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Quaternionic Geometry of Matroids

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Abstract: Building on a recent paper [8], here we argue that the combinatorics of matroids are intimately related to the geometry and topology of toric hyperkähler varieties. We show that just like toric varieties occupy a central role in Stanley's proof for the necessity of McMullen's conjecture (or g-inequalities) about the classification of face vectors of simplicial polytopes, the topology of toric hyperkähler varieties leads to new restrictions on face vectors of matroid complexes. Namely in this paper we will give two proofs that the injectivity part of the Hard Lefschetz theorem survives for toric hyperkähler varieties. We explain how this implies the g-inequalities for rationally representable matroids. We show how the geometrical intuition in the first proof, coupled with results of Chari [3], leads to a proof of the g-inequalities for general matroid complexes, which is a recent result of Swartz [20]. The geometrical idea in the second proof will show that a pure O-sequence should satisfy the g-inequalities, thus showing that our result is in fact a consequence of a long-standing conjecture of Stanley.

Keywords: Face vectors, matroids, toric varieties, hyperkähler manifolds, Hard Lefschetz Theorem
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1 Introduction

McMullen [14] conjectured[†] in 1971 that the face vector[‡] $(f_0, \dots, f_{k-1})$ of a k-dimensional simplicial polytope $P \subset \mathbb{R}^k$ should satisfy, the following *g-inequalities*:

$$\begin{aligned}
&g_i \geq 0, \text{for } 1 \leq i \leq \lfloor \tfrac{k}{2} \rfloor, \\
&\text{and, if one writes} \\
&g_i = \tbinom{n_i}{i} + \tbinom{n_{i-1}}{i-1} + \cdots + \tbinom{n_r}{r}, \\
&\text{with } n_i > n_{i-1} > \cdots > n_r \geq r \geq 1, \text{ then} \\
&g_{i+1} \leq \tbinom{n_i+1}{i+1} + \tbinom{n_{i-1}+1}{i} + \cdots + \tbinom{n_r+1}{r+1} \\
&\ \text{for } 1 \leq i < \lfloor \tfrac{k}{2} \rfloor,
\end{aligned} \tag{1}$$

where

$$g_i = h_i - h_{i-1}$$

and

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{k-j}{i-j} f_{j-1}. \tag{2}$$

Stanley [17] in 1980 proved this conjecture using toric varieties. In a nutshell the proof goes as follows. First one perturbs the vertices of P a little bit so that P becomes a rational polytope. Because P is simplicial this does not change the face vector of P. The next step is to take the corresponding k-dimensional toric orbifold $X(\Delta_P)$, where Δ_P is the fan of cones over the faces of P. It is a well-known fact (see e.g. [6]) that the ith h-number $h_i = b_{2i}(X(\Delta_P))$ agrees with the $2i$th Betti number of $X(\Delta_P)$. Now $X(\Delta_P)$ has an ample class $\omega \in H^2(X(\Delta_P), \mathbb{C})$, which induces a map

$$L : H^*(X(\Delta_P), \mathbb{C}) \to H^*(X(\Delta_P), \mathbb{C}),$$

by multiplication with ω. Using the injectivity part of the Hard Lefschetz theorem (see e.g. [4]), which implies that L is an injection below degree k, we get that the degree $2i$th part of the graded algebra $H^*(X(\Delta_P), \mathbb{C})/(\mathrm{im}(L)))$ has dimension

$$\dim(H^{2i}(X(\Delta_P), \mathbb{C})/(\mathrm{im}(L))) = h_i - h_{i-1} = g_i \tag{3}$$

for $2i < k$. Since $H^*(X(\Delta_P), \mathbb{C})$ is generated by $H^2(X(\Delta_P), \mathbb{C})$ we also get that the algebra $H^*(X(\Delta_P), \mathbb{C})/(\mathrm{im}(L))$ is generated in degree 2. Now, using (3), a well-known theorem of Macaulay (see e.g. [19, Theorem II.2.3]) proves the g-inequalities (1). See [6] or [19] for more details.

[†] He, in fact, conjectured a complete characterization, the sufficiency part of which was proven by an ingenious construction of Billera and Lee in [2].

[‡] f_i is the number of i-dimensional faces.

Our starting point is the observation [8, Corollary 1.2] that the h-vectors of a rationally representable matroid $\mathcal{M}_{\mathcal{B}}$ agree $h_i(\mathcal{M}_{\mathcal{B}}) = b_{2i}(Y(A,\theta))$ with the Betti numbers of a toric hyperkähler variety $Y(A,\theta)$, for a generic choice of θ, where the toric hyperkähler variety can be considered as a quaternionic analogue of a toric variety. Therefore any restriction on the cohomology of a toric hyperkähler variety will yield restrictions on the face vectors of rationally representable matroid complexes and vice versa any known restriction on the face vectors of (rationally representable) matroids yields cohomological restrictions on toric hyperkähler varieties. This two-way relationship between these two seemingly unrelated subjects, hyperkähler geometry on one hand and combinatorics of matroids on the other, is what we call the "Quaternionic geometry of matroids". A relationship of this flavor is exploited in a recent paper by Swartz and the author [9]. There the combinatorics of affine hyperplane arrangements yields the existence of many L^2 harmonic forms on the corresponding toric hyperkähler manifold, in harmony with conjectures by physicists in string theory. For details see [9].

In the present paper our purpose is to use intuition arising from the study of the geometry of toric hyperkähler varieties to prove results in the combinatorics of matroids. We will proceed as follows: In Section 2 and Section 3 we recall some basic notations and results from [19] and from [8]. Then we go on and in Section 4 give two different proofs for the injectivity part of the Hard Lefschetz theorem for toric hyperkähler varieties. The second one is basically taken from [19, Theorem 7.4], while the first proof could be easily generalized for other similar hyperkähler manifolds, such as Nakajima's quiver varieties [16] or Hitchin's moduli of Higgs bundles[§] [11]. In Section 5 we explain how the geometric idea in the first proof can be generalized to any matroid complexes, a result recently proven by Swartz in [20]. We show that the geometrical structure needed for the first proof for general matroids is provided by Chari's decomposition theorem [3]. In fact this proof is similar to Swartz's original proof in [20]. We conclude our paper by showing that the geometric structure which yielded the second proof of the injective Hard Lefschetz theorem is present for pure O-sequences. This way we find that the g-inequalities we proved in the previous section are in fact a consequence of a long standing conjecture of Stanley [18]. This last result is a strengthening of a result of Hibi in [10].

2 Simplicial and matroid complexes

We collect here some basic definitions and results on simplicial complexes and in particular matroid complexes from [19].

A simplicial complex Σ on a finite set $V = \{1, \ldots, n\}$ is a set of subsets of V, i.e. $\Sigma \subset 2^V$, such that $\{x\} \in \Sigma$ for any $x \in V$ and $F \in \Sigma$ and $F' \subset F$ implies $F' \in \Sigma$. We call $F \in \Sigma$ a *face* of Σ, the *dimension* of the face is one less than its size. The dimension of Σ is then the maximum dimension of its faces, while its *rank* is 1 more. A *facet* is a

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[§] A recent paper of the author [7] conjectures a strong version of the Hard Lefschetz theorem for the moduli space of Higgs bundles, generalizing the one in this paper; and also relates it to the Alvis-Curtis duality in the representation theory of finite groups of Lie-type.

face of maximal dimension. A simplicial complex is called *pure* if its maximal faces are all facets. The *f-vector* of a rank-k simplicial complex is $(f_0, f_1, \dots, f_{k-1})$, where f_i is the number of i-dimensional faces in Σ. The *h-vector* of the simplicial complex is $(h_0, \dots, h_k)$ given by (2).

Define the *Stanley-Reisner ring* of a rank-k simplicial complex Σ as the graded ring given by:

$$\mathbb{C}[\Sigma] = \mathbb{C}[x_1, \dots, x_n]/\langle x_F = \prod_{i \in F} x_i | F \notin \Sigma \rangle.$$

All our simplicial complexes in this paper will be *Cohen-Macaulay*, which will imply that we will always have a *linear system of parameters* or *l.s.o.p* for short, which is a sequence $(\theta) = (\theta_1, \dots, \theta_k)$ of linear combinations of the x_i, such that the graded ring

$$\mathbb{C}[\Sigma]/(\theta) := \mathbb{C}[\Sigma]/(\theta_1 \mathbb{C}[\Sigma] + \dots + \theta_k \mathbb{C}[\Sigma])$$

is finite dimensional as a vector space over $\mathbb{C}$ and that the h-numbers $h_i(\Sigma) = (\mathbb{C}[\Sigma]/(\theta))_i$ agree with the dimension of the corresponding graded piece of $\mathbb{C}[\Sigma]/(\theta)$.

We will use the following operation on simplicial complexes in Section 5. Given two simplicial complexes Σ with vertex set V and Θ with vertex set U we define their *poset-theoretic* product (or *join*) $\Sigma \times \Theta$ as a simplicial complex with vertex set $U \cup V$ and all faces of the form $F \cup F'$ where $F \in \Sigma$ and $F' \in \Theta$. The poset-theoretic product has the advantage that it behaves nicely after taking the corresponding Stanley-Reisner rings: $\mathbb{C}[\Sigma \times \Theta] \cong \mathbb{C}[\Sigma] \otimes \mathbb{C}[\Theta]$.

For examples of (Cohen-Macaulay) simplicial complexes we mention the boundary complex of a simplicial convex polytope, which was mentioned in the introduction. Another class for interest for us are *matroid complexes* or simply just *matroids*. A matroid complex $\mathcal{M}$ is a simplicial complex on a vertex set V such that for every $W \subset V$ the induced subcomplex $\mathcal{M}_W = \{F \in \mathcal{M} : F \subset W\}$ is pure. The *rank* of the matroid is 1 more than its dimension. A vertex $i \in V$ is a coloop of $\mathcal{M}$ if $\mathcal{M}_{V \setminus i}$ has rank smaller than the rank of $\mathcal{M}$.

The motivating example of a matroid complex $\mathcal{M}_{\mathcal{B}}$ on vertex set $V = \{1, \dots, n\}$ is obtained from a vector configuration $\mathcal{B} = (b_1, \dots, b_n) \in \mathbb{K}^k$ in a k-dimensional vector space over a field $\mathbb{K}$, defined by $F \in \mathcal{M}$ iff $\{b_i\}_{i \in F}$ is linearly independent. Such a matroid is called *representable* over $\mathbb{K}$. For example, if $\mathbb{K} = \mathbb{Q}$ then we call the matroid $\mathcal{M}$ rationally representable.

For more details on these definitions consult [19], the poset-theoretic product was used in [3].

3 Toric hyperkähler varieties

Here we collect notation and terminology from [8] which we will need in the present paper. For more details see [8].

Let $A = [a_1, \dots, a_n]$ be a $d \times n$-integer matrix whose $d \times d$-minors are relatively prime. We choose an $n \times (n-d)$-matrix $B = [b_1, \dots, b_n]^T$ which makes the following sequence exact:

$$0 \longrightarrow \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \longrightarrow 0.$$

Taking $\theta \in \mathbb{N}\mathcal{A}$, where $\mathcal{A} := \{a_1, \ldots, a_n\}$ is a vector configuration in $\mathbb{Z}^d$, [8] constructs a quasi-projective variety $Y(A,\theta)$ (which sometimes we abbreviate as Y), called a *toric hyperkähler variety.* (This construction is an algebraic geometric version of the original construction of Bielawski and Dancer in [1].) By [19, Proposition 6.2] if $\theta \in \mathbb{N}\mathcal{A}$ is generic $Y(A,\theta)$ is an orbifold, while if, in addition, A is unimodular then $Y(A,\theta)$ is a smooth variety.

The topology of $Y(A,\theta)$ is governed by an affine hyperplane arrangement denoted by $\mathcal{H}(B,\psi)$ of n planes in $\mathbb{R}^{n-d}$. For example a key result in [19, Corollary 6.6] claims that the h-numbers of the matroid of the vector configuration $\mathcal{B} = \{b_1, \ldots, b_n\}$ agree with the Betti numbers of Y:

$$h_i(\mathcal{M}_{\mathcal{B}}) = b_{2i}(Y(A,\theta)).$$

In the next section we will make use of a projective subvariety $C(A,\theta)$ of $Y(A,\theta)$, which is called the *core* of $Y(A,\theta)$. It is a reducible variety whose components are projective toric varieties, corresponding to top dimensional bounded regions in $\mathcal{H}(B,\psi)$. If the matroid of $\mathcal{B}$ is coloop-free than the core is a middle and pure dimensional projective subvariety of $Y(A,\theta)$.

Finally we need to mention a result from [5]. They construct and study a certain residual $U(1)$-action on $Y(A,\theta)$, which comes from an algebraic $\mathbb{C}^\times$-action. It follows from their results that, when $\mathcal{B}$ is coloop-free, one can always choose such a circle action, which makes $Y(A,\theta)$, a *hyper-compact* hyperkähler manifold. It means that the $U(1)$-action is Hamiltonian with proper moment map with a minimum, and also that the holomorphic symplectic form $\omega_{\mathbb{C}}$ is of homogeneity 1, meaning that for $\lambda \in \mathbb{C}^\times$

$$\lambda^* \omega = \lambda \omega. \tag{4}$$

For further results about the topology and geometry of toric hyperkähler varieties consult the papers [1], [5], [8], [9] and [12].

4 Injective Hard Lefschetz for hyperkähler manifolds

We are now ready to give two proofs of the following

Theorem 4.1. For a smooth toric hyperkähler variety $Y(A,\theta)$ of real dimension $4n-4d = 4k$, such that $\mathcal{B}$ is coloop-free, we have that

$$\begin{gathered} L^{k-2i} : H^{2i}(Y,\mathbb{C}) \to H^{2k-2i}(Y,\mathbb{C}) \\ L^{k-2i}(\alpha) = \alpha \wedge \omega^{k-2i} \end{gathered} \tag{5}$$

is injective if $2i < k$, where $\omega = [\omega_I]$ is the cohomology class of the Kähler form corresponding to the complex structure I.

Just like in Stanley's proof of the McMullen conjecture, we also have the following numerical consequences:

Corollary 4.2. The h-vector $(h_1(\mathcal{M}), \dots, h_k(\mathcal{M}))$ of a coloop-free and rank k matroid $\mathcal{M}$, which is (unimodularly and) rationally representable, satisfies

$$h_i(\mathcal{M}) \leq h_j(\mathcal{M}), \tag{6}$$

for $i \leq j \leq k-i$ and the g-inequalities (1).

Proof of Corollary. Let the (unimodular) vector configuration $\mathcal{B} = \{b_1, \dots, b_n\} \in \mathbb{Z}^k \subset \mathbb{Q}^k$ represent the matroid $\mathcal{M}$. Choosing a Gale dual configuration $\mathcal{A} = (a_1, \dots, a_n) \in \mathbb{Z}^d$ and a generic $\theta \in \mathbb{N}\mathcal{A}$, we can construct a smooth toric hyperkähler variety $Y(A, \theta)$, whose Betti numbers agree with the h-numbers of $\mathcal{M}$. Now Theorem 4.1 immediately implies (6). From Theorem 4.1 we can also deduce (1) exactly as in Stanley's argument for simplicial convex polytopes. See the introduction or for more details [19, Theorem III.1.1].

Proof 1 of Theorem 4.1. As explained above we have a $\mathbb{C}^\times$-action on $Y := Y(A, \theta)$, for which the corresponding $U(1) \subset \mathbb{C}^\times$-action is hyper-compact. Recall that this means that it is Hamiltonian with a proper moment $\mu_{\mathbb{R}} : Y \to \mathbb{R}$ map with respect to ω, and for which the holomorphic symplectic form $\omega_{\mathbb{C}}$ is of homogeneity 1 meaning (4). Suppose that the fixed point set of the circle action has f components, which are denoted by $F_1, \dots, F_f$. The numbering is such that $\mu_{\mathbb{R}}(F_m) > \mu_{\mathbb{R}}(F_l)$ implies $m > l$. Now we define the Bialynicki-Birula stratification of Y with respect to our $\mathbb{C}^\times$-action. Namely define $U_m = \{p \in Y | \lim_{\lambda\to 0} \lambda p \in F_m\}$, which is an affine bundle over F_m. Moreover we let $U_{\leq m} = \cup_{j\leq m} U_j$ and $U_{<m} = \cup_{j<m} U_j$, which are open subvarieties of Y. Because the moment map $\mu_{\mathbb{R}}$ is proper it follows that $U_{\leq f} = Y$, i.e. that we get this way a stratification of Y. Finally we denote by N_m the negative normal bundle of F_m. Because the holomorphic symplectic form is of homogeneity 1 with respect to our $\mathbb{C}^\times$-action, it follows (cf. [15, Proposition 7.1]) that

$$\mathrm{rank}_{\mathbb{C}}(N_m) + \dim_{\mathbb{C}}(F_m) = \frac{1}{2}\dim_{\mathbb{C}} Y = k. \tag{7}$$

By induction on m we prove that the map L^{k-2i} in (5), when restricted to $U_{\leq m}$, is injective for $2i < k$. For $m = 1$ the statement is clear because by (4) $U_1 = T^*F_1$ thus $\dim_{\mathbb{C}}(F_1) = k$ and the statement follows from the traditional Hard Lefschetz theorem for the compact Kähler manifold F_1. Now suppose we have the required injectivity of the map L^{k-2i} on $U_{<m}$. Then consider the decomposition $U_{\leq m} = U_{<m} \cup U_m$. ¿From this decomposition, using the Thom isomorphism

$$H^{2i}(U_{\leq m}, U_{<m}; \mathbb{C}) \cong H^{2i-2n_m}(U_m, \mathbb{C}), \tag{8}$$

we get the cohomology exact sequence:

$$0 \to H^{2i-2n_m}(U_m, \mathbb{C}) \xrightarrow{\tau} H^{2i}(U_{\leq m}, \mathbb{C}) \xrightarrow{r} H^{2i}(U_{<m}, \mathbb{C}) \to 0,$$

where $n_m = \mathrm{rank}_{\mathbb{C}}(N_m)$, τ is the Gysin map and r is the natural restriction map on cohomology. Now suppose $2i < k$ and $0 \neq \alpha \in H^{2i}(U_{\leq m}, \mathbb{C})$. If $r(\alpha) \neq 0$, then by induction we can deduce that $L^{k-2i}(\alpha) \neq 0$. If $r(\alpha) = 0$, then there is a $\beta \in H^{2i-2n_m}(U_m, \mathbb{C})$ such that $\tau(\beta) = \alpha$. However, U_m is homotopy equivalent with the smooth compact Kähler manifold F_m and $\omega|_{F_m}$ is a Kähler class. If we denote $f_m = \dim_{\mathbb{C}} F_m$, then the Hard Lefschetz theorem for F_m yields that $0 \neq \beta \wedge \omega^{f_m - 2(i-n_m)} = \beta \wedge \omega^{k-2i+n_m}|_{F_m}$, because $f_m + n_m = k$ by (7). Since, τ is injective we get that $\tau(\beta \wedge \omega^{k-2i}|_{F_m}) = \alpha \wedge \omega^{k-2i}|_{U_{\leq m}} \neq 0$.

The result follows.

Because we only used the hyper-compactness of the toric hyperkähler variety, the same proof also yields the following

Corollary 4.3. For a hyper-compact hyperkähler manifold M (such as toric hyperkähler varieties, Nakajima's quiver varieties [16] or moduli spaces of Higgs bundles [11]) we have that

$$L^{k-2i} : H^{2i}(M, \mathbb{C}) \to H^{2k-2i}(M, \mathbb{C})$$

$$L^{k-2i}(\alpha) = \alpha \wedge \omega^{k-2i}$$

is injective if $2i < k$, where $\omega = [\omega_I]$ is the class of the Kähler form corresponding to the complex structure I.

Remark 4.4. In a recent work [7] the author explains a conjecture for a strong version of the Hard Lefschetz theorem for the moduli space of Higgs bundles, which is a theorem for rank 2 Higgs bundles. This completely unexpected conjecture is a generalization of the corollary above and has some intriguing relationship with the representation theory of finite groups of Lie type.

We now recall our original proof of Theorem 4.1 from [8, Theorem 7.4] in the smooth case because we will use the idea in the final section.

Proof 2 of Theorem 4.1. Let $X_1, \ldots X_r$ denote the irreducible components of the core of Y. Let $\phi_i : H^*(Y, \mathbb{C}) \to H^*(X_i, \mathbb{C})$ denote the natural restrictions. The heart of the proof of [8, Theorem 7.4] is that

$$(\phi_1) \cap (\phi_2) \cap \ldots \cap \ker(\phi_r) \quad = \quad \{0\}. \tag{9}$$

In [8] we presented two proofs of this fact. One [8, Proposition 3.4] was a more general result for semi-projective toric orbifolds and the proof goes similarly to our first Proof 1 of Theorem 4.1 above, i.e. uses Morse theory type considerations with induction. It turns out that [8, Proposition 3.4] is equivalent with the fact that the bounded complex of the polytope (or in our case the bounded complex of the affine hyperplane arrangement $\mathcal{H}(B, \psi)$) is always contractible. The second proof was given after equation (34) of [8], which showed that (9) is in fact equivalent with Stanley's result [19, Proposition III.3.2] that the Stanley-Reisner ring of a matroid is level.

Now we proceed as follows. For $2i < k$ take $\alpha \in H^{2i}(Y, \mathbb{C})$. Then, because of (9), we have a j so that $\phi_j(\alpha) \in H^{2i}(X_j, \mathbb{C})$ is nonzero. But the traditional hard Lefschetz theorem for the smooth compact Kähler manifold X_j implies that $\phi_j(\alpha \wedge \omega^{k-2i}) \neq 0$.

The result follows.

Remark 4.5. [8, Theorem 7.4] proves the same result, in the way sketched above, for a rationally representable matroid, i.e. for toric hyperkähler orbifolds, not just for smooth toric hyperkähler varieties. Here we restricted our attention to the smooth case, because the other Proof 1 only works in this case. The reason is that (7) could be false in the orbifold case.

2. Proof 1 works for any hyper-compact hyperkähler manifold, however an extension of Proof 2 in the general case is not immediate. Indeed, the equivalent of (9) perhaps in intersection cohomology is not known for a general hyper-compact hyperkähler manifold.

3. Another consequence of (9), explained in [8, Section 7], is that one can present the cohomology ring of Y in terms of cogenerator polynomials corresponding to the X_i, the components of the core. Indeed this algebraic presentation is rather similar to a presentation of a pure O-sequence, the only difference will be that we replace the cogenerator polynomials by monomials. This similarity will lead to the proof of Theorem 6.3 below.

5 Proof of the g-inequalities for matroid complexes

In this section we will use the geometrical idea from our first proof of Theorem 4.1 to prove the following generalization:

Theorem 5.1. The h-vector $(h_1(\mathcal{M}), \ldots, h_k(\mathcal{M}))$ of a coloop-free and rank k matroid $\mathcal{M}$ satisfies (6) and the g-inequalities (1).

Remark 5.2. This was first proven by Swartz [20], by using an algebraic version of Chari's [3] decomposition theorem of matroids. Here we will show, that [3] gives us the geometrical structure for a general matroid so that we can repeat our Morse theory type first proof of Theorem 4.1. In fact this proof is similar to Swartz's original proof.

Proof 5.3. So let us first recall Chari's result [3, Theorem 3]:

Theorem 5.4 (Chari). A coloop-free matroid complex has a PS-ear decomposition.

A *PS-ear decomposition* of a pure rank-k simplicial complex Σ on a vertex set $\{1, \ldots, n\}$ is a covering by pure rank-k simplicial subcomplexes $\Sigma = \cup_{i=1}^{m} \Sigma_i$, such that

- Σ_1 is the poset-theoretic product of boundaries of simplices (a PS-k-sphere in the terminology of [3]), while for each $i = 2, \ldots, m$, Σ_i is the poset-theoretic product of a simplex and a PS-sphere (called a PS-ball in [3]), and

- For $i \geq 2$, $\Sigma_i \cap \left(\cup_{j=1}^{i-1}\Sigma_j\right) = \partial\Sigma_i$, where $\partial\Sigma_i$ denotes the pure rank-$(k-1)$ simplicial complex (which is just a PS-sphere in this case) whose facets are the rank-$(k-1)$ faces of Σ_i that are contained in only one facet of Σ_i.

We will show that Theorem 5.1 holds for simplicial complexes having a PS-ear decomposition, a result which was also mentioned by Swartz in [20]. We will see that this PS-ear decomposition is in fact a very good combinatorial analogue of the Morse stratification of Y (or rather its Lagrangian core) used in Proof 1 of Theorem 4.1.

We first make a

Definition 5.5. Let R be a ring and M be a graded R-module. Then we say that M satisfies injective hard Lefschetz (IHL for short) around degree $k/2$ for $\omega \in R_1$ if the map

$$L^{k-2i} : M_i \to M_{k-i}$$
$$L^{k-2i}(\alpha) = \alpha\omega^{k-2i}$$

is injective for $0 < i \leq k/2$.

We will proceed by induction on m to show that

$$\begin{gathered}\text{there is an l.s.o.p } (\theta_1, \ldots, \theta_k) \text{ so that the graded ring } \mathbb{C}[\Sigma]/(\theta) \\ \text{satisfies IHL around } k/2 \text{ with } \omega = \textstyle\sum_i x_i.\end{gathered} \tag{10}$$

When $m = 1$, then Σ is just a poset-theoretic product of boundaries of simplices. Therefore $\mathbb{C}[\Sigma]$ can be thought of as the torus equivariant cohomology ring of a product of projective spaces, while an l.s.o.p. (θ) can be chosen so that $\mathbb{C}[\Sigma]/(\theta)$ is just the cohomology ring of the product of projective spaces. Then $\omega = \sum x_i$ is just a Kähler class, so the classical Hard Lefschetz theorem proves (10).

Now suppose we know our statement for $m-1$ and consider a pure rank-k simplicial complex with a PS-ear-decomposition. Let us denote $\Sigma_{<m} = \cup_{j=1}^{m-1}\Sigma_j$. Consider the natural surjective map $\mathbb{C}[\Sigma] \to \mathbb{C}[\Sigma_{<m}]$. We think of the kernel of this map as a graded $\mathbb{C}[x_1, \ldots, x_n]$-module and denote it by $\mathbb{C}[\Sigma, \Sigma_{<m}]$. So we have the following exact sequence of graded $\mathbb{C}[x_1, \ldots, x_n]$-modules:

$$0 \to \mathbb{C}[\Sigma, \Sigma_{<m}] \to \mathbb{C}[\Sigma] \to \mathbb{C}[\Sigma_{<m}] \to 0.$$

We now claim that we can find an l.s.o.p $(\theta) = (\theta_1, \ldots, \theta_k)$ for $\mathbb{C}[\Sigma]$ such that in both graded $\mathbb{C}[x_1, \ldots, x_n]$-modules $\mathbb{C}[\Sigma_{<m}]/(\theta)$ and $\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta)$ the IHL for ω is satisfied around degree $k/2$.

By induction we know that the set of (θ) which is an l.s.o.p. for $\mathbb{C}[\Sigma_{<m}]$ and $\mathbb{C}[\Sigma_{<m}]/(\theta)$ satisfies IHL for ω is non-empty and clearly Zariski open in $\mathbb{C}^{nk}$. Because the set of (θ) which is l.s.o.p. for $\mathbb{C}[\Sigma]$ is also non-empty and Zariski open, the intersection of these two sets will also be non-empty and Zariski open. In summary we see that the set of (θ) which is an l.s.o.p for $\mathbb{C}[\Sigma]$ and $\mathbb{C}[\Sigma_{<m}]/(\theta)$ satisfies IHL for ω is non-empty and Zariski open.

It is also clear that the set of (θ) which is an l.s.o.p for $\mathbb{C}[\Sigma]$ and $\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta)$ satisfies IHL around degree $k/2$ for ω is Zariski open. We now prove that it is in fact non-empty. Take the natural map $\mathbb{C}[\Sigma_m] \to \mathbb{C}[\partial\Sigma_m]$ and denote by $\mathbb{C}[\Sigma_m, \partial\Sigma_m]$ the kernel. We think of this kernel as an $\mathbb{C}[x_1, \ldots, x_n]$-module by letting the variables x_j which correspond to vertices not in Σ_m acting trivially. Then it is easy to see that $\mathbb{C}[\Sigma_m, \partial\Sigma_m]$ and $\mathbb{C}[\Sigma, \Sigma_{<m}]$ are isomorphic as graded $\mathbb{C}[x_1, \ldots, x_n]$-modules (this is the analogue of excision in cohomology). But $\Sigma_m = \Delta \times \Phi$ is a poset-theoretic product of a k-simplex Δ with a poset-theoretic product of boundary of simplices Φ. Now it is clear that

$$\mathbb{C}[\Sigma_m, \partial\Sigma_m] \cong \mathbb{C}[\Phi] \otimes \mathbb{C}[\Delta, \partial\Delta]$$

as graded $\mathbb{C}[x_1, \ldots, x_n]$-modules (this corresponds to the Thom isomorphism (8) in cohomology). If $x_1, \ldots, x_l$ correspond to the vertices of Δ then $\mathbb{C}[\Delta, \partial\Delta]$ is just a free $\mathbb{C}[x_1, \ldots, x_l]$-module generated by a degree l element $x_1 x_2 \ldots x_l$ (which is the analogue of the Thom class).

Recall that the set of $(\theta) = (\theta_1, \ldots, \theta_k) \in (\mathbb{C}[\Sigma])_1^k = \mathbb{C}^{nk}$ for which

$$\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta) := \mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta_1 \mathbb{C}[\Sigma, \Sigma_{<m}] + \cdots + \theta_k \mathbb{C}[\Sigma, \Sigma_{<m}])$$

satisfies IHL around degree $k/2$ for $\omega = \sum_{i=1}^n x_i$ is clearly Zariski open in $\mathbb{C}^{nk}$. Now we show that it is non-empty. Take $(\theta) = (x_1, \ldots, x_l, \theta_{l+1}, \ldots, \theta_k)$, so that $(\theta_{l+1}, \ldots, \theta_k)$ is an *l.s.o.p* for $\mathbb{C}[\Phi]$ and $\mathbb{C}[\Phi]/(\theta_{l+1}, \ldots, \theta_k)$ satisfies IHL around $(k-l)/2$ with $\omega = \sum x_i$.

For this choice we have

$$\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta) = x_1 x_2 \ldots x_l \mathbb{C}[\Phi]/(\theta_{l+1}, \ldots, \theta_k),$$

and so IHL for $\mathbb{C}[\Phi]/(\theta_{l+1}, \ldots, \theta_k)$ around degree $(k-l)/2$ implies IHL for $\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta)$ around degree $k/2$ with $\omega = \sum x_i$.

As the intersection of non-empty Zariski subsets of $\mathbb{C}^{nk}$ is non-empty we can choose a $(\theta) = (\theta_1, \ldots, \theta_k)$, which is an l.s.o.p for $\mathbb{C}[\Sigma]$ and $\mathbb{C}[\Sigma_{<m}]$ and for which both $\mathbb{C}[\Sigma_{<m}]/(\theta)$ and $\mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta)$ satisfies IHL around $k/2$ with $\omega = \sum x_i$. Now using the short exact sequence:

$$0 \to \mathbb{C}[\Sigma, \Sigma_{<m}]/(\theta) \to \mathbb{C}[\Sigma]/(\theta) \to \mathbb{C}[\Sigma_{<m}]/(\theta) \to 0,$$

we can repeat the argument of Proof 1 of Theorem 4.1, to get that $\mathbb{C}[\Sigma]/(\theta)$ satisfies IHL around $k/2$ with $\omega = \sum x_i$.

Because Σ has a PS-ear decomposition it is shellable (see [3, Proposition 5]), and so Cohen-Macaulay, we have that $h_i(\Sigma) = \dim_{\mathbb{C}}((\mathbb{C}[\Sigma]/(\theta))_i)$ and so Theorem 5.1 follows.

Remark 5.6. Because we have Hard Lefschetz theorem for boundary complexes of simplicial convex polytopes the above proof would have worked equally well for simplicial complexes with a decomposition just like PS-ear-decomposition above, but changing *PS*-spheres, in the definition, with boundary complexes of simplicial convex polytopes. For a unimodularly and rationally representable matroid such a presentation always arises naturally. Namely we can consider the Morse stratification (for details on this see [5]) of a hyper-compact $U(1)$-action on the bounded complex of a generic hyperplane arrangement which represents our given matroid. In this case the above combinatorial proof of Theorem 5.1 would essentially agree with Proof 1 of Theorem 4.1.

6 Proof of the g-inequalities for pure O-sequences

First a definition:

Definition 6.1. A sequence of non-negative integers $(h_1, h_2, \dots, h_k)$ is called a pure O-sequence, if $h_k > 0$ and there exists monomials $m_1, \dots, m_{h_k}$ of degree k in the degree one variables $x_1, \dots, x_{h_1}$, so that h_l is the number of monomials m of degree l in variables $x_1, \dots, x_{h_1}$, such that $m|m_i$ for some $0 < i \leq h_k$.

Now we can state a long standing conjecture of Stanley [18]:

Conjecture 6.2 (Stanley). The h-vector $(h_1(\mathcal{M}), \dots, h_k(\mathcal{M}))$ of a rank k matroid $\mathcal{M}$ is a pure O-sequence.

This conjecture is still open for general matroids, although recently it has been proved for cographic matroids using [13], i.e. for the Betti numbers of toric quiver varieties [8, Section 8]. Another attack on Stanley's conjecture has been to deduce numerical inequalities between the numbers in a pure O-sequence and then prove these inequalities for the h-vector of a matroid complex. As an example, Hibi [10] proved that for a pure O-sequence one has

$$h_i \leq h_j, \tag{11}$$

where $i \leq j \leq k - i$ and in particular that

$$h_1 \leq h_2 \leq \cdots \leq h_{\lfloor \frac{k}{2} \rfloor},$$

this was in turn proven for h-vectors of matroid complexes by Chari [3].

Here we strengthen this result by proving the following

Theorem 6.3. A pure O-sequence $(h_1, h_2, \dots, h_k)$ satisfies (11) and the g-inequalities.

Corollary 6.4. Theorem 5.1 is a consequence of Stanley's Conjecture 6.2.

Proof of Theorem 6.3. We are going to follow the structure of Proof 2 of Theorem 4.1. Namely take a pure O-sequence $(h_1, h_2, \dots, h_k)$ with generating monomials $m_1, \dots, m_{h_k}$ in variables $x_1, \dots, x_{h_1}$. First we construct a graded ring

$$R = \tfrac{\mathbb{C}[\partial_1, \dots, \partial_{h_1}]}{I}$$

$$I = \operatorname{ann}(m_1) \cap \cdots \cap \operatorname{ann}(m_{h_k})$$

which will be the analogue of the cohomology ring $H^*(Y,\mathbb{C})$ of a toric hyperkähler manifold. Here ∂_i is a variable of degree one, which we think of as a differential operator, satisfying $\partial_i(x_j)=\delta_{ij}$. The ideal in the denominator is the ideal I of polynomials in the ∂_i which annihilate all the monomials m_j. Clearly $\dim R_j = h_j$. Then we construct graded rings

$$R^j = \frac{\mathbb{C}[\partial_1,\ldots,\partial_{h_1}]}{I_j}$$

$$I_j = \operatorname{ann}(m_j)$$

for each monomial m_j, which will be the analogue of $H^*(X_j,\mathbb{C})$ (in fact it is useful to think about R^j as the cohomology ring of the product of projective spaces of dimension given by the exponents in the monomial m_j). Because $I\subset I_j$, we have a natural map $p_j : R\to R^j$. The equation $I=\cap_j I_j$ now implies the analogue of (4), i.e. that the map

$$p = p_1\times\cdots\times p_{h_k} : R\to R^1\times\cdots\times R^{h_k}$$

is injective. Now take the degree 1 class $\omega=\sum_j \partial_j$. It is clear that the map $L_j^{k-2i} : R_i^j \to R_{k-i}^j$ given by $L_j^{k-2i}(\alpha)=\alpha p_j(\omega^{k-2i})$ is injective for $2i<k$. Indeed, think of R^j as the cohomology ring of the product of projective spaces. Then $p_j(\omega)$ corresponds to the natural ample class, so the hard Lefschetz theorem implies injectivity of L_j^{k-2i}. Of course in this case one can check this result by hand for the explicitly defined rings R^j. The injectivity of p and of L_j^{k-2i} implies the injectivity of $L^{k-2i} : R_i \to R_{k-i}$, $L^{k-2i}(\alpha)=\alpha\omega^{k-2i}$ for $2i<k$. The result follows.

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