Mirror symmetry and Langlands duality in the non-Abelian Hodge theory of a curve

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Summary. The paper surveys the mirror symmetry conjectures of Hausel–Thaddeus and Hausel–Rodriguez-Villegas concerning the equality of certain Hodge numbers of $SL(n,\mathbb{C})$ vs. $PGL(n,\mathbb{C})$ flat connections and character varieties for curves, respectively. Several new results and conjectures and their relations to works of Hitchin, Gothen, Garsia–Haiman and Earl–Kirwan are explained. These use the representation theory of finite groups of Lie-type via the arithmetic of character varieties and lead to an unexpected conjecture for a Hard Lefschetz theorem for their cohomology.

1 Introduction

Non-Abelian Hodge theory [29], [44] of a genus g smooth complex projective curve C studies three moduli spaces attached to C and a reductive complex algebraic group G, which in this paper will be either $GL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$ or $PGL(n,\mathbb{C})$. They are

- $\mathcal{M}^d_{Dol}(G)$, the moduli space of semistable G-Higgs bundles on C;
- $\mathcal{M}^d_{DR}(G)$, the moduli space of flat G-connections on C and
- $\mathcal{M}^d_{B}(G)$ the character variety, i.e., the moduli space of representations of $\pi_1(C)$ into G modulo conjugation.

Under certain assumptions these moduli spaces are smooth varieties (or orbifolds when $G = PGL(n,\mathbb{C})$) with the underlying differentiable manifolds canonically identified and endowed with a natural hyperkähler metric.

The cohomology of this underlying manifold has been studied mostly from the perspective of $\mathcal{M}^d_{Dol}(G)$. Using a natural circle action on it [29] and [16] calculated the Poincaré polynomials for $G = SL(2,\mathbb{C})$ and $G = SL(3,\mathbb{C})$ respectively; while [25] and [39] found a simple set of generators for the cohomology ring for $G = PGL(2,\mathbb{C})$, respectively $G = PGL(n,\mathbb{C})$. The paper [26] then calculated the cohomology ring explicitly for $G = PGL(2,\mathbb{C})$. The techniques used in these papers do not seem to generalize easily to higher n.

A new point of view was introduced in [27] and [28]. It was shown that hyperkähler metrics and Hitchin systems [30] for $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_{DR}(G^L)$, with $G = SL(n,\mathbb{C})$ and Langlands dual $G^L = PGL(n,\mathbb{C})$, realize the geometrical setup for mirror symmetry proposed in [49]. Based on this observation, [28] conjectured the existence of a topological version of mirror symmetry, i.e., the equality of certain Hodge numbers of $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_{DR}(G^L)$. This was checked for $G = SL(2,\mathbb{C})$ and $SL(3,\mathbb{C})$ using [29] and [16].

This mirror symmetry conjecture suggests to study not only the cohomology of $\mathcal{M}^d_{DR}(G)$, $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_{B}(G)$ but also its mixed Hodge structure. It was shown in [28] that the mixed Hodge structures of $\mathcal{M}^d_{Dol}(G)$ and $\mathcal{M}^d_{DR}(G)$ agree, and are pure (see Theorem 2.1 or [38]). However, the mixed Hodge structure of the character variety $\mathcal{M}^d_{B}(G)$ has not been investigated until recently.

In this paper we study this mixed Hodge structure, more precisely,

- the *mixed Hodge polynomial* $H(x, y, t)$;
- the E-polynomial

$$E(x, y) := x^n y^n H(1/x, 1/y, -1),$$

  where $n = \dim_{\mathbb{C}} \mathcal{M}^d_B(G)$ and
- the Poincaré polynomial $H(1, 1, t)$.

Here the H-polynomial encodes the dimensions of the graded pieces of the mixed Hodge structure on $\mathcal{M}^d_B(G)$ (see Section 2.2).

In [23] an arithmetic method was used to calculate the E-polynomial of $\mathcal{M}^d_B(G)$. The idea was to count the $\mathbb{F}_q$-rational points of $\mathcal{M}^d_B(G(\mathbb{F}_q))$, for the variety $\mathcal{M}^d_B(G)$ over the finite field $\mathbb{F}_q$. Using a result of [37] we found a closed formula, resembling the famous Verlinde formula [51], as a simple sum over irreducible representations of $G(\mathbb{F}_q)$. In particular, the representation theory behind the E-polynomial of the character variety is that of *finite* groups of Lie type. This could be considered as an analog of Nakajima's principle [40], stating that the representation theory of a Kac–Moody algebra is encoded in the cohomology of (hyperkähler) quiver varieties.

The shape of the E-polynomials of the various character varieties lead us to conjecture [23] that mirror symmetry also holds for the pair $\mathcal{M}^d_B(G)$ and $\mathcal{M}^d_B(G^L)$, at least for $G = SL(n,\mathbb{C})$. Calculating Hodge numbers via number theory, we were able to check this conjecture for $n = 4$ or a prime. Since the two mirror symmetry conjectures of [28] and [23] are equivalent on the level of Euler characteristics, we get a proof of the original mirror symmetry conjecture of [28] on the level of Euler characteristics as well.

More interestingly, [23] arrives at explicit formulas, in terms of a simple generating function, for E-polynomials of $\mathcal{M}_B^d(GL(n,\mathbb{C}))$. For example, the Euler characteristic of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ equals $\mu(n)n^{2g-3}$, where μ is a basic number theoretic function: the Möbius function, i.e., the sum of all primitive nth roots of unity. This hints at an interesting link between number theory and topology of the above hyperkähler manifolds. Furthermore, the E-polynomials turn out to be palindromic, i.e., they satisfy an unexpected Poincaré duality-type symmetry. We can trace back this symmetry to the Alvis–Curtis duality [1, 5] in the representation theory of finite groups of Lie type.

We present a deformation of the E-polynomial of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$, which, conjecturally [23], should agree with the H-polynomial. Modifying this formula we obtain the conjectural H-polynomial of the Higgs moduli space $\mathcal{M}_{Dol}^d(GL(n,\mathbb{C}))$. We then explain how, using our mirror symmetry conjectures as a guide, one arrives at conjectures regarding H-polynomials of the varieties associated to $SL(n,\mathbb{C})$. These conjectures imply a conjecture on Poincaré polynomials of $\mathcal{M}_{Dol}^d(PGL(n,\mathbb{C}))$. The latter resembles Lusztig's conjecture [36] on Poincaré polynomials of Nakajima's quiver varieties, also hyperkähler manifolds, similar to the Higgs moduli space $\mathcal{M}_{Dol}^d(G)$. We should also mention Zagier's [52] formula for the Poincaré polynomial of the moduli space $\mathcal{N}^d$ of stable bundles (the "Kähler version" of $\mathcal{M}_{Dol}^d(SL(n,\mathbb{C}))$), where the formula is a similar sum, but is parametrized by ordered partitions of n.

We discuss in detail several checks of these conjectures, showing how they imply results of Hitchin [29], Gothen [16] and Earl–Kirwan [8]. The combinatorics of these formulas are quite non-trivial. Surprisingly, the calculus of Garsia–Haiman [12] is used to check the conjecture for $g = 0$.

The curious Poincaré duality satisfied by the conjectured Hodge numbers of $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$, leads to a conjecture that a version of the Hard Lefschetz theorem is satisfied for the non-compact varieties under consideration. This can be thought of as a generalization of a result in [21] on the quaternionic geometry of matroids, and as an analogue of Faber's conjecture [9] on the moduli space of curves.

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2 Abelian and non-Abelian Hodge theory

In this section we recall some basic definitions on Abelian and non-Abelian Hodge theory.

2.1 Hodge–De Rham theory

There are various cohomology theories associating a graded anti-commutative ring to a smooth complex algebraic variety M. First of all, the singular, or Betti, cohomology $H^*_B(M,\mathbb{C})$ of M with complex coefficients, defined for any reasonable topological space. The dimension $b_k(M) := \dim H^k_B(M,\mathbb{C})$ is called the k-th Betti number. The Poincaré polynomial is

$$P(t;M) := \sum_k b_k(M)t^k.$$

Next, the De Rham cohomology $H^*_{DR}(M,\mathbb{C})$, the space of closed differential forms modulo exact forms, defined for any differentiable manifold. The De Rham theorem establishes the isomorphism:

$$H^*_B(M,\mathbb{C}) \cong H^*_{DR}(M,\mathbb{C}). \tag{1}$$

For projective M we have the Dolbault cohomology

$$H^k_{Dol}(M,\mathbb{C}) = \bigoplus_{p+q=k} H^q(M,\Omega^p_M).$$

The Hodge theorem establishes a natural isomorphism

$$H^k_{DR}(M,\mathbb{C}) \cong H^k_{Dol}(M,\mathbb{C}). \tag{2}$$

The above isomorphisms imply the Hodge decomposition theorem:

$$H^k_B(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M), \tag{3}$$

where $H^{p,q}(M) := H^p(M,\Omega^q_M)$. The numbers $h^{p,q}(M) := \dim H^{p,q}(M)$ are called Hodge numbers of M. The Hodge polynomial is:

$$H(x,y;M) := \sum_{p,q} h^{p,q}(M)x^py^q.$$

For more details on these cohomology theories see [15].

2.2 Mixed Hodge structures

Deligne [7] generalized the Hodge decomposition theorem (3) to any complex variety M, not necessarily smooth or projective, by introducing a so-called *mixed Hodge structure* on $H^*_B(M,\mathbb{C})$. This implies a decomposition [1]

$$H^k_B(M,\mathbb{C}) \cong \bigoplus_{p,q} H^{p,q;k}(M),$$

where $p+q$ is called the weight of $H^{p,q;k}(M)$. For a smooth projective variety we have $H^{p,q;p+q}(M) = H^{p,q}(M)$, i.e. the weight of $H^{p,q;k}(M)$ is always k (called the *pure weight*). In general, other weights appear in the mixed Hodge structure; we will see such examples later. The dimensions $h^{p,q;k}(M) := H^{p,q;k}(M)$ are called mixed Hodge numbers of M. Form the three variable polynomial:

$$H(x,y,t;M) := \sum_{p,q,k} h^{p,q;k}(M)x^p y^q t^k. \tag{4}$$

Similarly, Deligne [7] constructs a mixed Hodge structure on the compactly supported $H^*_{B,cpt}(M,\mathbb{C})$ singular cohomology of a complex algebraic variety M. This yields the decomposition

$$H^k_{B,cpt}(M,\mathbb{C}) \cong \bigoplus_{p,q} H^{p,q;k}_{cpt}(M),$$

and compactly supported mixed Hodge numbers $h^{p,q;k}_{cpt}(M) := \dim H^{p,q;k}_{cpt}(M)$. One introduces the e-numbers

$$e^{p,q}(M) = \sum_k (-1)^k h^{p,q;k}_{cpt}(M)$$

and the E-polynomial:

$$E(x,y;M) := \sum_{p,q} e^{p,q}(M)x^p y^q. \tag{5}$$

Clearly, for M smooth projective, $E(x,y) = H(-x,-y)$. Moreover, for a smooth variety Poincaré duality implies that

$$E(x,y) = (xy)^n H(1/x,1/y,-1),$$

where n is the complex dimension of M. The significance of the E-polynomial is that it is additive for decompositions and multiplicative for Zariski locally trivial fibrations.

For more details see [7] or [3].

[1] In fact what one gets from a mixed Hodge structure are two filtrations on the cohomology, and the decomposition in question is the associated graded.

2.3 Stringy cohomology

Let Γ be a finite group acting on M. By the naturality of the mixed Hodge structure, Γ will act on $H^{p,q,k}(M)$ and we have

$$H^{p,q;k}(M/\Gamma) \cong \left(H^{p,q;k}(M)\right)^{\Gamma}.$$

However, for a Calabi–Yau M and a Γ preserving the Calabi–Yau structure string theorists [50, 53] introduced different Hodge numbers on the Calabi–Yau orbifold M/Γ: the so-called stringy Hodge numbers, which are the "right" Hodge numbers for mirror symmetry. Their mathematical significance is highlighted by a theorem of Kontsevich [34] which says that stringy Hodge numbers agree with ordinary Hodge numbers of any crepant resolution. Following [3] we can define the stringy E-polynomials:

$$E_{\mathrm{st}}(x,y;M/\Gamma) := \sum_{[\gamma]} E(x,y;M^{\gamma})^{C(\gamma)}(xy)^{F(\gamma)},$$

where the sum runs over the conjugacy classes of Γ; $C(\gamma)$ is the centralizer of γ; M^{γ} is the subvariety fixed by γ; and $F(\gamma)$ is an integer, called the *fermionic shift*, which is defined as follows. The group element γ has finite order, so it acts on $TM|_{M^{\gamma}}$ as a linear automorphism with eigenvalues $e^{2\pi i w_1}, \ldots, e^{2\pi i w_n}$, where each $w_j \in [0,1)$. Let $F(\gamma) = \sum w_j$; this is an integer since, by hypothesis, γ acts trivially on the canonical bundle.

The last cohomology theory needed is the stringy cohomology of a Calabi–Yau orbifold twisted by a B-field. Following [31] we let $B \in H^2_{\Gamma}(M, U(1))$, i.e., an isomorphism class of a Γ-equivariant flat unitary gerbe. For any $\gamma \in \Gamma$ this B-field induces a $C(\gamma)$-equivariant local system $[L_{B,\gamma}] \in H^1_{C(\gamma)}(M^{\gamma}, U(1))$ on the fixed point set M^{γ} and we can twist the stringy E-polynomial:

$$E^B_{\mathrm{st}}(x,y;M/\Gamma) := \sum_{[\gamma]} E(x,y;M^{\gamma};L_{B,\gamma})^{C(\gamma)}(xy)^{F(\gamma)}. \tag{6}$$

For more information about stringy cohomology see [3], for twisting with a B-field see [28].

2.4 Non-Abelian Hodge theory

The starting point of non-Abelian Hodge theory is the identification of the space $H^1_B(M, \mathbb{C}^{\times})$ with the space of homomorphisms from $\pi_1(M) \to \mathbb{C}^{\times}$; the space $H^1_{DR}(M, \mathbb{C}^{\times})$ with algebraic local systems on M and the space

$$H_{Dol}(M, \mathbb{C}^{\times}) \cong H^1(M, \mathcal{O}^{\times}) \oplus H^0(M, \Omega^1)$$

with pairs of a holomorphic line bundle and a holomorphic one-form.

This can be generalized to any non-Abelian complex reductive group G. We define $H^1_B(M,G)$ to be conjugacy classes of representations of $\pi_1(M) \to G$. I.e.,

$$H^1_B(M,G) := \mathrm{Hom}(\pi_1(M),G)//G,$$

the affine GIT quotient of the affine variety $\mathrm{Hom}(\pi_1(M),G)$ by the conjugation action of G, called the *character variety*. The space $H^1_{DR}(M,G)$ can be identified as the moduli space of algebraic G-local systems on M. Finally, $H^1_{Dol}(M,G)$ is the moduli space of certain semistable G-Higgs bundles on M. We will give a precise definition in the case of a curve below. The identification between $H^1_B(M,G)$ and $H^1_{DR}(M,G)$, which is analogous to the De Rham map (1), is given by the Riemann–Hilbert correspondence [6, 47], while the identification between $H^1_{DR}(M,G)$ and $H^1_{Dol}(M,G)$, analogous to the Hodge decomposition (2), is given in [4, 45] by the theory of harmonic bundles, the non-Abelian generalization of Hodge theory.

For an introduction to non-Abelian Hodge theory see [44], and ([33], Section 3), for more details on the construction of the spaces appearing in non-Abelian Hodge theory and the maps between them see [45, 46, 47].

2.5 The case of a curve

We fix a smooth projective complex curve C of genus g and specify our spaces in the case when $M=C$ and $G=GL(n,\mathbb{C})$. We have:

$$\begin{aligned}&\mathcal{M}_B(GL(n,\mathbb{C})) := H^1_B(C,GL(n,\mathbb{C}))\\ &=\{A_1,B_1,\dots,A_g,B_g\in GL(n,\mathbb{C})|[A_1,B_1]\cdot\dots\cdot[A_g,B_g]=Id\}//GL(n,\mathbb{C}).\end{aligned}$$

There is a natural way to twist these varieties. This will be needed for $PGL(n,\mathbb{C})$ and we introduce these twists below. For $d\in\mathbb{Z}$, consider:

$$\begin{aligned}\mathcal{M}^d_B(GL(n,\mathbb{C})) := \{&A_1,B_1,\dots,A_g,B_g\in GL(n,\mathbb{C})|\\ &[A_1,B_1]\cdot\dots\cdot[A_g,B_g]=e^{\frac{2\pi i d}{n}}Id\}//GL(n,\mathbb{C}).\end{aligned}$$

The De Rham space looks like

$$\begin{aligned}\mathcal{M}_{DR}(GL(n,\mathbb{C})) &:= H^1_{DR}(C,GL(n,\mathbb{C}))\\ &=\{\text{moduli space of flat } GL(n,\mathbb{C})\text{-connections on } C\}\end{aligned}$$

and in the twisted case we need to fix a point $p\in C$, and define

$$\mathcal{M}^d_{DR}(GL(n,\mathbb{C})) := \left\{\begin{array}{c}\text{moduli space of flat } GL(n,\mathbb{C})\text{-connections on } C\setminus\{p\},\\ \text{with holonomy } e^{\frac{2\pi i d}{n}}Id \text{ around } p\end{array}\right\}.$$

Finally, the Dolbeault spaces are:

$$\mathcal{M}_{Dol}(GL(n,\mathbb{C})) := H^1_{Dol}(C, GL(n,\mathbb{C})) = \{\text{moduli space of semistable rank } n \text{ degree } 0 \text{ Higgs bundles on } C\},$$

where a rank n Higgs bundle is a pair (E,ϕ) of a rank n algebraic vector bundle E on C, with degree 0 and Higgs field $\phi \in H^0(C, K_C \otimes \mathrm{End}(E))$. A Higgs bundle is called semistable if for any Higgs subbundle (F,ψ) (i.e., a subbundle with compatible Higgs fields) we have

$$\frac{\deg(F)}{\mathrm{rank}(F)} \leq \frac{\deg(E)}{\mathrm{rank}(E)} = 0.$$

The twisted version of $\mathcal{M}_{Dol}(GL(n,\mathbb{C}))$ is defined:

$$\mathcal{M}^d_{Dol}(GL(n,\mathbb{C})) := \{\text{moduli space of semistable rank } n \text{ degree } d \text{ Higgs bundles on } C\}.$$

The varieties above for $GL(n,\mathbb{C})$ have dimension $n^2(2g-2)+2$. The Betti space is affine, while the De Rham space is analytically (but not algebraically) isomorphic, via the Riemann–Hilbert correspondence, to the Betti space, so that the De Rham space is a Stein manifold as a complex manifold but not an affine variety as an algebraic variety. Finally, the Dolbeault space is a quasi-projective variety with large projective subvarieties.

From now on we fix a d with $(n,d)=1$. In this case the corresponding twisted spaces are smooth, have a diffeomorphic underlying manifold $\mathcal{M}^d(GL(n,\mathbb{C}))$ which carries a complete hyperkähler metric [29]. The complex structures of $\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))$ and $\mathcal{M}^d_{DR}(GL(n,\mathbb{C}))$ appear in the hyperkähler structure.

We started this subsection by determining these spaces for $GL(1,\mathbb{C}) \cong \mathbb{C}^\times$. With the identifications explained, we see that

$$\begin{aligned} {\mathcal{M}^d}_B(GL(1,\mathbb{C})) &\cong (\mathbb{C}^\times)^{2g}, \\ \mathcal{M}^d_{Dol}(GL(1,\mathbb{C})) &\cong T^* Jac^d(C) \end{aligned} \tag{7}$$

and $\mathcal{M}^d_{DR}$ is a certain affine bundle over $Jac^d(C)$.

Interestingly, for $d=0$ they are all algebraic groups and they act on the corresponding spaces for $GL(n,\mathbb{C})$ and any d by tensorization.

We can consider the map

$$\begin{aligned} \lambda_{Dol} : \mathcal{M}^d_{Dol}(GL(n,\mathbb{C})) &\to \mathcal{M}^d_{Dol}(GL(1,\mathbb{C})), \\ (E,\Phi) &\mapsto (\det(E), \mathrm{tr}(\phi)). \end{aligned}$$

The fibres of this map can be shown to be isomorphic using the above tensorization action. It follows that up to isomorphism it is irrelevant which fibre we take, but we usually take a point $(\Lambda, 0) \in \mathcal{M}^d_{Dol}(GL(1,\mathbb{C}))$ and define

$$\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})) := \lambda^{-1}_{Dol}((\Lambda,0)).$$

For the other two spaces we have:

$$\mathcal{M}^d_{DR}(SL(n,\mathbb{C})) = \left\{ \begin{array}{c} \text{moduli space of flat } SL(n,\mathbb{C})\text{-connections on } C \setminus \{p\} \\ \text{with holonomy } e^{\frac{2\pi i d}{n}} Id \text{ around } p \end{array} \right\},$$

and

$$\mathcal{M}^d_B(SL(n,\mathbb{C})) = \{A_1, B_1, \dots, A_g, B_g \in SL(n,\mathbb{C})| \\ [A_1,B_1]\cdot \dots \cdot [A_g,B_g] = e^{\frac{2\pi i d}{n}} Id\}//SL(n,\mathbb{C}).$$

The varieties $\mathcal{M}^d_B(SL(n,\mathbb{C}))$, $\mathcal{M}^d_{DR}(SL(n,\mathbb{C}))$ and $\mathcal{M}^d_{DR}(SL(n,\mathbb{C}))$ are smooth of dimension $(n^2-1)(2g-2)$, with diffeomorphic underlying manifold $\mathcal{M}^d(SL(n,\mathbb{C}))$. The Betti space is affine, and the Betti and De Rham spaces are again analytically, but not algebraically, isomorphic.

We see that the finite subgroup $Jac[n] \cong \mathbb{Z}_n^{2g} \subset \mathcal{M}_{Dol}(GL(1,\mathbb{C}))$ preserves the fibration λ_{Dol} and thus acts on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. The quotient then is:

$$\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C})) := \mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))/Jac[n]$$

and similarly

$$\mathcal{M}^d_{DR}(PGL(n,\mathbb{C})) := \mathcal{M}^d_{DR}(SL(n,\mathbb{C}))/Jac[n],$$

and

$$\mathcal{M}^d_B(PGL(n,\mathbb{C})) = \mathcal{M}^d_B(SL(n,\mathbb{C}))/\mathbb{Z}_n^{2g}.$$

This shows that all the three spaces $\mathcal{M}^d_B(PGL(n,\mathbb{C}))$, $\mathcal{M}^d_{DR}(PGL(n,\mathbb{C}))$ and $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$ are hyperkähler orbifolds of dimension $(n^2-1)(2g-2)$. As they are orbifolds we can talk about their stringy mixed Hodge numbers as defined above in Section 2.3. Moreover, they carry natural orbifold B-fields, constructed as follows: Consider a universal Higgs pair $(\mathbf{E},\mathbf{\Phi})$ on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})) \times C$; it exists because $(d,n)=1$. Restrict $\mathbf{E}$ to $\mathcal{M}^d_{Dol} \times \{p\}$ to get the vector bundle $\mathbf{E}_p$ on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. Now we can consider the projective bundle $\mathbb{P}\mathbf{E}_p$ of $\mathbf{E}_p$ which is a $PGL(n,\mathbb{C})$-bundle. The bundle $\mathbf{E}_p$ is a $GL(n,\mathbb{C})$-bundle but not a $SL(n,\mathbb{C})$-bundle, because it has a non-trivial determinant. The obstruction class to lifting the $PGL(n,\mathbb{C})$-bundle $\mathbb{P}\mathbf{E}$ to an $SL(n,\mathbb{C})$-bundle is a class

$$B \in H^2(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}),\mathbb{Z}_n)) \subset H^2(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})),U(1)),$$

which gives us a B-field on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. By ([28], Section 3), this field has a natural equivariant extension $\hat{B} \in H^2_{\Gamma}(\mathcal{M}^d_{Dol}(SL(n,\mathbb{C})),U(1))$, giving a B-field on $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$. This B-field will appear in our mirror symmetry discussions below.

Non-Abelian Hodge theory on a curve is explained in [29], via a gauge-theoretical approach. This yields natural hyperkähler metrics on our spaces. The case $GL(1,\mathbb{C})$ is treated in [13]. Further information about the geometry and cohomology of $\mathcal{M}^1_{Dol}(SL(2,\mathbb{C}))$ is in [20].

2.6 Mixed Hodge structure on non-Abelian Hodge cohomologies

The main subject of this paper is the mixed Hodge polynomial of the (stringy, sometimes with a B-field) cohomology of $\mathcal{M}^d_{Dol}(G)$, $\mathcal{M}^d_{DR}(G)$ and $\mathcal{M}^d_B(G)$, for $G = GL(n,\mathbb{C}), PGL(n,\mathbb{C})$ or $SL(n,\mathbb{C})$. For notational convenience, we omit G and simply write $\mathcal{M}^d_B$, $\mathcal{M}^d_{DR}$ and $\mathcal{M}^d_{Dol}$.

Consider first $G = GL(1,\mathbb{C})$. From (7) we calculate:

$$\begin{aligned} H(x,y,t;\mathcal{M}^d_B(GL(1,\mathbb{C}))) &= (1+xyt)^{2g}, \\ H(x,y,t;\mathcal{M}^d_{Dol}(GL(1,\mathbb{C}))) &= H(x,y,t;\mathcal{M}^d_{DR}(GL(1,\mathbb{C}))) \\ &= (1+xt)^g(1+yt)^g. \end{aligned}$$

It is remarkable that $H(x,y,t;\mathcal{M}^d_B(GL(1,\mathbb{C}))) \neq H(x,y,t;\mathcal{M}^d_{DR}(GL(1,\mathbb{C})))$ even though the spaces are analytically isomorphic. Furthermore, we can explicitly see that the mixed Hodge structure on $H^k(\mathcal{M}^d_{Dol}(GL(1,\mathbb{C})),\mathbb{C})$ and on $H^k(\mathcal{M}^d_{DR}(GL(1,\mathbb{C})),\mathbb{C})$ is pure, while on $H^k(\mathcal{M}^d_B(GL(1,\mathbb{C})),\mathbb{C})$ it is not.

A Künneth argument implies that:

$$\begin{aligned} &H(x,y,t;\mathcal{M}^d_{Dol}(GL(n,\mathbb{C}))) \\ &\qquad = H(x,y,t;\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C})))H(x,y,t;\mathcal{M}^d_{Dol}(GL(1,\mathbb{C}))), \end{aligned}$$

and similarly for the other two spaces. Thus the calculation for $GL(n,\mathbb{C})$ is equivalent to the calculation for $PGL(n,\mathbb{C})$.

Now we list what is known about the cohomologies $H^*(\mathcal{M}^d,\mathbb{C})$. The Poincaré polynomials $P(t;\mathcal{M}^1(SL(2,\mathbb{C})))$ and $P(t;\mathcal{M}^1(PGL(2,\mathbb{C})))$ were calculated in [29], while $P(t;\mathcal{M}^1(SL(3,\mathbb{C})))$ and $P(t;\mathcal{M}^1(PGL(3,\mathbb{C})))$ have been calculated in [16]. Both papers used Morse theory for a natural $\mathbb{C}^\times$-action on $\mathcal{M}^d_{Dol}$ (acting by multiplication on the Higgs field). The idea was to calculate the Poincaré polynomial of the various fixed point components of this action, and then sum them up with a certain shift. The largest of the fixed point components, when $\phi = 0$, is the important and well-studied space:

$$\begin{aligned} \mathcal{N}^d(SL(n,\mathbb{C})) := \{&\text{the moduli space of stable vector bundles} \\ &\text{of fixed determinant bundle of degree } d\}. \end{aligned} \tag{8}$$

Its Poincaré polynomial was calculated in [18] by arithmetic and in [2] by gauge-theoretical methods, with explicit formulas given in [52]. Thus its contribution to $P(t;\mathcal{M}^d(SL(n,\mathbb{C})))$ is easy to handle. However, the other components of the fixed point set of the circle action are more cumbersome to determine already for $n = 4$. Consequently, the Morse theory approach has not been completed for $n \geq 4$.

As a representative example we calculate from [29] the Poincaré polynomial $P(t; \mathcal{M}^1(PGL(2, \mathbb{C})))$, when $g = 3$:

$$3\,t^{12} + 12\,t^{11} + 18\,t^{10} + 32\,t^9 + 18\,t^8 + 12\,t^7 + 17\,t^6 + 6\,t^5 + 2\,t^4 + 6\,t^3 + t^2 + 1. \quad (9)$$

The cohomology ring of $\mathcal{M}^1_{Dol}(PGL(2, \mathbb{C}))$ has been described explicitly by generators [25] and relations [26]. This information was essential for our main Conjecture 5.1. Finally, Markman [39] showed that for $PGL(n, \mathbb{C})$ the universal cohomology classes do generate the cohomology ring.

The following result first appeared in [38] using a construction of [25]. Here we present a simple proof.

Theorem 2.1. *The Hodge structure on $H^k(\mathcal{M}^d_{Dol}, \mathbb{C})$ is pure of weight k.*

Proof. The compactification $\overline{\mathcal{M}}^d_{Dol}$ of $\mathcal{M}^d_{Dol}$ constructed in [19] is a projective orbifold so that the Hodge structure on $H^k(\overline{\mathcal{M}}^d_{Dol}, \mathbb{C})$ is pure of weight k. Now [19] also implies that the natural map $H^*(\overline{\mathcal{M}}^d_{Dol}, \mathbb{C}) \to H^*(\mathcal{M}^d_{Dol}, \mathbb{C})$ is surjective. The claim follows from the functoriality of mixed Hodge structures [7]. □

One can similarly prove the same result for $\mathcal{M}^d_{DR}$.

Theorem 2.2. *The Hodge structure on $H^k(\mathcal{M}^d_{DR}, \mathbb{C})$ is pure of weight k.*

Proof. As explained in ([28], Theorem 6.2) one can deform the complex structure of $\overline{\mathcal{M}}^d_{Dol}$ to the projective orbifold $\overline{\mathcal{M}}^d_{DR}$, which is the compactification of $\mathcal{M}^d_{DR}$ given by Simpson in [48]. This way we see that the natural map $H^*(\overline{\mathcal{M}}^d_{DR}, \mathbb{C}) \to H^*(\mathcal{M}^d_{DR}, \mathbb{C})$ is a surjection, and we conclude as above. □

In fact the argument in ([28], Theorem 6.2) shows that

Theorem 2.3 (HT4). *The mixed Hodge structure on $H^*(\mathcal{M}^d_{Dol}, \mathbb{C})$ is isomorphic to the mixed Hodge structure on $H^*(\mathcal{M}^d_{DR}, \mathbb{C})$.*

However, the Hodge structure on $\mathcal{M}^d_B$ has not been studied in the literature. We will see later that it is not pure anymore. In the following section we explain our interest in the Hodge structures on $\mathcal{M}^d_{Dol}$, $\mathcal{M}^d_{DR}$ and $\mathcal{M}^d_B$. Our motivation is mirror symmetry.

3 Mirror symmetry conjectures

The starting point in [28] was the observation that the pairs $\mathcal{M}^d_{DR}(SL(n,\mathbb{C}))$ together with the B-field B^e and $\mathcal{M}^d_{DR}(PGL(n,\mathbb{C}))$ with the B-field $\hat{B}^d$ give a geometric realization for mirror symmetry as proposed in Strominger–Yau–Zaslow [49] and modified for B-fields by Hitchin in [31]. This geometric picture predicts the existence of a special Lagrangian fibration on each space, with dual fibres. In [28] it is shown that the so-called Hitchin map [30] provides the required special Lagrangian fibration on our spaces, with dual Abelian varieties as fibers. For details see ([28], Section 3).

Our focus in this survey is on the topological implications of this manifestation of mirror symmetry. The following conjecture can be called the topological mirror test for our SYZ-mirror partners.

Conjecture 3.1 ([28]). *For $d,e\in\mathbb{Z}$, with $(d,n)=(e,n)=1$, we have*

$$E^{B^e}_{\mathrm{st}}\Big(x,y;\mathcal{M}^d_{DR}(SL(n,\mathbb{C}))\Big)=E^{\hat{B}^d}_{\mathrm{st}}\Big(x,y;\mathcal{M}^e_{DR}(PGL(n,\mathbb{C}))\Big).$$

Remark 3.2. Since $M^d_{DR}(SL(n,\mathbb{C}))$ is smooth, the left-hand side actually equals the E-polynomial $E\Big(x,y;M^d_{DR}(SL(n,\mathbb{C}))\Big)$, which is independent of e. This motivates the following:

Conjecture 3.3 (HT4). *For $d_1.d_2\in\mathbb{Z}$ with $(d_1,n)=(d_2,n)=1$ we have:*

$$E\Big(x,y;\mathcal{M}^{d_1}_{Dol}(SL(n,\mathbb{C}))\Big)=E\Big(x,y;\mathcal{M}^{d_2}_{Dol}(SL(n,\mathbb{C}))\Big).\qquad(10)$$

This is quite interesting since the Betti numbers of $\mathcal{N}^d(SL(n,\mathbb{C}))$, the moduli space of stable vector bundles, with fixed determinant of degree d (the "Kähler version" of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$), are known to depend on d. Already for $n=5$, Zagier's explicit formulas [52] for $P(t;\mathcal{N}^1(SL(5,\mathbb{C})))$ and $P(t;\mathcal{N}^3(SL(5,\mathbb{C})))$ are different. We will see evidence for this Conjecture 3.3 in Corollary 3.11.

Remark 3.4. Conjecture 3.1 was proved for $n=2$ and $n=3$ in [28]. The proof proceeds by first transforming the calculation to $\mathcal{M}^d_{Dol}$ via Theorem 2.3 and then using the Morse theoretic method of [29] and [16]. It is unclear how to extend this method to $n\geq 4$.

Remark 3.5. An important ingredient of the proofs was a modification of a result of Narasimhan–Ramanan [41] to Higgs bundles. It describes the fixed points of the action of elements of $Jac[n]$ on $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$. The fixed point sets have lower rank $m|n$ Higgs moduli spaces $\mathcal{M}^d_{Dol}(SL(m,\mathbb{C});\tilde{C})$ for a certain covering $\tilde{C}$ of C. Their cohomology enters in the stringy contribution to the right-hand side of Conjecture 3.1 (recall (6)).

3.1 Number theory to the rescue

Although the mirror symmetry Conjecture 3.1 is still open for $n \geq 4$, recently some evidence for its validity has been achieved in form of progress on a related conjecture.

Conjecture 3.6 (HRV). *For $d, e \in \mathbb{Z}$ with $(d,n) = (e,n) = 1$, we have*

$$E_{\mathrm{st}}^{B^e}\Big(x, y, \mathcal{M}_B^d(SL(n,\mathbb{C}))\Big) = E_{\mathrm{st}}^{\hat{B}^d}\Big(x, y, \mathcal{M}_B^e(PGL(n,\mathbb{C}))\Big).$$

This conjecture has been proved [23] when n is a prime and when $n = 4$; which implies Conjecture 3.1 on the level of Euler characteristic in these cases. The method is arithmetic: one counts rational points on the variety $\mathcal{M}_B^d$ over a finite field $\mathbb{F}_q$, when n divides $q-1$, where $q = p^r$ is a prime power. We get:

Theorem 3.7 ([23]). *The E-polynomial of $\mathcal{M}_B^d$ has only $x^k y^k$ type terms, and*

$$E(q) = \#(\mathcal{M}_B^d(G)(\mathbb{F}_q)).$$

The problem reduces to the count of solutions of the equation:

$$[A_1, B_1] \cdot \dots \cdot [A_g, B_g] = \xi_n,$$

in the finite group of Lie type $G(\mathbb{F}_q)$, i.e., so that $A_i, B_i \in G(\mathbb{F}_q)$, where $\xi_n \in G$ is a central element of order n. A simple modification of a theorem of Mednykh [37], (which goes back to Frobenius–Schur [11], and has since been rediscovered by many authors, (see Freed–Quinn [10], (5.19)), implies:

Theorem 3.8. *Let $G = SL(n,\mathbb{C})$ or $G = GL(n,\mathbb{C})$. Then the number of rational points on $\mathcal{M}_B^d(G)$ over a finite field $\mathbb{F}_q$, where $q = p^r$ is a prime power, with $n|(q-1)$ is given by:*

$$\#(\mathcal{M}_B^d(G)(\mathbb{F}_q)) = \sum_{\chi \in \mathrm{Irr(G(\mathbb{F}_q))}} \frac{|G|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n),$$

where the sum is over all irreducible characters of $G(\mathbb{F}_q)$.

The two theorems above imply the following

Corollary 3.9 ([23]). *The E-polynomial of $\mathcal{M}_B^d(G)$ is given by:*

$$E(q) = \sum_{\chi \in \mathrm{Irr(G(\mathbb{F}_q))}} \frac{|G|^{2g-2}}{\chi(1)^{2g-1}} \chi(\xi_n). \tag{11}$$

Remark 3.10. An immediate consequence of this formula is the Betti analog of Conjecture 3.3. This follows from Corollary 3.9 since that character formula transforms by a Galois automorphism when one changes from d_1 to d_2. Moreover, because our $\mathcal{M}_B^{d_1}(G)$ and $\mathcal{M}_B^{d_2}(G)$ are Galois-conjugate, we can deduce that their Betti numbers agree, and presumably, that their mixed Hodge structures also agree. In summary, we have

Corollary 3.11 ([23]). *For* $d_1, d_2 \in \mathbb{Z}$ *with* $(d_1, n) = (d_2, n) = 1$ *we have*

$$E\Big(x, y; \mathcal{M}_B^{d_1}(G)\Big) = E\Big(x, y; \mathcal{M}_B^{d_2}(G)\Big) \tag{12}$$

and

$$P\Big(t; \mathcal{M}_B^{d_1}(G)\Big) = P\Big(t; \mathcal{M}_B^{d_2}(G)\Big). \tag{13}$$

This gives an affirmative answer to Conjecture 3.3 on the level of Poincaré polynomials. In general, Galois conjugate varieties tend to be (although need not be, see, e.g. [43]) homeomorphic over $\mathbb{C}$.

Problem 3.12. Are the underlying topological spaces of the varieties $\mathcal{M}_B^{d_1}(G)$ and $\mathcal{M}_B^{d_2}(G)$ homeomorphic for $(n, d_1) = (n, d_2) = 1$? Are they birationally isomorphic? Are $\mathcal{M}_{Dol}^{d_1}(G)$ and $\mathcal{M}_{Dol}^{d_2}(G)$ birationally isomorphic?

Remark 3.13. In order to calculate the character formula in Corollary 3.9, we will need to know the values of irreducible characters of G on central elements. Fortunately, for $GL(n, \mathbb{F}_q)$ this has been done by Green [14]. For $SL(n, \mathbb{F}_q)$ the required information, i.e., the value of characters on central elements, was obtained by Lehrer in [35]. In the next section we show an explicit result for the character formula for $GL(n, \mathbb{F}_q)$.

Remark 3.14. Our mirror symmetry Conjecture 3.6 can be translated to a complicated formula which is valid for the character tables of $PGL(n, \mathbb{F}_q)$ and $SL(n, \mathbb{F}_q)$. In particular, we believe that by introducing punctures for our Riemann surfaces, a similar mirror symmetry conjecture would in fact capture the exact difference between the full character tables of $PGL(n, \mathbb{F}_q)$ and $SL(n, \mathbb{F}_q)$ (not just on central elements as above). This way our mirror symmetry proposal could be phrased as follows: *the differences between the character tables of* $PGL(n, \mathbb{F}_q)$ *and its Langlands dual* $SL(n, \mathbb{F}_q)$ *are governed by mirror symmetry.* It is particularly enjoyable to see the effect of mirror symmetry on the differences between the character tables of $GL(2, \mathbb{F}_q)$ and $SL(2, \mathbb{F}_q)$, which were first calculated in 1907 by Jordan [32] and by Schur [42].

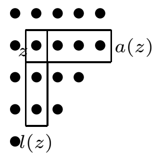
4 Explicit formulas for the E-polynomials

Here we calculate the E-polynomials of $\mathcal{M}_B^d(PGL(n, \mathbb{C}))$, which we denote by $E_n(q)$.

We start with partitions. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ be a partition of n, so that $\sum \lambda_i = n$. The *Ferrers diagram* $d(\lambda)$ of λ is the set of lattice points

$$\{(i,j) \in \mathbb{Z}_{\leq 0} \times \mathbb{N} : j < \lambda_{-i+1}\}. \tag{14}$$

The *arm length* $a(z)$ and *leg length* $l(z)$ of a point $z \in d(\lambda)$ denote the number of points strictly to the right of z and below z, respectively, as indicated in this example:



where $\lambda = (5,5,4,3,1)$, $z = (-1,1)$, $a(z) = 3$ and $l(z) = 2$. The *hook length* then is defined as

$$h(z) = l(z) + a(z) + 1.$$

Let

$$V_n(q) = E_n(q) q^{(1-g)n(n-1)} (q-1)^{2g-2},$$

and

$$Z_n(q,T) = \exp\left(\sum_{r\geq 1} V_n(q^r)\frac{T^r}{r}\right).$$

We define the Hook polynomials for a partition λ as follows :

$$\mathcal{H}^\lambda(q) = \prod_{z\in d(\lambda)} q^{-l(z)}(1-q^{h(z)}).$$

Theorem 4.1 ([23]). *For $n = 1,2,3,\ldots$ one has*

$$\prod_{n=1}^{\infty} Z_n(q,T^n) = \sum_{\lambda\in\mathcal{P}} (\mathcal{H}^\lambda(q))^{2g-2} T^{|\lambda|}, \tag{15}$$

where $\mathcal{P}$ is the set of all partitions.

One simple corollary of this is a new topological result:

Corollary 4.2 ([23]). *The Euler characteristic of $\mathcal{M}^d(PGL(n,\mathbb{C}))$ equals $\mu(n)n^{2g-3}$, where μ is the Möbius function, i.e., $\mu(n)$ is the sum of primitive nth root of unities.*

Another interesting application of the theorem is the following:

Corollary 4.3. *The E-polynomial* $E_n(q) = E(q; \mathcal{M}_B^d(PGL(n,\mathbb{C})))$ *is palindromic, i.e., it satisfies, what we call, the* curious *Poincaré duality:*

$$q^{2N} E_n(1/q) = E_n(q),$$

where $2N = (n^2-1)(2g-2)$ *is the complex dimension of* $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$.

Remark 4.4. This result originates in the so-called Alvis–Curtis duality [1], [5] in the character theory of $GL(n,\mathbb{F}_q)$, which is a duality between irreducible representations of $GL(n,\mathbb{F}_q)$. In particular, if $\chi, \chi' \in \mathrm{Irr}(GL(n,\mathbb{F}_q))$ are dual, then the dimension $\chi(1)$ is a polynomial in q which satisfies

$$q^{\frac{n(n-1)}{2}} \chi(1)(1/q) = \chi'(1)(q).$$

For example when $n=2$, Theorem 4.1 gives:

$$E_2(q) = (q^2-1)^{2g-2} + q^{2g-2}(q^2-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q-1)^{2g-2} - \frac{1}{2} q^{2g-2}(q+1)^{2g-2}, \quad (16)$$

when $g=3$ this gives

$$E(x,y;\mathcal{M}_B^1(PGL(2,\mathbb{C}))) = q^{12} - 4\,q^{10} + 6\,q^8 - 14\,q^6 + 6\,q^4 - 4\,q^2 + 1, \quad (17)$$

which is a palindromic polynomial. Note also that there does not seem to be much in common with the Poincaré polynomial (9).

5 A conjectured formula for mixed Hodge polynomials

Here we present the conjecture of [23] on the H-polynomials of the spaces $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$. As usual we fix the curve C and its genus g and the group $PGL(n,\mathbb{C})$ and write $\mathcal{M}_B^d$ for $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ and $H_n(x,y,t)$ for $H(x,y,t;\mathcal{M}_B^d)$.

Let

$$V_n(q,t) = H_n(q,t) \frac{(qt^2)^{(1-g)n(n-1)}(qt+1)^{2g}}{(qt^2-1)(q-1)},$$

and

$$Z_n(q,t,T) = \exp\left(\sum_{r\geq 1} V_n(q^r, -(-t)^r) \frac{T^r}{r} \right).$$

We define the t-deformed Hook polynomials for genus g and partition λ:

$$\mathcal{H}_g^\lambda(q,t) = \prod_{x\in d(\lambda)} \frac{(qt^2)^{(2-2g)l(x)}(1+q^{h(x)}t^{2l(x)+1})^{2g}}{(1-q^{h(x)}t^{2l(x)+2})(1-q^{h(x)}t^{2l(x)})}.$$

The following generating function then defines our rational functions $H_n(q,t)$:

$$\prod_{n=1}^{\infty} Z_n(q,t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}_g^{\lambda}(q,t)T^{|\lambda|}. \tag{18}$$

Because for the character variety we have that $h^{i,j;k}(\mathcal{M}_B^d) = 0$ provided that $i \neq j$, the following conjecture describes $H_n(x,y,t)$ completely.

Conjecture 5.1 ([23]). *The mixed Hodge polynomials of the character varieties $\mathcal{M}_B^d(PGL(n,\mathbb{C}))$ are given by the generating function (18):*

$$H_n(\sqrt{q},\sqrt{q},t) = H_n(q,t).$$

Thus $H_n(q,t)$, which is a priori only a rational function, is conjectured to be the H-polynomial of the character variety, so in the next conjecture we formalize our expectations from $H_n(q,t)$, with the addition of a curious, Poincaré duality-type of symmetry, which was in fact our most important guide in the derivation of these formulas:

Conjecture 5.2. *The rational functions $H_n(q,t)$ defined in the generating function of (18) satisfy the following properties:*

- *$H_n(q,t)$ is a polynomial in q and t.*
- *The q and the t degree of $H_n(q,t)$ equal $2N = 2(n^2-1)(g-1)$. The largest degree monomial in both variables is $(qt)^{2(n^2-1)(g-1)}$.*
- *All coefficients of $H_n(q,t)$ are non-negative integers.*
- *The coefficients of $H_n(q,t) = \sum h_j^i q^j t^i$ satisfy what we call the* curious Poincaré duality*:*

$$h_{N-j}^{i-j} = h_{N+j}^{i+j} \tag{19}$$

Now we list some checks and implications of the above conjectures:

Remark 5.3. Computer calculations with Maple gives $H_n(q,t)$ from the above generating function when $n = 2,3,4$. In all these cases for small g we do get a polynomial in q and t with the expected degree and positive coefficients, satisfying the curious symmetry (19).

Remark 5.4. The paper [26] contains a monomial basis, in the tautological generators, for the cohomology ring $H^*(\mathcal{M}_B^1(PGL(2,\mathbb{C})),\mathbb{C})$. Understanding the action of the Frobenius on these generators leads to a formula for the mixed Hodge polynomial of $\mathcal{M}_B^1(PGL(2,\mathbb{C}))$ of the form

$$\begin{aligned} H_2(\sqrt{q},\sqrt{q},t) = {} & \frac{(q^2t^3+1)^{2g}}{(q^2t^2-1)(q^2t^4-1)} + \frac{q^{2g-2}t^{4g-4}(q^2t+1)^{2g}}{(q^2-1)(q^2t^2-1)} \\ & - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt+1)^{2g}}{(qt^2-1)(q-1)} - \frac{1}{2}\frac{q^{2g-2}t^{4g-4}(qt-1)^{2g}}{(q+1)(qt^2+1)}, \end{aligned} \tag{20}$$

which agrees with the conjectured one through (18), and clearly reduces to (4.1), when $t=-1$. For example when $g=3$, this gives

$$\begin{aligned} &H(\sqrt{q},\sqrt{q},t;\mathcal{M}^d(PGL(2,\mathbb{C})))\\ &= t^{12}q^{12}+t^{12}q^{10}+6\,t^{11}q^{10}+t^{12}q^{8}+t^{10}q^{10}+6\,t^{11}q^{8}+16\,t^{10}q^{8}+6\,t^{9}q^{8}+t^{10}q^{6}\\ &+t^{8}q^{8}+26\,t^{9}q^{6}+16\,t^{8}q^{6}+6\,t^{7}q^{6}+t^{8}q^{4}+t^{6}q^{6}+6\,t^{7}q^{4}+16\,t^{6}q^{4}+6\,t^{5}q^{4}+t^{4}q^{4}+t^{4}q^{2}\\ &+6\,t^{3}q^{2}+t^{2}q^{2}+1, \end{aligned} \tag{21}$$

which is a common refinement of (9) when $q=1$ and of (17) when $t=-1$. Note also how the curious Poincaré duality appears when one refines the Poincaré polynomial (9), which does not possess any kind of symmetry, to the mixed Hodge polynomial (21).

Remark 5.5. Note that $P_n(t)=H_n(1,t)$ should be the Poincaré polynomial of the character variety, which is the same as the Poincaré polynomial of the diffeomorphic Higgs moduli space $\mathcal{M}^d_{Dol}$. For $n=2$, Hitchin in [29] calculated the Poincaré polynomial of this latter space, and an easy calculation shows that if one substitutes $q=1$ into (20) we get $P_2(t)=H_2(1,t)$, the Poincaré polynomial of Hitchin. For $n=3$, Gothen in [16] calculated $P_3(t)$. Since it is a pleasure to work with a formula like (20), we write down what our Conjecture 5.1 gives for $n=3$:

$$\begin{aligned} &H_3(q,t)=\\ &\frac{\left(q^3t^5+1\right)^{2g}\left(q^2t^3+1\right)^{2g}}{(q^3t^6-1)(q^3t^4-1)(q^2t^4-1)(q^2t^2-1)}+\frac{q^{6g-6}t^{12g-12}\left(q^3t+1\right)^{2g}\left(q^2t+1\right)^{2g}}{(q^3t^2-1)(q^3-1)(q^2t^2-1)(q^2-1)}\\ &+\frac{q^{4g-4}t^{8g-8}\left(q^3t^3+1\right)^{2g}(qt+1)^{2g}}{(q^3t^4-1)(q^3t^2-1)(qt^2-1)(q-1)}+\frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left((qt+1)^{2g}\right)^2}{(qt^2-1)^2(q-1)^2}\\ &-\frac{1}{3}\,\frac{q^{6g-6}t^{12g-12}\left(q^2t^2-qt+1\right)^{2g}}{(q^2t^4+qt^2+1)(q^2+q+1)}-\frac{q^{4g-4}t^{8g-8}\left(q^2t^3+1\right)^{2g}(qt+1)^{2g}}{(q^2t^4-1)(q^2t^2-1)(qt^2-1)(q-1)}\\ &-\frac{q^{6g-6}t^{12g-12}\left(q^2t+1\right)^{2g}(qt+1)^{2g}}{(q^2t^2-1)(q^2-1)(qt^2-1)(q-1)}. \end{aligned}$$

It is a nice exercise to show that $H_3(1,t)$ does produce (the corrected version[2] of) Gothen's complicated looking formula in [16].

It is also worth noting that many terms in $H_n(q,t)$ have poles at $q=1$, which somehow cancel, according to our conjecture.

Remark 5.6. When $g=0$, we know from the definitions that $H_1(x,y,t)=1$ and $H_n(x,y,t)=0$ otherwise. One can deduce the same from Conjecture 5.1 by applying Theorem 2.10 in [12] to calculate the right-hand side of (18). Moreover Conjecture 5.2 has the same flavour as the main conjecture in [12] about q,t Catalan numbers, which was in turn proved by Haiman in [17] using subtle intersection theory on the Hilbert scheme of n points on $\mathbb{C}^2$.

[2] One accidental mistake in the calculation of [16] was pointed out in (10.3) of [28].

Apart from the fact that this Hilbert scheme is also a hyperkähler manifold, the similarities between the two conjectures are rather surprising.

Remark 5.7. When $g = 1$ we have $H_n(x, y, t) = 1$ for every n, but this we could not prove from (18) for $H_n(q, t)$.

Remark 5.8. Let us look at the conjecture (19). Recall that H^2 of our varieties are exactly one dimensional, generated by a class, call it $[\omega]$, which is the Kähler class in the complex structure of $\mathcal{M}^d_{Dol}$. This carries the weight q^2t^2 in the mixed Hodge structure. The following Hard Lefschetz type conjecture enhances the curious Poincaré duality of the conjecture (19):

Conjecture 5.9. *If L denotes the map by multiplication with $[\omega]$, then the map*

$$L^k : H^{N-k,N-k;i-k}(\mathcal{M}^d_B(PGL(n,\mathbb{C}))) \to H^{N+k,N+k;i+k}(\mathcal{M}^d_B(PGL(n,\mathbb{C})))$$

is an isomorphism.

Interestingly this conjecture implies a theorem of [21] that the Lefschetz map $L^k : H^{N-k} \to H^{N+k}$ is injective for $\mathcal{M}^d_{Dol}$, and it is explained there how this weak version of Hard Lefschetz, when applied to toric hyperkähler varieties, yields new inequalities for the h-numbers of matroids. See also [24] for the original argument on toric hyperkähler varieties. Furthermore, this conjecture can be proved when $n = 2$ using the explicit description of the cohomology ring in [26]. The general case can be thought of as an analog of Faber's conjecture [9] on the cohomology of the moduli space of curves, which is another non-compact variety whose cohomology ring is conjectured to satisfy some form of Hard Lefschetz theorem.

Remark 5.10. There are two subspaces of the cohomology $H^*(\mathcal{M}^d_B, \mathbb{C})$ which are particularly interesting. One of them is the middle dimensional cohomology $H^{2N}(\mathcal{M}^d_B, \mathbb{C})$, which is the top non-vanishing cohomology. The mixed Hodge structure breaks up into parts with respect to the q-degree. The curious Poincaré dual (19) of these spaces are also interesting: it is easy to see that they are exactly the pure part of the mixed Hodge structure, i.e., spaces of the form $H^{i,i;2i}$. (Another significance of the pure part is that if there is a smooth projective compactification of the variety, then its image is in this pure part.) Thus it would already be interesting to get the pure part of $H_n(q, t)$. It is easy to identify the pure part in our case with what we call the *Pure* ring, which is the subring of $H^*(\mathcal{M}^d, \mathbb{C})$ generated by the tautological classes $a_i \in H^{2i}(\mathcal{M}^d, \mathbb{C})$ for $i = 2, \ldots, n$ (the other tautological classes, which generate the cohomology ring, are not pure classes).

For example, when $n = 2$, it was known [29] that the middle degree cohomology of the Higgs moduli space $\mathcal{M}^d_{Dol}(PGL(2, \mathbb{C}))$ is g dimensional. The Pure ring was determined in [26], and it was found to be g dimensional due to the relation $\beta^g = 0$ (where $\beta = a_2$). Thus these two seemingly unrelated observations are dual to each other via our curious Poincaré duality (19). To see this curious duality in action let us recall the formula (21). The terms which contain the top degree 12 in t are $t^{12}q^{12}$, $t^{12}q^{10}$ and $t^{12}q^{8}$, which are curious Poincaré dual via (19) to the terms 1, t^4q^2 and t^8q^4, which is exactly the ring generated by the degree-four class β, which has additive basis 1, β and β^2.

The analogous ring, generated by the corresponding classes $a_2,\dots,a_n \in H^*(\mathcal{N}^d,\mathbb{C})$, which a priori is a quotient of our Pure ring (as $\mathcal{N}^d \subset \mathcal{M}^d_{Dol}$ naturally), was studied for the moduli space $\mathcal{N}^d$ of rank n, degree-d stable bundles (with $(n,d)=1$) in [8], where it was found that the top non-vanishing degree of this ring is $2n(n-1)(g-1)$. Computer calculations for our conjecture for $n=2,3,4$ also show that our conjectured Pure ring has the same 1-dimensional top degree. This and the known situation for $n=2$ (see [26]), yields the following

Conjecture 5.11. *The Pure rings of $\mathcal{M}^d_{Dol}$ and $\mathcal{N}^d$, i.e., the subrings of the cohomology rings generated by the classes $a_2,\dots,a_n$ are isomorphic. In particular, unlike the whole cohomology ring of $\mathcal{N}^d$, it does not depend on d.*

Now we explain a combinatorial consequence of this conjecture. First we extract a conjectured formula for $PP_n(t)$ the Poincaré polynomial of the Pure ring. Indeed we only have to deal with monomials in Conjecture 5.1 whose t-degree is double of their q-degree.

Let

$$PV_n(t) = PP_n(t)\frac{t^{2(1-g)n(n-1)}}{(t^2-1)},$$

and

$$PZ_n(t,T) = \exp\left(\sum_{r\geq 1} PV_n(t^r)\frac{T^r}{r}\right).$$

We now define the pure part of the t-deformed Hook polynomials for genus g and partition λ as follows:

$$\mathcal{PH}_g^\lambda(t) = t^{4(1-g)n(\lambda')} \prod_{x\in d(\lambda);a(x)=0} \frac{1}{(1-t^{2h(x)})},$$

where

$$n(\lambda') := \sum_{z\in d(\lambda)} l(z).$$

We get the conjecture that $PP_n(t)$ is given by

$$\prod_{n=1}^{\infty} PZ_n(t,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{PH}_g^\lambda(t)T^{|\lambda|}. \tag{22}$$

Combining the two conjectures above we can formulate:

Conjecture 5.12. *The rational functions $PP_n(t)$ defined in (22) satisfy*

- *$PP_n(t)$ is a polynomial in t;*
- *all coefficients of $PP_n(t)$ are non-negative integers;*
- *The degree of $PP_n(t)$ is $2n(n-1)(g-1)$, and the coefficient of the leading term is 1.*

For example, when $n = 3$ the Poincaré polynomial of the Pure ring should be:

$$PP_3(t) = \frac{1}{(t^6-1)(t^4-1)} + t^{12g-12} - \frac{t^{8g-8}}{t^2-1} + \frac{1}{3}\frac{t^{12g-12}}{(t^2-1)^2} - \frac{1}{3}\frac{t^{12g-12}}{t^4+t^2+1} - \frac{t^{8g-8}}{(t^4-1)(t^2-1)} + \frac{t^{12g-12}}{t^2-1}.$$

Remark 5.13. The formula of Conjecture 5.1 can be modified to give a conjectured formula for the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}$. Recall from Theorem 2.1 that the mixed Hodge structure on $H^k(\mathcal{M}^d_{Dol}, \mathbb{C})$ is pure of weight k; thus this mixed Hodge polynomial coincides with the E-polynomial.

We now introduce polynomials $H_n(q,x,y)$ of three variables. Let

$$V_n(q,x,y) = H_n(q,x,y)\frac{(qxy)^{(1-g)n(n-1)}(qx+1)^g(qy+1)^g}{(qxy-1)(q-1)},$$

and

$$Z_n(q,x,y,T) = \exp\left(\sum_{r\geq 1} V_n(q^r, -(-x)^r, -(-y)^r)\frac{T^r}{r}\right).$$

Define the (x,y)-deformed Hook polynomials for genus g and partition λ:

$$\mathcal{H}^\lambda_g(q,x,y) = \prod_{z\in d(\lambda)} \frac{(qxy)^{(2-2g)l(z)}(1+q^{h(z)}y^{l(z)}x^{l(z)+1})^g(1+q^{h(z)}x^{l(z)}y^{l(z)+1})^g}{(1-q^{h(z)}(xy)^{l(z)+1})(1-q^{h(z)}(xy)^{l(z)})}. \quad (23)$$

The following generating function defines $H_n(q,x,y)$:

$$\prod_{n=1}^{\infty} Z_n(q,x,y,T^n) = \sum_{\lambda\in\mathcal{P}} \mathcal{H}^\lambda_g(q,x,y)T^{|\lambda|}. \quad (24)$$

Clearly we have $H_n(q,t,t) = H_n(q,t)$ which says that a specialization of $H_n(q,x,y)$ gives the mixed Hodge polynomial $H_n(q,t)$ of $\mathcal{M}^d_B$. The following conjecture says that another specialization gives the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}$ and $\mathcal{M}^d_{DR}$.

Conjecture 5.14. *$H_n(q,x,y)$ is a polynomial with non-negative integer coefficients with specialization $H_n(1,x,y)$ equal to the E-polynomial of the Higgs moduli space $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$.*

Thus we have a mysterious formula $H_n(q,x,y)$ which specializes, on one hand to the H-polynomial of the character variety, and on the other hand to the mixed Hodge polynomial of the Higgs (or equivalently flat connection) moduli space. It would be very interesting to find a geometrical meaning for $H_n(q,x,y)$.

Checks on this Conjecture 5.14 include a proof for $n=2$ and $n=3$, (one can easily modify Hitchin's and Gothen's argument to get the Hodge polynomial instead of the Poincaré polynomial of the Higgs moduli space) and also computer checks that the shape of the polynomial $H_n(1,x,y)$ is the expected one when $n=4$.

Consider now the specification $H_n(q,-1,y)$. Interestingly, the corresponding specification of the (x,y)-deformed Hook polynomials (23) becomes a polynomial, showing that at least $H_n(q,-1,y)$ is a polynomial. We get an even nicer formula if we make the further specification $H_n(1,-1,y)$ which by Conjecture 5.14 should be the Hirzebruch y-genus of the moduli space of Higgs bundles $\mathcal{M}^d_{Dol}$. Namely, for $g>1$, most of the (x,y) deformed Hook polynomials vanish, when one substitutes first $x=-1$ and then $q=1$. Indeed, the only partitions which will have a non-zero contribution to the y-genus are the partitions of the form $n=1+1+\cdots+1$; when $l(z)=0$ only for once. This in turn gives the following closed formula for the conjectured y-genus of $\mathcal{M}^d_{Dol}$:

Conjecture 5.15. *The Hirzebruch y-genus of $\mathcal{M}^d_{Dol}(PGL(n,\mathbb{C}))$, for $g>1$, equals*

$$(1-y+\cdots+(-y)^{n-1})^{g-1}\sum_{m|n}\frac{\mu(m)}{m}\left((-y)^{n(n-n/m)}m\prod_{i=1}^{n/m-1}(1-(-y)^{mi})^2\right)^{g-1}.$$

Note that the term corresponding to $m=1$ is exactly the known y-genus of $\mathcal{N}^d$ (see [41]). The rest should be thought of as the contribution of the other fixed point components of the circle action on $\mathcal{M}^d_{Dol}$. Of course, this conjectured formula gives the known specialization of Corollary 4.2 at $y=-1$, while the $y=1$ specialization gives $\mu(n)n^{g-2}$ when n is odd, and 0 when n is even. The specialization at $y=1$ can be thought of as the signature of the pairing on the rationalized circle equivariant cohomology of $\mathcal{M}^d_{Dol}$ as defined in [22].

Remark 5.16. Finally we discuss how to obtain a conjecture for the mixed Hodge polynomial of $\mathcal{M}^d_B(SL(n,\mathbb{C}))$. For the mixed Hodge polynomial of $\mathcal{M}^d_{Dol}(SL(n,\mathbb{C}))$ the mirror symmetry Conjecture 3.1, together with Conjecture 5.14 imply a conjecture. For $\mathcal{M}^d_B(SL(n,\mathbb{C}))$ the mixed Hodge polynomial contains more information than the E-polynomial. In order to have a conjecture on $H_n(x,y,t;\mathcal{M}^d_B(SL(n,\mathbb{C})))$ a mirror symmetry conjecture is needed on the level of the H-polynomial. We finish by formulating such a conjecture, generalizing Conjecture 3.6 for H-polynomials:

Conjecture 5.17. *For all $d, e \in \mathbb{Z}$, with $(d,n) = (e,n) = 1$ we have*

$$H_{\mathrm{st}}^{B^e}\Big(x, y, t; \mathcal{M}_B^d(SL(n,\mathbb{C}))\Big) = H_{\mathrm{st}}^{\hat{B}^d}\Big(x, y, t; \mathcal{M}_B^e(PGL(n,\mathbb{C}))\Big),$$

where H_{st}^B is the stringy mixed Hodge polynomial twisted with a B-field, which can be defined identically as E_{st}^B in (6).

References

1. D. Alvis – "The duality operation in the character ring of a finite Chevalley group", *Bull. Amer. Math. Soc. (N.S.)* **1** (1979), no. 6, 907–911.
2. M.F. Atiyah and R. Bott – "The Yang-Mills equations over Riemann surfaces", *Philos. Trans. Roy. Soc. London Ser. A* **308** (1982), 523–615.
3. V.V. Batyrev and D. Dais – "Strong McKay correspondence, string-theoretic Hodge numbers and mirror symmetry", *Topology* **35** (1996), 901–929.
4. K. Corlette – "Flat G-bundles with canonical metrics", *J. Differential Geom.* **28** (1988), no. 3, 361–382.
5. C. Curtis – "Truncation and duality in the character ring of a finite group of Lie type", *J. Algebra* **62** (1980), no. 2, 320–332.
6. P. Deligne – "Équations différentielles á points singuliers réguliers", Lecture Notes in Math., vol. 163, Springer, Berlin-New York, 1970.
7. — , "Théorie de Hodge II, *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), 5–57.
8. R. Earl and F. Kirwan – "The Pontryagin rings of moduli spaces of arbitrary rank holomorphic bundles over a Riemann surface", *J. London Math. Soc.* (2) **60** (1999), no. 3, 835–846.
9. C. Faber – "A conjectural description of the tautological ring of the moduli space of curves", in "Moduli of curves and abelian varieties", 109–129, Aspects Math., E33, Vieweg, Braunschweig, 1999.
10. D. Freed and F. Quinn – "Chern-Simons theory with finite gauge group ", *Comm. Math. Phys.* **156** (1993), no. 3, 435–472.
11. G. Frobenius and I. Schur – "Über die reellen Darstellungen der endlichen Gruppen", *Sitzungsberichte der königlich preussichen Akademie der Wissenschaften* (1906), 186-208
12. A.M. Garsia and M. Haiman – "A remarkable q,t-Catalan sequence and q-Lagrange inversion", *J. Algebraic Combin.* **5** (1996) no. 3, 191-244.
13. W.M. Goldman and E.Z. Xia – "Rank One Higgs Bundles and Representations of Fundamental Groups of Riemann Surfaces", 2004, `math.DG/0402429`.
14. J.A Green – "The characters of the finite general linear groups", *Trans. Amer. Math. Soc.* **80** (1955), 402–447.
15. P. Griffiths, J. Harris – *Principles of algebraic geometry.* New York, Wiley 1978.
16. P.B. Gothen – "The Betti numbers of the moduli space of rank 3 Higgs bundles", *Internat. J. Math.* **5** (1994), 861–875.
17. M. Haiman – "t, q-Catalan numbers and the Hilbert scheme", *Discrete Math.* **193** (1998) no. 1-3, 201–224.
18. G. Harder and M.S. Narasimhan – "On the cohomology groups of moduli spaces of vector bundles over curves", *Math. Ann.* **212** (1975), 215–248.

19. T. Hausel – "Compactification of moduli of Higgs bundles", *J. Reine Angew. Math.* **503** (1998), 169–192.
20. — , *Geometry of the moduli space of Higgs bundles"*, Ph.D. thesis, University of Cambridge, 1998, `math.AG/0107040`.
21. — , "Quaternionic Geometry of Matroids", 2003, `math.AG/0308146`.
22. T. Hausel and N. Proudfoot – "Abelianization for hyperkähler quotients", (to appear in *Topology*), `math.SG/0310141`.
23. T. Hausel and F. Rodriguez-Villegas – "Mirror symmetry, Langlands duality and representations of finite groups of Lie type", (in preparation).
24. T. Hausel and B. Sturmfels – "Toric hyperkähler varieties", *Doc. Math.* **7** (2002), 495–534.
25. T. Hausel and M. Thaddeus – "Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles", *Proc. London Math. Soc.* **88** (2004), 632–658.
26. — , "Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles", *J. Amer. Math. Soc.*, **16** (2003), 303–329.
27. — , "Examples of mirror partners arising from integrable systems", *C. R. Acad. Sci. Paris*, **333** (4) (2001), 313–318.
28. — , "Mirror symmetry, Langlands duality and Hitchin systems", *Invent. Math.*, **153**, no. 1 (2003), 197–229.
29. N. Hitchin – "The self-duality equations on a Riemann surface", *Proc. London Math. Soc.* (3) **55** (1987), 59–126.
30. — , "Stable bundles and integrable systems", *Duke Math. J.* **54** (1987), 91–114.
31. — , "Lectures on special Lagrangian submanifolds", *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999)*, 151–182, Amer. Math. Soc., 2001.
32. H. Jordan – "Group characters of various types of linear groups", *Amer. J. Math*, **29** (1907), 387.
33. L. Katzarkov and T. Pantev – "Nonabelian (p,p) classes ", (in Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998)), *Int. Press Lect. Ser.*, 3, II, 625–715.
34. M. Kontsevich – "Motivic Integration", *Lecture at Orsay*, 1995.
35. G. Lehrer – "The characters of the finite special linear groups", *J. Algebra* **26** (1973), 564–583.
36. G. Lusztig – "Fermionic form and Betti numbers", `math.QA/0005010`.
37. A.D. Mednykh – "Determination of the number of nonequivalent coverings over a compact Riemann surface", *Soviet Mathematics Doklady* **19** (1978), 318–320.
38. M. Mehta – "Hodge structure on the cohomology of the moduli space of Higgs bundles", `math.AG/0112111`.
39. E. Markman – "Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces", *J. Reine Angew. Math.* **544** (2002), 61–82.
40. H. Nakajima – "Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras", *Duke Math. J.* **76** (1994) no. 2, 365–416.
41. M.S. Narasimhan and S. Ramanan – "Generalised Prym varieties as fixed points", *J. Indian Math. Soc.* **39** (1975), 1–19.
42. I. Schur – "Untersuchungen über die Darstellung der endlichen Gruppen durch Gebrochene Lineare Substitutionen", *J. Reine Angew. Math.* **132** (1907), 85.
43. J-P. Serre – "Exemples de variétés projectives conjuguées non homéomorphes", *C. R. Acad. Sci. Paris* **258** (1964), 4194–4196.

44. C.T. Simpson – "Nonabelian Hodge theory", *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, 747–756, Math. Soc. Japan, Tokyo, 1991.
45. — , "Higgs bundles and local systems", *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.
46. — , "Moduli of representations of the fundamental group of a smooth projective variety", I, *Inst. Hautes Études Sci. Publ. Math.* **79** (1994), 47–129.
47. — , "Moduli of representations of the fundamental group of a smooth projective variety", II, *Inst. Hautes Études Sci. Publ. Math.* **80** (1995), 5–79.
48. — , "The Hodge filtration on nonabelian cohomology", *Algebraic geometry—Santa Cruz 1995*, Proc. Symp. Pure Math. 62, ed. J. Kollár, R. Lazarsfeld, and D. Morrison, American Math. Soc., 1997.
49. A. Strominger, S.-T. Yau, and E. Zaslow – "Mirror symmetry is T-duality", *Nuclear Phys. B* **479** (1996), 243–259.
50. C. Vafa – "String vacua and orbifoldized LG models", *Modern Phys. Lett. A* **4** (1989), 1169–1185.
51. E. Verlinde – "Fusion rules and modular transformations in 2D conformal field theory", *Nucl. Phys. B* **300** (1988), 360.
52. D. Zagier – "Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula", (in Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 445–462, *Israel Math. Conf. Proc.*, 9, Bar-Ilan Univ., Ramat Gan, 1996.
53. E. Zaslow – "Topological orbifold models and quantum cohomology rings", *Comm. Math. Phys.* **156** (1993), 301–331.