M}}_k$ the compactification of $\widetilde{\mathcal{M}}_k$ and their direct limit $\overline{\mathcal{M}}_\infty$.

The rational cohomology of $\widetilde{Z}_\infty$ is generated by the universal classes and an extra degree 2 class: $c_1(L_{\widetilde{Z}_\infty})$, the first Chern class of the contact line bundle on $\widetilde{Z}_\infty$. Its Poincaré polynomial can be shown to be equal to the Poincaré polynomial of the free graded commutative algebra on these generators, showing that $H^*(\widetilde{Z}_\infty)$ is a free graded commutative algebra. It follows that it is isomorphic to $H^*(B\mathcal{G})$ the rational cohomology of the classifying space of the *whole* gauge group. We suggest that this is because they are both homotopy equivalent to $\overline{\mathcal{M}}_\infty$:

$$\overline{\mathcal{M}}_\infty \sim \widetilde{Z}_\infty \sim B\mathcal{G}. \tag{8.20}$$

The main technical difficulty which arises in attempting to prove this is the fact that the spaces $\widetilde{Z}_k$ and $\overline{\mathcal{M}}_k$ are not smooth, they have $\mathbb{Z}_2$-orbifold singularities, thus the calculation of their homology with integer coefficients is a bit subtle.

To avoid this problem one may want to consider the highest level Kähler quotient and the compactification only homotopically. In other words let us take $\widetilde{Z}'_k$ to be $(\widetilde{\mathcal{M}}_k \setminus \tilde{N}^k)_{U(1)}$, i.e. take the homotopy quotient instead of the singular topological quotient. In an analogue way we form the homotopy-simplectic cut $\overline{\mathcal{M}}'_k$. Then it can be shown, without problems coming from torsion, that we have

$$\widetilde{Z}'_\infty \sim \overline{\mathcal{M}}'_\infty \sim (\widetilde{\mathcal{M}}_\infty)_{U(1)}.$$

Now (2.1) gives the fibration

$$\widetilde{\mathcal{M}}_\infty \to (\widetilde{\mathcal{M}}_\infty)_{U(1)} \to BU(1),$$

which we propose to be homotopy equivalent to the fibration (3.10), in particular, that it is a product.

Since the codimension of the singular locus in $\widetilde{Z}_\infty$ is ∞, one hopes to conclude that $\widetilde{Z}_\infty \sim \widetilde{Z}'_\infty$ and similarly $\overline{\mathcal{M}}_\infty \sim \overline{\mathcal{M}}'_\infty$ yielding (8.20).

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