XVI

S-DUALITY IN HYPERKÄHLER HODGE THEORY

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To Nigel Hitchin for his 60th birthday

16.1 Introduction

In this chapter we survey the motivations, related results, and progress made towards the following problem, raised by Hitchin in 1995:

Problem 16.1 *What is the space of L^2 harmonic forms on the moduli space of Higgs bundles on a Riemann surface?*

The moduli space $\mathcal{M}^d_{Dol}(SL_n)$ of stable rank n Higgs bundles with fixed determinant of degree d on a Riemann surface was introduced and studied in Hitchin (1987), Nitsure (1991), and Simpson (1991). The Betti numbers of this space for $n = 2$ were determined in Hitchin (1987*b*) while for $n = 3$ in Gothen (1994). The above problem raised two new directions to study. First is the Riemannian geometry of $\mathcal{M}^d_{Dol}(SL_n)$, or more precisely the asymptotics of the natural hyperkähler metric, and its connection with Hodge theory. The second one, which can be considered the topological side of Problem 16.1, is to determine the intersection form on the middle-dimensional compactly supported cohomology of $\mathcal{M}^d_{Dol}(SL_n)$. While the first question seems still out of reach, although we will report on some modest progress below, the second is more approachable and we offer a conjecture at the end of this survey.

Problem 16.1 was motivated by S-duality conjectures emerging from the string theory literature about Hodge theory on certain hyperkähler moduli spaces, which are close relatives of $\mathcal{M}^d_{Dol}(SL_n)$.

In the physics literature S-duality stands for a *strong–weak duality* between two quantum field theories. The interest from the physics point of view is that it gives a tool to study physical theories with a large coupling constant via a conjectured equivalence with a theory with a small coupling constant where perturbative methods give a good understanding. The S-duality conjecture relevant for us is based on the Montonen–Olive electromagnetic duality proposal from 1977 in four-dimensional Yang–Mills theory (Montonen and Olive 1977). It was noted in (Witten and Olive 1978) that this duality proposal is more likely to hold in a supersymmetric version of the theory, and in Osborn (1979) it was argued that $N = 4$ supersymmetry is a good candidate. Hyperkähler Hodge theory is relevant in $N = 4$ supersymmetry as the space of differential forms on a hyperkähler manifold possesses an action of the $N = 4$ supersymmetry algebra via the various operators in hyperkähler Hodge theory.

In this chapter our interest lies in the mathematical predictions of such S-duality conjectures in physics. Sen (1994), using S-duality arguments in $N = 4$ supersymmetric Yang–Mills theory, predicted the dimension of the spaces $\mathcal{H}^d\left(\widetilde{M}_k^0\right)$ of L^2 harmonic d-forms on the universal cover $\widetilde{M}_k^0$ of the hyperkähler moduli space M_k^0 of certain $SU(2)$ magnetic monopoles on $\mathbb{R}^3$. In the interpretation of Sen (1994) the L^2 harmonic forms on $\widetilde{M}_k^0$ can be thought of as bound states of the theory, and the conjectured S-duality implies an action of $SL(2,\mathbb{Z})$ on $\bigoplus_k \mathcal{H}^*\left(\widetilde{M}_k^0\right)$. By further physical arguments Sen managed to predict this representation of $SL(2,\mathbb{Z})$ completely, implying the following:

Conjecture 16.1 *The dimension of the space of L^2 harmonic forms on $\widetilde{M}_k^0$ is*

$$\dim\left(\mathcal{H}^d\left(\widetilde{M}_k^0\right)\right) = \begin{cases} 0 & d \neq mid \\ \phi(k) & d = mid, \end{cases}$$

where $\phi(k) = \sum_{i=1}^k \delta_{1(i,k)}$ is the Euler ϕ function, and $mid = 2k-2$ is half of the dimension of $\widetilde{M}_k^0$.

Similar S-duality arguments led Vafa and Witten (1994) to get a conjecture on the space of L^2 harmonic forms on a certain smooth completion M_ϕ^{k,c_1}, constructed in Kronheimer (1990) and Nakajima (1998), of the moduli space of $U(n)$ Yang–Mills instantons of first Chern class c_1, energy k, and framing ϕ on one of Kronheimer's ALE spaces, which are four-dimensional complete hyperkähler manifolds, with an asymptotically locally Euclidean metric.

Conjecture 16.2 *The dimension of the space of L^2 harmonic forms on M_ϕ^{k,c_1} is*

$$\dim\left(\mathcal{H}^d\left(M_\phi^{k,c_1}\right)\right) = \begin{cases} 0 & d \neq mid \\ \dim\left(\operatorname{im}\left(H^{mid}_{cpt}\left(M_\phi^{k,c_1}\right) \to H^{mid}\left(M_\phi^{k,c_1}\right)\right)\right) & d = mid, \end{cases}$$

where mid now denotes half of the dimension of M_ϕ^{k,c_1}.

Vafa and Witten (1994) further argue that Conjecture 16.2 implies, via the work of Nakajima (1998) and Kac (1990), that

$$Z_\phi(q) = \sum_{c_1,k} q^{k-c/24} \dim\left(\mathcal{H}^{mid}\left(M_\phi^{k,c_1}\right)\right) \tag{16.1}$$

is a modular form, which, as was speculated in Vafa and Witten (1994), might be a consequence of S-duality.

This chapter will introduce the reader to various mathematical aspects of these three problems and offer mathematical techniques and results relating to them.

16.2 Hyperkähler quotients

A Riemannian manifold (M, g) is hyperkähler if it is Kähler with respect to three integrable complex structures $I, J, K \in \Gamma(\mathrm{End}(TM))$, which satisfy $I^2 = J^2 = K^2 = IJK = -1$, with Kähler forms ω_I, ω_J, and ω_K. Known compact examples are scarce (see e.g. Joyce 2000, section 7). Non-compact complete examples however are much more abundant. This is mostly because there is a widely applicable[1] *hyperkähler quotient construction*, due to Hitchin *et al.* (1987). The construction itself is an elegant quaternionization of the Marsden–Weinstein symplectic (or more precisely Kähler) quotient construction (see Mumford *et al.* 1994, chapter 8 for an introduction for the latter).

Let $\mathbb{M}$ be a hyperkähler manifold, $\mathcal{G}$ a Lie group, with Lie algebra $\mathfrak{g}$, and assume $\mathcal{G}$ acts on $\mathbb{M}$ preserving the hyperkähler structure (i.e. it acts by triholomorphic isometries). Let us further assume that we have moment maps $\mu_I : \mathbb{M} \to \mathfrak{g}^*$, $\mu_J : \mathbb{M} \to \mathfrak{g}^*$, and $\mu_K : \mathbb{M} \to \mathfrak{g}^*$ with respect to the symplectic forms ω_I, ω_J, and ω_K, respectively. We combine them into a single hyperkähler moment map:

$$\mu_{\mathbb{H}} = (\mu_I, \mu_J, \mu_K) : \mathbb{M} \to \mathbb{R}^3 \otimes \mathfrak{g}^*.$$

One takes $\xi \in \mathbb{R}^3 \otimes (\mathfrak{g}^*)^G$ and constructs the *hyperkähler quotient* at level ξ by

$$\mathbb{M}////_{\xi}\mathcal{G} := \mu_{\mathbb{H}}^{-1}(\xi)/\mathcal{G}.$$

The main result of Hitchin *et al.* (1987) is that the natural Riemannian metric on the smooth points of this quotient is hyperkähler.

Now we list three important examples of this construction, where the original hyperkähler manifold $\mathbb{M}$ and Lie group $\mathcal{G}$ are both infinite dimensional.

16.2.1 *Moduli of Yang–Mills instantons on* $\mathbb{R}^4$

Here we follow Hitchin (1987*a*, I example 3.6), compare also with Atiyah (1978).

Let G be a compact connected Lie group, which will be $U(n)$ or $SU(n)$ in this chapter. Let $P \to \mathbb{R}^4$ be a G-principal bundle over $\mathbb{R}^4$. Let $\mathbb{M}$ be the space of G-connections A on P of class C^∞, such that the energy

$$\left| \int_{\mathbb{R}^4} \mathrm{Tr}(F_A \wedge *F_A) \right| < \infty$$

is finite. Write

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4$$

in a fixed gauge, where $A_i \in \Omega^0(\mathbb{R}^4, \mathrm{ad}(P))$. Let $\mathcal{G} = \Omega(\mathbb{R}^4, Ad(P))$ be the gauge group of P. An element $g \in \mathcal{G}$ acts on $A \in \mathbb{M}$ by the formula $g(A) = g^{-1}Ag + g^{-1}dg$, preserving the hyperkähler structure. One finds that the hyperkähler

[1] Some colleagues even suggest, due to the success of this construction, that HyperKähLeR is in fact just a pronouncable version of the acronym HKLR.

moment map equation

$$\mu_{\mathbb{H}}(A) = 0 \Leftrightarrow F_A = *F_A$$

is just the self-dual Yang–Mills equation. Define the hyperkähler quotient $\mathcal{M}(\mathbb{R}^4, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the moduli space of finite-energy self-dual Yang–Mills instantons on P. By its construction it has a natural hyperkähler metric.

A similar construction (Kronheimer and Nakajima 1990) for $G = U(n)$ yields a hyperkähler metric on moduli spaces of $U(n)$ Yang–Mills instantons on certain four-dimensional complete hyperkähler manifolds, the ALE spaces of Kronheimer (1989). These moduli spaces will have natural completions and various components of them will be the spaces M_{ϕ}^{k,c_1} which were mentioned in the introduction. They will resurface later as examples for Nakajima quiver varieties.

16.2.2 *Moduli space of magnetic monopoles on* $\mathbb{R}^3$

The following construction can be considered as a dimensional reduction of the previous example. Here we follow Hitchin (1987*a*, I example 3.5) and Atiyah and Hitchin (1988).

Assume that $G = SU(2)$ and the matrices A_i are independent of x_4. Then we have

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

a connection on $\mathbb{R}^3$ and $A_4 = \phi \in \Omega^0(\mathbb{R}^3, \mathrm{ad}(P))$ becomes the *Higgs field*. The gauge group now is $\mathcal{G} = \Omega(\mathbb{R}^3, Ad(P))$ and $\mathbb{M}$ is the space of configurations (A, ϕ) satisfying certain boundary conditions. (The boundary condition is chosen to ensure finite energy.) The gauge group $\mathcal{G}$ acts on $\mathbb{M}$ by gauge transformations, preserving the natural hyperkähler metric on $\mathbb{M}$. The corresponding hyperkähler moment map equation

$$\mu_{\mathbb{H}}(A, \phi) = 0 \Leftrightarrow F_A = *d_A \phi$$

is equivalent to the Bogomolny equation.

Now by construction $M = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the moduli space of magnetic monopoles on $\mathbb{R}^3$ has a natural hyperkähler metric. It has infinitely many components $M = \cup_{k=1}^{\infty} M_k$ labeled by the magnetic charge k of the monopole.

M_k is acted upon by $\mathbb{R}^3$ by translations and by $U(1)$ by rotating the phase of the monopole. The quotient M_k^0 is still a smooth complete hyperkähler manifold of dimension $4k - 4$, with fundamental group $\mathbb{Z}_k$. We will denote by $\widetilde{M_k^0}$ its universal cover. In Atiyah and Hitchin (1985) they find the hyperkähler metric explicitly on the four-manifold M_2^0 and subsequently describe the scattering of two monopoles.

16.2.3 *Hitchin moduli space*

This example can be considered as a two-dimensional reduction of Section 16.2.1. We follow Hitchin (1987*b*, section 1; 1987*a* I example 3.3).

Now we assume that $G = U(n)$ and the matrices A_i in Section 16.2.1 are independent of x_3, x_4. We have now the connection $A = A_1 dx_1 + A_2 dx_2$ on the $U(n)$ principal bundle P on $\mathbb{R}^2$. We introduce $\Phi = (A_3 - A_4 i)dz \in \Omega^{1,0}(\mathbb{R}^2, \mathrm{ad}(P) \otimes \mathbb{C})$ the *complex Higgs field*. The gauge group now is $\mathcal{G} = \Omega(\mathbb{R}^2, Ad(P))$, which acts by gauge transformations on the space $\mathbb{M}$ of configurations (A, Φ) preserving the natural hyperkähler metric on $\mathbb{M}$. The moment map equations

$$\mu_{\mathbb{H}}(A, \Phi) = 0 \Leftrightarrow \begin{array}{c} F(A) = -[\Phi, \Phi^*], \\ d''_A \Phi = 0 \end{array}$$

are then equivalent with Hitchin's self-duality equations. There are no solutions of finite energy on $\mathbb{R}^2$, but as the equations are conformally invariant, we can replace $\mathbb{R}^2$ with a genus g compact Riemann surface C in the above definitions, and define $\mathcal{M}(C, P) = \mu_{\mathbb{H}}^{-1}(0)/\mathcal{G}$, the Hitchin moduli space, which has a natural hyperkähler metric by construction. There are different ways to think about this space with the different complex structures, which will be explained in Section 16.5.2.

16.3 Hodge theory

16.3.1 L^2 *harmonic forms on complete manifolds*

Let M be a complete Riemannian manifold of dimension n. We say that a smooth differential k-form $\alpha \in \Omega^k(M)$ is harmonic if and only if $d\alpha = d{*}\alpha = 0$, where $* : \Omega^k(M) \to \Omega^{n-k}(M)$ is the Hodge star operator. It is L^2 if and only if

$$\int_M \alpha \wedge *\alpha < \infty.$$

We denote by $\mathcal{H}^*(M)$ the space of L^2 harmonic forms.

A fundamental theorem of Hodge theory is the Hodge (orthogonal) decomposition theorem of de Rham (1984, section 32 theorem 24, Section 35 theorem 26):

$$\Omega^*_{L^2} = \overline{d\left(\Omega^*_{cpt}\right)} \oplus \mathcal{H}^* \oplus \overline{\delta\left(\Omega^*_{cpt}\right)}, \tag{16.2}$$

where δ is the adjoint of d. When M is compact this implies the celebrated Hodge theorem, which says that $\mathcal{H}^*(M) \cong H^*(M)$, that is, that there is a unique harmonic representative in every de Rham cohomology class. When M is non-compact we only have a topological lower bound. Namely, the Hodge decomposition theorem implies that the composite map

$$H^*_{cpt}(M) \to \mathcal{H}^*(M) \to H^*(M)$$

is just the forgetful map. (In the compact case these maps are isomorphisms, which gives the Hodge theorem mentioned above.) Thus

$$\mathrm{im}(H^*_{cpt}(M) \to H^*(M)) \tag{16.3}$$

is a "topological lower bound" for $\mathcal{H}^*(M)$. By Poincaré duality the map $H^*_{cpt}(M) \to H^*(M)$ is equivalent with the intersection pairing on $H^*_{cpt}(M)$.

In the cases most relevant for us M will be a hyperkähler manifold (sometimes orbifold) so $\dim(M) = 4k$ and we will additionally have $H^i(M) = 0$ for $i > 2k$. Therefore the possible non-trivial image in $\mathrm{im}(H^*_{cpt}(M) \to H^*(M))$ will be concentrated in the middle $2k$ dimension. (We will use the notation $mid = \dim(M)/2$ for the middle dimension of a manifold.) For such a hyperkähler manifold we denote

$$\begin{aligned} \chi_{L^2}(M) &= \dim\left(\mathrm{im}\left(H^{mid}_{cpt}(M) \to H^{mid}(M)\right)\right) \\ &= \dim\left(\mathrm{im}\left(H^*_{cpt}(M) \to H^*(M)\right)\right) \end{aligned} \tag{16.4}$$

the dimension of this image. $\chi_{L^2}(M)$ can be thought of either as a "topological lower bound" for $\dim(\mathcal{H}^*(M))$ or the Euler characteristic of topological L^2 cohomology.

16.3.2 *Results on L^2 harmonic forms*

There were few general theorems on describing $\mathcal{H}^*(M)$ for a non-compact complete manifold M (see however Hausel *et al.* 2004, introduction for an overview). It was thus a surprising development when Sen (1994), using arguments from S-duality, managed to predict the dimension of L^2 harmonic forms on $\widetilde{M}^k_0$ as was explained in Conjecture 16.1 in the Introduction. In particular, according to Sen's Conjecture 16.1 the space $\mathcal{H}^2(\widetilde{M}^0_2)$ should be one dimensional. Using the explicit description of Atiyah and Hitchin (1985) of the metric on $\mathcal{M}^0_2$ in Sen (1994) he was able to find an explicit L^2 harmonic two-form, called the *Sen two-form*, on $\widetilde{M}^0_2$. This was perhaps the strongest mathematical support exhibited for Conjecture 16.1 in Sen (1994).

More general mathematical support for Conjecture 16.1 came in 1996. Segal and Selby (1996) showed that the intersection form on $H^{mid}_{cpt}(\widetilde{M}^0_k)$ is definite. Moreover they obtained for the topological lower bound (16.3) for $\mathcal{H}^{mid}(\widetilde{M}^0_k)$

$$\chi_{L^2}(\widetilde{M}^0_k) = \dim(H^{mid}(\widetilde{M}^0_k)) = \phi(k).$$

This agrees with the predicted dimension of $\mathcal{H}^{mid}(\widetilde{M}^0_k)$ in Sen's Conjecture 16.1.

Motivated by Problem 16.1 and Segal–Selby's topological lower bound for Conjecture 16.1, the author calculated in Hausel (1998) that the intersection pairing on the g-dimensional space $H^{mid}_{cpt}(\mathcal{M}^1_{Dol}(SL_2))$ is trivial, in other words

$$\chi_{L^2}\left(\mathcal{M}^1_{Dol}(SL_2)\right) = 0 \tag{16.5}$$

for $g > 1$. This thus gave the surprising result that there are no L^2 harmonic forms on $\mathcal{M}^1_{Dol}(SL_2)$ plainly by topological reasons. The technique used in the proof of (16.5) was imitating Kirwan's proof (1992) of Mumford's conjecture on the cohomology ring of the moduli space of stable rank 2 bundles of degree 1 on the Riemann surface C. Therefore the extension of (16.5) to higher rank Higgs bundle moduli spaces $\mathcal{M}^d_{Dol}(SL_n)$ was not straightforward.

The next advance towards Sen's Conjecture 16.1 came in 2000. Hitchin (2000) showed that $\mathcal{H}^d(M) = 0$ unless $d = \dim(M)/2$ for a complete hyperkähler manifold M of linear growth. Examples include all our hyperkähler quotients discussed in this chapter. The proofs in Hitchin (2000) use techniques inspired by Jost and Zuo's extension (2000) of ideas of Gromov (1991). It is interesting to note that some of the proofs in Hitchin (2000) also exploit the operators in hyperkähler Hodge theory, which are relevant in $N = 4$ supersymmetry. Using the symmetries of the Atiyah–Hitchin metric (Hitchin 2000) proves Sen's conjecture for $k = 2$, that up to a scalar the only L^2 harmonic form on $\widetilde{M}^0_2$ is Sen's two-form.

A more topological approach was introduced in Hausel *et al.* (2004). Hausel *et al.* (2004) proves for fibered boundary manifolds M

$$\mathcal{H}^{mid}(M) \cong \operatorname{im}\left(IH^{mid}_{\underline{m}}(\overline{M}) \to IH^{mid}_{\bar{m}}(\overline{M})\right), \tag{16.6}$$

where $\overline{M}$ is a certain compactification of M, dictated by the asymptotics of the fibered boundary metric on M. Moreover $IH^{mid}_{\underline{m}}(\overline{M})$ denotes the intersection cohomology in dimension $mid = \dim(M)/2$ with lower middle perversity $\underline{m}$ and $IH^{mid}_{\bar{m}}(\overline{M})$ denotes the intersection cohomology in the middle dimension with upper middle perversity $\bar{m}$ of the possibly badly singular (i.e. not necessarily a Witt space) compactification $\overline{M}$. To illustrate (16.6) we take the compactification of $\widetilde{M}^0_2$, which happens to be the smooth space $\mathbb{CP}^2$ (with the non-standard orientation), where the above cohomologies in (16.6) all coincide, giving $\mathcal{H}^2(\widetilde{M}^0_2) \cong H^2(\mathbb{CP}^2)$. This provides a topological explanation for the existence and uniqueness of the Sen two-form.

The assumption that the metric is fibered boundary in Hausel *et al.* (2004) is fairly restrictive. Among hyperkähler quotients only a few examples satisfy this property (see the discussion in Hausel *et al.* 2004, section 7). Examples include all ALE gravitational instantons of Kronheimer (1989) and all known ALF (see Cherkis and Kapustin 1999) and some ALG gravitational instantons (see Cherkis and Kapustin 2002). In general our hyperkähler quotients have some kind of stratified asymptotic behaviour at infinity. For example, the metric on M^0_k is fibered boundary only when $k = 2$, for higher k it is known to behave differently at different regions of infinity. The first result which could handle Hodge theory on Riemannian manifolds with such a stratified behaviour at infinity appeared recently in a work by Carron. It proves for a QALE space M that:

$$\mathcal{H}^{mid}(M) \cong \operatorname{im}\left(H^{mid}_{cpt}(M) \to H^{mid}(M)\right).$$

A QALE space (Joyce 2000, section 9) by definition is a certain Calabi–Yau metric on a crepant resolution of $\mathbb{C}^k/\Gamma$, where $\Gamma \subset SU(k)$ is a finite subgroup. The asymptotics of the metric on such a QALE space is reminiscent to the asymptotics of the natural hyperkähler metric on M_ϕ^{k,c_1} appearing in the Vafa–Witten Conjecture 16.2. It is thus reasonable to hope that the Vafa–Witten Conjecture 16.2 will be decided soon.

As there have been extensive studies starting with Gibbons and Manton (1995) and more recently Bielawski (2008) on the asymptotics of the Riemannian metric on M_k^0, it is conceivable that we will have a precise understanding of the asymptotic behaviour of this metric, and in turn the Hodge theory of L^2 harmonic forms on $\widetilde{M}_k^0$, perhaps extending techniques from Carron. Thus one may be optimistic that Sen's Conjecture 16.1 will be decided in the foreseeable future.

Finally, one must admit that the description of the asymptotics of the metric at infinity on $\mathcal{M}_{Dol}^d(SL_n)$ is still lacking, thus calculation of $\mathcal{H}^*\left(\mathcal{M}_{Dol}^d(SL_n)\right)$ is presently hopeless. The topological side of Problem 16.1, that is, to determine $\chi_{L^2}\left(\mathcal{M}_{Dol}^d(SL_n)\right)$, when $(d,n)=1$, is more reasonable. After introducing a new arithmetic technique to study Hodge structures on the cohomology of our hyperkähler manifolds, we will be able to offer a general conjecture on the intersection form on Higgs moduli spaces, in particular that (16.5) holds for any n.

16.4 Mixed Hodge theory

As explained above there have been some limited successes of calculating $\mathcal{H}^*(M)$ for a hyperkähler quotient and understanding its relation to the cohomology $H^*(M)$ or more generally the cohomology of an appropriate compactification $H^*(\bar{M})$. Another extension of Hodge theory yields some different and in some ways more detailed insight into the cohomology of our hyperkähler quotients. This technique is Deligne's mixed Hodge structure on the cohomology of any complex algebraic variety. Instead of the global analysis on the Riemannian geometry of the complex algebraic variety it will relate to the arithmetic of the variety over finite fields.

16.4.1 *Mixed Hodge structure of Deligne*

Motivated by the (then still unproven) Weil conjectures and Grothendieck's "yoga of weights", which drew cohomological conclusions about complex varieties from the truth of those conjectures, Deligne (1971, 1974) proved the existence of mixed Hodge structures on the cohomology $H^*(M,\mathbb{Q})$ of a complex algebraic variety M. Here we give a quick introduction, for more details see Hausel and Rodriguez-Villegas (section 2.2) and the references therein. Deligne's mixed Hodge structure entails two filtrations on the rational cohomology of M. The increasing weight filtration

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2j} = H^j(X,\mathbb{Q})$$

and a decreasing Hodge filtration

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0.$$

We can define mixed Hodge numbers obtained from these two filtrations by the following formula:

$$h^{p,q;j}(X) := \dim_{\mathbb{C}} \left(Gr_p^F Gr_{p+q}^W H^j(X)_{\mathbb{C}} \right). \tag{16.7}$$

From these numbers we form

$$H(M; x, y, t) = \sum_{p,q,k} h^{p,q;k}(M) x^p y^q t^k,$$

the *mixed Hodge polynomial*. By virtue of its definition it has the property that the specialization

$$P(M; t) = H(M; 1, 1, t)$$

gives the *Poincaré polynomial* of M. When M is smooth of dimension n we take another specialization

$$E(M; x, y) := x^n y^n H(1/x, 1/y, -1), \tag{16.8}$$

the so-called *E-polynomial* of a smooth variety M.

Deligne's construction of mixed Hodge structure is complex geometrical: for a smooth variety M it is defined by the log geometry of a compactification $\overline{M}$ with normal crossing divisors. In particular a global analytical description, like the Hodge theory of harmonic forms on a smooth complex projective manifold, of the mixed Hodge structure on a smooth variety is missing, which causes some difficulty in finding the meaning of mixed Hodge numbers in physical contexts (see the remark after Conjecture 16.3).

16.4.2 *Arithmetic and topological content of the E-polynomial*

The connection of the E-polynomial to the arithmetic of the variety is provided by the following theorem of Katz (Hausel and Rodriguez-Villegas, Appendix). Here we give an informal version of Katz's result for precise formulation (see Hausel and Rodriguez-Villegas 2008, theorem 6.1.2(3), theorem 2.1.8):

Theorem 16.1 *Let M be a smooth quasi-projective variety defined over $\mathbb{Z}$ (i.e. given by equations with integer coefficients). Assume that the number of points of M over a finite field $\mathbb{F}_q$, that is,*

$$E(q) := \#\{M(\mathbb{F}_q)\}$$

is a polynomial in q. Then the E-polynomial can be obtained from the count polynomial as follows:

$$E(M; x, y) = E(xy).$$

This theorem is especially useful when we further have $h^{p,q;k}(M) = 0$ unless $p + q = k$. In this case we say that the mixed Hodge structure on $H^*(M)$ is *pure*. In this case

$$H(M; x, y, t) = (xyt^2)^n E\left(\frac{-1}{xt}, \frac{-1}{yt}\right)$$

and so the Poincaré polynomial can be recovered from the E-polynomial as follows:

$$P(M; t) = H(M; 1, 1, t) = t^{2n} E\left(\frac{-1}{t}, \frac{-1}{t}\right).$$

Examples of varieties with pure MHS on their cohomology include smooth projective varieties (in this case we get the traditional Hodge structure, which is by definition pure), the moduli space of Higgs bundles $\mathcal{M}_{Dol}$, the moduli space of flat connections $\mathcal{M}_{DR}$ on a Riemann surface, and Nakajima's quiver varieties.

In general we can define the *pure part* of $H(M; x, y, t)$ as

$$PH(M; x, y) = \mathrm{Coeff}_{T^0}\left(H(M; xT, yT, tT^{-1})\right).$$

More generally we can define the *pure part* of the cohomology of M as

$$PH^*(M) := W_n H^n(M) \subset H^*(M),$$

which is a subring $PH^*(M) \subset H^*(M)$ of the cohomology of M. For a smooth M, the pure part of $H^*(M)$ is always the image of the cohomology of a smooth compactification (see Deligne 1971, corollaire 3.2.17). It is in fact this result which can be used to show that the spaces mentioned in the previous paragraph have pure mixed Hodge structure. That is, one can prove that they admit a smooth compactification which surjects on cohomology. Prototypes of such compactifications were constructed in Simpson (1997) for $\mathcal{M}_{DR}$ and in Hausel (1998) for $\mathcal{M}_{Dol}$.

16.5 Applications of mixed Hodge theory

Using the method sketched in the previous section the strongest results on cohomology can be achieved when the variety has a pure MHS on its cohomology, consequently the E-polynomial determines the mixed Hodge polynomial, and additionally it has polynomial-count so that Theorem 16.1 gives an arithmetic way to determine the E-polynomial. This is the case for Nakajima quiver varieties, where our method gives complete results.

16.5.1 *Nakajima quiver varieties*

Nakajima quiver varieties are constructed (Nakajima 1998) by a finite-dimensional hyperkähler quotient construction. Here we review the affine algebraic-geometric version of this construction.

Let Γ be a quiver (oriented graph) with vertex set $I = \{1, \dots, n\}$ and edges $E \subset I \times I$. Let

$$\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n), \quad \mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_n) \in \mathbb{N}^I$$

be two-dimensional vectors and V_i and W_i corresponding complex vector spaces, that is, $\dim(V_i) = \mathbf{v}_i$ and $\dim(W_i) = \mathbf{w}_i$. We define the vector spaces

$$\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{a \in E} \mathrm{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in I} \mathrm{Hom}(V_i, W_i)$$

of framed representations of the quiver Γ , and the action

$$\rho : GL(\mathbf{v}) := \prod_{i \in I} GL(V_i) \to GL(\mathbb{V}_{\mathbf{v}}),$$

with derivative

$$\varrho : \mathfrak{gl}(\mathbf{v}) := \prod_{i \in I} \mathfrak{gl}(V_i) \to \mathfrak{gl}(\mathbb{V}_{\mathbf{v}}).$$

The complex moment map

$$\mu : \mathbb{V}_{\mathbf{v},\mathbf{w}} \times \mathbb{V}^*_{\mathbf{v},\mathbf{w}} \to \mathfrak{gl}^*_{\mathbf{v}}$$

of ρ is given at $X \in \mathfrak{gl}_{\mathbf{v}}$ by

$$\langle \mu(v, w), X \rangle = \langle \varrho(X) v, w \rangle. \tag{16.9}$$

For $\xi = 1_{\mathbf{v}} \in \mathfrak{gl}(\mathbf{v})^{GL(\mathbf{v})}$ we define the (always smooth) Nakajima quiver variety by

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mu^{-1}(\xi) // GL(\mathbf{v}) = \mathrm{Spec}\left(\mathbb{C}[\mu^{-1}(\xi)]^{\mathrm{GL}(\mathbf{v})}\right)$$

as an affine GIT quotient. Alternatively one can construct the manifold underlying $\mathcal{M}(\mathbf{v}, \mathbf{w})$ as a hyperkähler quotient of $\mathbb{V}_{\mathbf{v},\mathbf{w}} \times \mathbb{V}^*_{\mathbf{v},\mathbf{w}}$ by the maximal compact subgroup $U(\mathbf{v}) \subset GL(\mathbf{v})$. This shows that $\mathcal{M}(\mathbf{v}, \mathbf{w})$ possesses a hyperkähler metric. The holomorphic symplectic quotient we presented above is the one where the arithmetic technique of Section 16.4 is applicable. Before we explain that, let us recall the following fundamental theorem of Nakajima (1998) about the cohomology of these Nakajima quiver varieties:

Theorem 16.2 *Assume that the quiver Γ has no edge-loops. Then there is an irreducible representation of the Kac–Moody algebra $\mathfrak{g}(\Gamma)$ of highest weight $\mathbf{w}$ on $\oplus_{\mathbf{v}} H^{mid}(\mathcal{M}(\mathbf{v}, \mathbf{w}))$. In particular the Weyl–Kac character formula gives the middle Betti numbers of Nakajima quiver varieties. Furthermore the intersection form on $H^{mid}_c(\mathcal{M}(\mathbf{v}, \mathbf{w}))$ is definite, thus $\chi_{L^2}(\mathcal{M}(\mathbf{v}, \mathbf{w}))$ equals the middle Betti number of $\mathcal{M}(\mathbf{v}, \mathbf{w})$.*

Remark 16.1 When Γ is an affine Dynkin diagram $\mathcal{M}(\mathbf{v}, \mathbf{w})$ could be identified with one of the spaces M^{k,c_1}_{ϕ} of certain Yang–Mills instantons on a ALE space X_Γ. In Kac (1990) he explains that the Weyl–Kac character formula for an affine Dynkin diagram has certain modular properties. This was the line of argument in Vafa and Witten (1994) that (16.1) is a modular form provided Conjecture 16.2 holds.

In Hausel (2006) a simple Fourier transform technique was found to enumerate the rational points of $\mathcal{M}(\mathbf{v}, \mathbf{w})$ over a finite field $\mathbb{F}_q$. The corresponding count function $E(q)$ turned out to be polynomial, and as the mixed Hodge structure is pure on $H^*(\mathcal{M}(\mathbf{v}, \mathbf{w}))$ the technique of Section 16.4 applies in its full strength to give a formula for the Betti numbers of the varieties $\mathcal{M}(\mathbf{v}, \mathbf{w})$. The result is the following formula from Hausel (2006):

Theorem 16.3 *For any quiver Γ, the Betti numbers of the Nakajima quiver varieties are given by the following generating function, with the notation as in Hausel (2006, theorem 3):*

$$\sum_{\mathbf{v}\in\mathbb{N}^I} P_t(\mathcal{M}(\mathbf{v},\mathbf{w}))t^{-d(\mathbf{v},\mathbf{w})}T^{\mathbf{v}} = \frac{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\left(\prod_{(i,j)\in E} t^{-2\langle\lambda^i,\lambda^j\rangle}\right)\left(\prod_{i\in I} t^{-2\langle\lambda^i,(1^{\mathbf{w}_i})\rangle}\right)}{\prod_{i\in I}\left(t^{-2\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}{\displaystyle\sum_{\mathbf{v}\in\mathbb{N}^I} T^{\mathbf{v}} \sum_{\lambda\in\mathcal{P}(\mathbf{v})} \frac{\prod_{(i,j)\in E} t^{-2\langle\lambda^i,\lambda^j\rangle}}{\prod_{i\in I}\left(t^{-2\langle\lambda^i,\lambda^i\rangle}\prod_k\prod_{j=1}^{m_k(\lambda^i)}(1-t^{2j})\right)}}, \tag{16.10}$$

Remark 16.2 When Γ has no edge-loops Nakajima's Theorem 16.1 implies that the right-hand side of (16.10) is a deformation of the Weyl–Kac character formula. Simple reasoning gives the same result about the denominator of the right-hand side of (16.10) and the Kac denominator. Moreover, Kac's denominator formula and Hua's formula (2000, theorem 4.9) expressing the denominator of (16.10) as an infinite product imply a conjecture of Kac (cf. Hua 2000, corollary 4.10). Namely, if $A_\Gamma(\mathbf{v}, q)$ denotes the number of absolutely indecomposable representations of Γ of dimension vector $\mathbf{v}$ over the finite field $\mathbb{F}_q$, then it turns out to be a polynomial in q and Kac's conjecture 1 (1983) says that the constant coefficient

$$A_\Gamma(\mathbf{v}, 0) = m_{\mathbf{v}} \tag{16.11}$$

equals with the multiplicity of the weight $\mathbf{v}$ in the Kac–Moody algebra $\mathfrak{g}(\Gamma)$. This can be proved, as sketched above and announced in Hausel (2006), to be a consequence of (16.10) and the above-mentioned results of Nakajima and Hua.

Remark 16.3 When the quiver is affine ADE and the RHS becomes an infinite product (indications that this can happen are the infinite product in Hausel 2006, section 3 and the infinite products in the recent Sasaki) we could get an alternative proof of the modularity of (16.1) in the Vafa–Witten S-duality conjecture.

In the remaining part of this survey we will motivate and study another application of the technique in Section 16.4, which will be less powerful as the mixed Hodge structure will fail to be pure, but will also open new interesting directions by the study of this more complicated mixed Hodge structure.

16.5.2 *Spaces diffeomorphic to the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$*

Among the spaces discussed in this chapter it is the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$ as defined in Section 16.2.2 which exhibits perhaps the most plentiful structures many of which are rooted in its hyperkähler quotient origin. In particular there are three distinct complex algebraic variety structures on $\mathcal{M}(C, P_{U(n)})$. These can be thought of (Simpson 1997) as the three types of non-Abelian (first) cohomology: Dolbeault, De Rham, and Betti, of the Riemann surface C. The survey paper Hausel (2005) gives a quick introduction to these spaces and some of the cohomological implications to be discussed below.

In this chapter the ground field is always $\mathbb{C}$ unless otherwise indicated. Following Hitchin (1987*b*) and Simpson (1997) we define a component of the twisted $GL_n = GL_n(\mathbb{C})$ Dolbeault cohomology of C as

$$\mathcal{M}^d_{Dol}(GL_n) := \left\{ \begin{matrix} \text{Moduli space of semistable rank } n \\ \text{degree } d \text{ Hitchin pairs on } C \end{matrix} \right\}$$

the GL_n De Rham cohomology as

$$\mathcal{M}^d_{DR}(GL_n) := \left\{ \begin{matrix} \text{Moduli space of flat } GL_n\text{-connections} \\ \text{on } C \setminus \{p\}, \text{ with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{matrix} \right\}$$

and the GL_n Betti cohomology

$$\mathcal{M}^d_B(GL_n) := \Big\{ A_1, B_1, \ldots, A_g, B_g \in GL_n |$$
$$A_1^{-1} B_1^{-1} A_1 B_1 \cdots A_g^{-1} B_g^{-1} A_g B_g = e^{\frac{2\pi i d}{n}} Id \Big\} // GL_n$$

as a twisted GL_n character variety of C.

When $d=0$ these three varieties are diffeomorphic to the Hitchin moduli space $\mathcal{M}(C, P_{U(n)})$. However we prefer to consider the twisted versions, when $(d,n)=1$, because then all the varieties are smooth. In this case these three varieties are all diffeomorphic to a twisted version $\mathcal{M}^d(C, P_{U(n)})$ of Hitchin moduli space and so to each other. The mixed Hodge structure is pure on $H^*\left(\mathcal{M}^d_{Dol}(GL_n)\right)$ and $H^*\left(\mathcal{M}^d_{DR}(GL_n)\right)$, while it is not pure on $H^*\left(\mathcal{M}^d_B(GL_n)\right)$. As the mixed Hodge structures are different on $H^*\left(\mathcal{M}^d_{DR}(GL_n)\right)$ and $H^*\left(\mathcal{M}^d_B(GL_n)\right)$, the spaces $M^d_{DR}(GL_n)$ and $\mathcal{M}^d_B(GL_n)$ cannot be isomorphic as complex algebraic varieties. Nevertheless as complex analytic manifolds the Riemann–Hilbert monodromy map

$$\mathcal{M}^d_{DR}(GL_n) \stackrel{RH}{\to} \mathcal{M}^d_B(GL_n) \tag{16.12}$$

sending a flat connection to its holonomy gives an isomorphism.

We will also consider the varieties $\mathcal{M}^d_{Dol}(SL_n)$, $\mathcal{M}^d_{DR}(SL_n)$, and $\mathcal{M}^0_B(SL_n)$, which can be defined by replacing GL_n with SL_n in the above definitions. Moreover $\mathcal{M}^0_{Dol}(GL_1)$, $\mathcal{M}^0_{DR}(GL_1)$, and $\mathcal{M}^0_B(GL_1)$ turn out to be Abelian groups. Then $\mathcal{M}^0_{Dol}(GL_1)$, $\mathcal{M}^0_{DR}(GL_1)$, and $\mathcal{M}^0_B(GL_1)$ will act on $\mathcal{M}^d_{Dol}(GL_n)$, $\mathcal{M}^d_{DR}(GL_n)$, and $\mathcal{M}^d_B(GL_n)$, respectively, by an appropriate form of tensorization. Finally we denote the corresponding (affine GIT) quotients by $\mathcal{M}^d_{Dol}(PGL_n)$, $\mathcal{M}^d_{DR}(PGL_n)$ and $\mathcal{M}^d_B(PGL_n)$. In our case, when $(d,n)=1$, they will turn out to be orbifolds. For more details on the construction of these varieties see Hausel (2005).

In the next section we explain the original motivation to consider the E-polynomials of these three complex algebraic varieties. The motivation is mirror symmetry, and most probably the same S-duality we discussed in the Introduction in connection with the Hodge cohomology of the moduli spaces of Yang–Mills instantons in four dimension and magnetic monopoles in three. S-duality ideas relating to mirror symmetry for Hitchin spaces have appeared in the physics literature (Bershadsky *et al.* 1995; Kapustin and Witten 2007).

16.5.3 *Topological mirror test*

For our mathematical considerations the relationship to mirror symmetry stems from the following observation of Hausel and Thaddeus (2003). It uses the famous *Hitchin map* (Hitchin 1987*c*), which makes the moduli space of Higgs bundles $\mathcal{M}_{Dol}$ into a completely integrable Hamiltonian system, so that the generic fibers are Abelian varieties.

Theorem 16.4 *In the following diagram*

$$\begin{array}{ccc} \mathcal{M}^d_{Dol}(PGL_n) & & \mathcal{M}^d_{Dol}(SL_n) \\ \downarrow{\scriptstyle \chi_{PGL_n}} & & \downarrow{\scriptstyle \chi_{SL_n}} \\ \mathcal{H}_{PGL_n} & \cong & \mathcal{H}_{SL_n} \end{array}$$

the generic fibers of the Hitchin maps χ_{PGL_n} and χ_{SL_n} are dual Abelian varieties.

Remark 16.4 If we change complex structures and consider $\mathcal{M}^d_{DR}(PGL_n)$ and $\mathcal{M}^d_{DR}(SL_n)$, then the Hitchin map on them becomes special Lagrangian fibrations, and consequently the pair of $\mathcal{M}^d_{DR}(PGL_n)$ and $\mathcal{M}^d_{DR}(SL_n)$ satisfies the requirements of the SYZ construction (Strominger *et al.* 1996) for a pair of mirror symmetric Calabi–Yau manifolds (see Hausel and Thaddeus 2001, 2003 for more details).

This motivates the calculation of Hodge numbers of $\mathcal{M}^d_{DR}(PGL_n)$ and $\mathcal{M}^d_{DR}(SL_n)$ to see if there is any relationship between them, which one would expect in mirror symmetry. Based on calculations in the $n=2,3$ cases Hausel and Thaddeus (2003) proposed

Conjecture 16.3 *For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$,*

$$E_{st}^{B^e}\left(x, y; \mathcal{M}_{DR}^d(SL_n)\right) = E_{st}^{\hat{B}^d}\left(x, y; \mathcal{M}_{DR}^e(PGL_n)\right),$$

where B^e and $\hat{B}^d$ are certain gerbes on the corresponding Hitchin spaces and the E-polynomials above are stringy E-polynomials for orbifolds twisted by the relevant gerbe as defined in Hausel and Thaddeus (2003).

Morally, this conjecture should be related to the S-duality considerations of Kapustin and Witten (2007) and in turn to the geometric Langlands programme of Beilinson and Drinfeld (1995). However the lack of global analytical interpretation of the mixed Hodge numbers (16.7) appearing in Conjecture 16.3 prevents a straightforward physical interpretation. Nevertheless the agreement of certain Hodge numbers for Hitchin spaces for Langlands dual groups is an interesting direction from a purely mathematical point of view. In particular, if we change our focus from $\mathcal{M}_{DR}$ and $\mathcal{M}_{Dol}$ to $\mathcal{M}_B$ we will uncover some surprising connections to the representation theory of finite groups of Lie type.

16.5.4 *Mirror symmetry for finite groups of Lie type*

As $\mathcal{M}_{DR}$ and $\mathcal{M}_B$ are complex analytically identical via the Riemann–Hilbert map (16.12), the complex analytical structure of dual special Lagrangian fibrations of Theorem 16.4 are present on the pair $\mathcal{M}_B^d(SL_n)$ and $\mathcal{M}_B^e(PGL_n)$. We might as well try to think of this pair as mirror symmetric in the SYZ picture. The mixed Hodge numbers of $\mathcal{M}_B$ are however different from the mixed Hodge numbers of $\mathcal{M}_{DR}$ so the corresponding topological mirror test (Hausel 2005) will also be different from Conjecture 16.3:

Conjecture 16.4 *For all $d, e \in \mathbb{Z}$, satisfying $(d,n) = (e,n) = 1$,*

$$E_{st}^{B^e}\left(x, y, \mathcal{M}_B^d(SL_n)\right) = E_{st}^{\hat{B}^d}\left(x, y, \mathcal{M}_B^e(PGL_n)\right).$$

For this conjecture however there is a powerful arithmetic method to calculate these E-polynomials. Using this technique we have already managed to check this conjecture (Hausel 2005) when n is a prime and $n = 4$. This arithmetic method is based on the technique explained in Section 16.4 and the following character formula from Hausel and Rodriguez-Villegas (2008):

Theorem 16.5 *Let $G = SL_n$ or GL_n, let $G(\mathbb{F}_q)$ be the corresponding finite group of Lie type*

$$E\left(\sqrt{q}, \sqrt{q}, \mathcal{M}_B^d(G)\right) = \#\left\{\mathcal{M}_B^d(G(\mathbb{F}_q))\right\} = \sum_{\chi \in Irr(G(\mathbb{F}_q))} \frac{|G(\mathbb{F}_q)|^{2g-2}}{\chi(1)^{2g-1}} \chi\left(\xi_n^d\right),$$

where the sum is over all irreducible characters of the finite group of Lie type $G(\mathbb{F}_q)$.

This character formula combined with Conjecture 16.4 implies certain relationships between the character tables of $PGL_n(\mathbb{F}_q)$ and $SL_n(\mathbb{F}_q)$. An intriguing way to formulate it is to say that *certain differences between the character tables of $PGL_n(\mathbb{F}_q)$ and its Langlands dual $SL_n(\mathbb{F}_q)$ are governed by mirror symmetry.* This kind of consideration could be interesting because the character tables of $PGL_n(\mathbb{F}_q)$ or more generally those of $GL_n(\mathbb{F}_q)$ have been known for a long time starting with the work of Green (1955), while the character tables of $SL_n(\mathbb{F}_q)$ have just recently been completed (Bonnafé 2006; Shoji 2006). It is especially enjoyable to follow the effect of the mirror symmetry proposal of Conjecture 16.4 by comparing the character tables of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$ first calculated a hundred years ago by Jordan 1907 and Schur 1907.

16.5.5 *Conjectural answer*

Finally, we can put all our observations and conjectures together to state a conjectural answer to the topological side of Problem 16.1.

As we already noted the mixed Hodge structure on $H^*(\mathcal{M}_B)$ is not pure. Therefore we are losing information by considering only $E(\mathcal{M}_B; x, y)$. It turns out that it is interesting to consider the full mixed Hodge polynomial $H(\mathcal{M}_B; x, y, t)$. When $n = 2$ it can be calculated via the explicit description of $H^*(\mathcal{M}_B)$ in Hausel and Thaddeus (2003). We get Hausel and Rodriguez-Villegas (Theorem 1.1.3):

$$\begin{aligned}
&H(\mathcal{M}_B(PGL_2); x, y, t)\\
&\quad = \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)}\\
&\quad\quad - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)},
\end{aligned}$$

where $q = xy$ and the four terms correspond to the four types of irreducible characters of $GL(2, \mathbb{F}_q)$. When $g = 3$ this equals

$$\begin{aligned}
&t^{12}q^{12} + t^{12}q^{10} + 6\,t^{11}q^{10} + t^{12}q^8 + t^{10}q^{10} + 6\,t^{11}q^8 + 16\,t^{10}q^8 + 6\,t^9q^8 + t^{10}q^6\\
&\quad + t^8q^8 + 26\,t^9q^6 + 16\,t^8q^6 + 6\,t^7q^6 + t^8q^4 + t^6q^6 + 6\,t^7q^4 + 16\,t^6q^4\\
&\quad + 6\,t^5q^4 + t^4q^4 + t^4q^2 + 6\,t^3q^2 + t^2q^2 + 1.
\end{aligned}$$

In particular we see that the pure part is $1 + q^2t^4 + q^4t^8$. These terms correspond to the cohomology classes 1, β, and β^2, and the term q^6t^{12} is not present because by the Newstead relation $\beta^g = \beta^3 = 0$ holds (Hausel and Thaddeus 2003). In particular there is no pure part in the middle = 12-dimensional cohomology. The same argument holds for all g, which shows that there is no pure part in the middle-dimensional cohomology of $\mathcal{M}_B^1(PGL_2)$. It is however easy to see that the intersection form on middle cohomology can only be non-trivial on the pure part and so this implies Hausel and Rodriguez-Villegas (2008, Corollary 5.4.1):

Corollary 16.1 *The intersection form on $H^*_{cpt}\left(\mathcal{M}_B^1(PGL_2)\right)$ is trivial.*

This gives an alternative proof of (16.5) as the equation

$$\chi_{L^2}\left(\mathcal{M}_B^1(SL_2)\right) = \chi_{L^2}\left(\mathcal{M}_B^1(PGL_2)\right)$$

is easy to prove. Moreover this approach is more promising to generalize for any n. We will offer a conjecture about the pure part of the cohomology of $\mathcal{M}_B^d(PGL_n)$ below and in turn that will yield a conjecture for the intersection form on the middle-dimensional compactly supported cohomology, answering the topological side of Problem 16.1.

To state our conjecture in its full generality we introduce character varieties on Riemann surfaces with k punctures and parabolic type $\mu = (\mu^1, \ldots, \mu^k)$ at the punctures, where μ^i is a partition of n. In other words we fix semisimple conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_k \subset GL_n$, which are generic and have type μ (in other words μ^i_j is the multiplicity of the jth eigenvalue of a matrix in $\mathcal{C}_i$). One can prove as in Hausel *et al.* (2008, lemma 2.1.2) that there exists generic semisimple conjugacy classes for every type $\mu = (\mu^1, \ldots, \mu^k)$. For a generic $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ of type μ we define

$$\mathcal{M}_B^\mu := \{A_1, B_1, \ldots, A_g, B_g \in GL_n, C_1 \in \mathcal{C}_1, \ldots, C_k \in \mathcal{C}_k | \\ [A_1, B_1] \cdots [A_g, B_g] C_1 \cdots C_k = I_n\} // GL_n$$

as an affine GIT quotient by the diagonal adjoint action of GL_n. The generic choice of the semisimple conjugacy classes implies that $\mathcal{M}_B^\mu$ is smooth. The torus GL_1^{2g} acts on $\mathcal{M}_B^\mu$ by multiplying the matrices A_i and B_i by a scalar. We can define the quotient

$$\bar{\mathcal{M}}_B^\mu := \mathcal{M}_B^\mu // GL_1^{2g}$$

as the corresponding PGL_n character variety. The variety $\bar{\mathcal{M}}_B^\mu$ is an orbifold.

By studying the Riemann–Hilbert map on the level of cohomologies we are led (Hausel, in preparation) to consider the comet-shaped quiver Γ associated to g and μ. Namely, we can put g loops on a central vertex, and k legs of length $l(\mu^j)$. We also equip Γ with a dimension vector $\mathbf{v}$, which has dimension $\sum_{i=1}^{l} \mu^j_i$ at the lth vertex on the ith leg. Consider now the number $A_\Gamma(q, \mathbf{v})$ of absolutely indecomposable representations of Γ of dimension $\mathbf{v}$ over the finite field $\mathbb{F}_q$. Kac (1983, proposition 1.15) proved that $A_\Gamma(q, \mathbf{v})$ is a polynomial in q with integer coefficients. We have the following conjecture from Hausel (in preparation):

Conjecture 16.5 *The pure part of the cohomology of $\bar{\mathcal{M}}_B^\mu$ is given by*

$$PH\left(\bar{\mathcal{M}}_B^\mu, x, y\right) = (xy)^{d_\mu/2} A_\Gamma(\mathbf{v}, 1/(xy)),$$

where $(\Gamma, \mathbf{v})$ is the star-shaped quiver and dimension vector given by the parabolic type μ, and d_μ is the dimension of $\mathcal{M}_B^\mu$.

This conjecture gives a cohomological interpretation of $A_\Gamma(\mathbf{v}, q)$ and in particular implies that it has non-negative coefficients confirming (Kac 1983, conjecture 2) in the case when Γ is comet-shaped. When μ is indivisible Conjecture 16.5 can be proved to follow from the master conjecture in Hausel *et al.* (in preparation), which expresses the mixed Hodge polynomials of all the character varieties $\bar{\mathcal{M}}^{\mu}_{B}$ as a generating function generalizing the Cauchy formula for Macdonald polynomials. It also has the following consequence on the topological L^2 cohomology $\chi_{L^2}\left(\bar{\mathcal{M}}^{\mu}_{B}\right)$ of (16.4).

Conjecture 16.6 *The topological L^2 cohomology of the manifold $\bar{\mathcal{M}}^{\mu}_{B}$ is given by*

$$\chi_{L^2}(\bar{\mathcal{M}}^{\mu}_{B}) = 0, \ \textit{when } g > 1 \tag{16.13}$$

$$\chi_{L^2}(\bar{\mathcal{M}}^{\mu}_{B}) = 1, \ \textit{when } g = 1 \tag{16.14}$$

$$\chi_{L^2}(\bar{\mathcal{M}}^{\mu}_{B}) = m_{\mathbf{v}}, \ \textit{when } g = 0, \tag{16.15}$$

where $m_{\mathbf{v}}$ is the multiplicity of the weight $\mathbf{v}$ in the Kac–Moody algebra $\mathfrak{g}(\Gamma)$, which are encoded by the Kac denominator formula for the star-shaped quiver Γ.

When $g > 1$ and the parabolic type is $\mu = ((n))$, that is, we have only one puncture with central conjugacy class, then one can identify $\bar{\mathcal{M}}^{\mu}_{B} = \mathcal{M}^{d}_{B}(PGL_n)$, with some d such that $(d, n) = 1$. In this case (16.13) says that

$$\chi_{L^2}\left(\mathcal{M}^{d}_{B}(PGL_n)\right) = 0,$$

which appeared as (Hausel and Rodriguez-Villegas Conjecture 4.5.1). It follows from the mirror symmetry Conjecture 16.3 that

$$H^{mid}_{cpt}\left(\mathcal{M}^{d}_{B}(SL_n)\right) \cong H^{mid}_{cpt}\left(\mathcal{M}^{d}_{B}(PGL_n)\right)$$

and then the intersection forms also agree. This and (16.15) then imply that (16.5) holds for any n, that is, that the intersection form on the compactly supported cohomology of $\mathcal{M}^{d}_{B}(SL_n)$ is trivial. This gives a conjectural answer to the topological side of Problem 16.1.

When $g = 1$ the conjectured (16.14) follows from Conjecture 16.5 and the observation that the coefficient of q in the A-polynomial $A_{\Gamma}(q)$ for a $g = 1$ comet-shaped quiver Γ is always 1.

When $g = 0$ the varieties $\mathcal{M}^{\mu}_{B} = \bar{\mathcal{M}}^{\mu}_{B}$ coincide. Conjecture 16.5 then implies that

$$\chi_{L^2}\left(\mathcal{M}^{\mu}_{B}\right) = A_{\Gamma}(\mathbf{v}, 0).$$

Conjecture (16.15) is a combination of this and the equality $A_{\Gamma}(\mathbf{v}, 0) = m_{\mathbf{v}}$, that is, Kac's conjecture 1, in Kac (1983), which, as discussed in Remark 16.2, follows from Theorem 16.3.

Finally one can define $\bar{\mathcal{M}}^{\mu}_{Dol}$ the moduli space of stable parabolic PGL_n-Higgs bundles with quasi-parabolic type $\mu^j \in \mathcal{P}(n)$ and generic weights at the jth puncture on the Riemann surface (Boden and Yokogawa 1996, García-Prada *et al.* 2007). Then one can prove that $\bar{\mathcal{M}}^{\mu}_{B}$ is diffeomorphic to $\bar{\mathcal{M}}^{\mu}_{Dol}$. Thus Conjecture 16.6 also calculates the intersection form on the compactly supported cohomology of the moduli space $\bar{\mathcal{M}}^{\mu}_{Dol}$ of stable parabolic PGL_n-Higgs bundles of any rank.

Example 16.1 Consider the genus 0 Riemann surface $\mathbb{P}^1$ with four punctures. Consider the moduli space $\mathcal{M}_{toy}$ of stable rank 2 parabolic Higgs bundles on $\mathbb{P}^1$, with generic parabolic weights on the full parabolic flag at the punctures (see Boden and Yokogawa 1996). This is a complex surface and the intersection form on $H^2_c(\mathcal{M}_{toy})$ was discussed in Hausel (1998, example 2 for theorem 7.13). $H^2_c(\mathcal{M}_{toy})$ is five-dimensional but $\chi_{L^2}(\mathcal{M}_{toy})$ is only four. (The cohomology class of the generic fiber of the Hitchin map is the one in the kernel.)

$\mathcal{M}_{toy}$ is diffeomorphic to the character variety $\bar{\mathcal{M}}^{\mu}_{B}$ where $g = 0$ and $\mu = ((1,1),(1,1),(1,1),(1,1))$. Thus by Conjecture 16.6 we should be able to calculate $\chi_{L^2}\left(\bar{\mathcal{M}}^{\mu}_{B}\right)$ in terms of the representation theory of the corresponding quiver Γ. The corresponding quiver Γ in this case will be the affine $\tilde{D}_4$ Dynkin diagram, with $\mathbf{v} = (2,1,1,1,1)$ the minimal positive imaginary root. Its multiplicity $m_{\mathbf{v}}$ in the affine Kac–Moody algebra associated to Γ is known to be 4. Alternatively it is known (Kac 1983 example b to conjecture 2) that $A_{\Gamma}(\mathbf{v}, q) = q + 4$, which by (16.11) gives $m_{\mathbf{v}} = 4$. Thus indeed $\chi_{L^2}\left(\mathcal{M}^{\mu}_{B}\right) = m_{\mathbf{v}} = 4$ checking (16.15) in this case via Hausel (1998, Example 2 for Theorem 7.13).

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