ON THE NUMBER OF LATTICE HYPERPLANES WHICH ARE NEEDED TO COVER THE LATTICE POINTS OF A CONVEX BODY

Károly Bezdek[1] **and Tamás Hausel**

Abstract. Let $\mathbf{K}$ be a convex body of $\mathbf{E}^d$ and $\mathbf{L}$ be a $d-$dimensional lattice of $\mathbf{E}^d$, where $d \geq 2$. Assume that the union of n lattice hyperplanes of $\mathbf{L}$ covers the lattice points in $\mathbf{K} \cap \mathbf{L} \neq \emptyset$. In this note we prove that $n \geq c \cdot d^{-3} \cdot w_{\mathbf{L}}(\mathbf{K})$, where $w_{\mathbf{L}}(\mathbf{K})$ denotes the lattice width of $\mathbf{K}$ with respect to $\mathbf{L}$ and c is an absolute constant.

1. Introduction

A *convex body* of the $d-$dimensional Euclidean space $\mathbf{E}^d$ is a compact convex set with a non-empty interior. A *$d-$dimensional lattice* $\mathbf{L}$ of $\mathbf{E}^d$ is the set of all integral linear combinations of d linearly independent vectors in $\mathbf{E}^d$. A *lattice hyperplane* of $\mathbf{L}$ is a hyperplane of $\mathbf{E}^d$ the intersection of which with the $d-$dimensional lattice $\mathbf{L}$ is a $(d-1)-$dimensional sublattice. The *polar lattice* $\mathbf{L}^*$ of $\mathbf{L}$ is the lattice $\{x \in \mathbf{E}^d | \langle x, y\rangle \in \mathbf{Z}$ for all $y \in \mathbf{L}\}$, where $\mathbf{Z}$ is the set of integers and $\langle\ ,\ \rangle$ denotes the usual inner product of $\mathbf{E}^d$. Now, let $\mathbf{K}$ be a convex body of $\mathbf{E}^d$ and let $\mathbf{L}$ be a $d-$dimensional lattice of $\mathbf{E}^d$. Then the *lattice width* $w_{\mathbf{L}}(\mathbf{K})$ of $\mathbf{K}$ with respect to the lattice $\mathbf{L}$ is defined by

$$\min_{v \in \mathbf{L}^*} \left[\max\{\langle v, x\rangle | x \in \mathbf{K}\} - \min\{\langle v, x\rangle | x \in \mathbf{K}\}\right].$$

Many important properties of the lattice width are discussed in [8]. The problem which we want to raise and partially discuss in this note can be formulated as follows.

Problem. Take a convex body $\mathbf{K}$ of $\mathbf{E}^d$ and a $d-$dimensional lattice $\mathbf{L}$ of $\mathbf{E}^d$, where $d \geq 2$. Assume that the union of n lattice hyperplanes of $\mathbf{L}$ covers the lattice points in $\mathbf{K} \cap \mathbf{L} \neq \emptyset$. Prove or disprove that $n \geq c^* \cdot d^{-1} \cdot w_{\mathbf{L}}(\mathbf{K})$, where

[1]The work was supported by Hung. Nat. Found. for Sci. Research No. 326–0313.

$w_{\mathbf{L}}(\mathbf{K})$ denotes the lattice width of $\mathbf{K}$ with respect to the lattice $\mathbf{L}$ and $c^* > 0$ is some absolute constant.

Let $\mathbf{K}$ be a convex body of $\mathbf{E}^d$ containing the origin O of $\mathbf{E}^d$ as an interior point. Then the set $\mathbf{K}^* = \{x \in \mathbf{E}^d | \langle x, y\rangle \leq 1 \text{ for all } y \in \mathbf{K}\}$, which is also a convex body is called the polar body of $\mathbf{K}$. If $\mathbf{K}$ is symmetric about O, then

$$\frac{c_1^d}{d^d} \leq \text{Vol } (\mathbf{K}) \cdot \text{Vol } (\mathbf{K}^*) \leq \frac{c_2^d}{d^d},$$

where Vol ( ) stands for the volume of the corresponding set and $c_1 > 0$ and $c_2 > 0$ are absolute constants. The upper bound is due to Blaschke [2] and Santaló [9]. The lower bound is due to Bourgain and Milman [4]. Let $c = \frac{c_1}{8}$. (Clearly, we may assume that $0 < c \leq \frac{1}{2}$.) In this note we prove the following

Theorem 1. Let $\mathbf{K}$ be a convex body of $\mathbf{E}^d$ and $\mathbf{L}$ be a $d-$dimensional lattice of $\mathbf{E}^d$, where $d \geq 2$. If the union of n lattice hyperplanes of $\mathbf{L}$ covers the lattice points in $\mathbf{K} \cap \mathbf{L} \neq \emptyset$, then $n \geq c \cdot d^{-3} \cdot w_{\mathbf{L}}(\mathbf{K})$, where $w_{\mathbf{L}}(\mathbf{K})$ denotes the lattice width of $\mathbf{K}$ with respect to $\mathbf{L}$ and $c = \frac{c_1}{8}$ is an absolute constant.

Remark. Kannan and Lovász conjecture [8] that if $\mathbf{L}$ is a $d-$dimensional lattice of $\mathbf{E}^d$ and $\mathbf{B}$ is a $d-$dimensional ball in $\mathbf{E}^d$ with $\mathbf{B} \cap \mathbf{L} = \emptyset$, then $w_{\mathbf{L}}(\mathbf{B}) \leq c_3 \cdot d$, where $w_{\mathbf{L}}(\mathbf{B})$ denotes the lattice width of $\mathbf{B}$ with respect to $\mathbf{L}$ and $c_3 > 0$ is an absolute constant. This and the proof below would yield the inequality $n \geq \frac{1}{2 \cdot c_3} \cdot d^{-2} \cdot w_{\mathbf{L}}(\mathbf{K})$ in Theorem 1.

It is worth mentioning that for a centrally symmetric convex body $\mathbf{K}$ one can prove a stronger statement than Theorem 1. Namely, if $\mathbf{K}$ is a centrally symmetric convex body of $\mathbf{E}^d$, then the ellipsoid concentric and homothetic with factor $\frac{1}{\sqrt{d}}$ to the John-Löwner ellipsoid of $\mathbf{K}$ lies in $\mathbf{K}$ (see [6] and [7]), where $d \geq 2$. Thus, combining this result with the following proof of Theorem 1 one can get

Theorem 2. Let $\mathbf{K}$ be a centrally symmetric convex body of $\mathbf{E}^d$ and let $\mathbf{L}$ be a $d-$dimensional lattice of $\mathbf{E}^d$, where $d \geq 2$. If the union of n lattice hyperplanes of $\mathbf{L}$ covers the lattice points in $\mathbf{K} \cap \mathbf{L} \neq \emptyset$, then $n \geq c \cdot d^{-\frac{5}{2}} \cdot w_{\mathbf{L}}(\mathbf{K})$, where $w_{\mathbf{L}}(\mathbf{K})$ denotes the lattice width of $\mathbf{K}$ with respect to $\mathbf{L}$ and $c = \frac{c_1}{8}$ is an absolute constant.

Hence, it is sufficient to prove Theorem 1.

2. Proof of Theorem 1

Let $H_1, H_2, \dots, H_n$ be the n lattice hyperplanes of $\mathbf{L}$ the union of which covers the lattice points in $\mathbf{K} \cap \mathbf{L}$. The idea of the proof is the following. We approximate $\mathbf{K}$ by an ellipsoid $\mathbf{B} \subset \mathbf{K}$ and construct n congruent strips symmetric about the lattice hyperplanes $H_1, H_2, \dots, H_n$ such that the union of them covers $\mathbf{B}$. Then a theorem of Bang [1] (see also [3], [5]) which solves Tarski's plank problem implies that the sum of the widths of our strips is at least the width of $\mathbf{B}$ yielding the required inequality of Theorem 1. The details are as follows.

A well-known application of the John-Löwner ellipsoid ([6] and [7]) yields that there are concentric ellipsoids $\mathbf{B}$ and $d \cdot \mathbf{B}$ of $\mathbf{E}^d$ with the property that $\mathbf{B} \subset \mathbf{K} \subset d \cdot \mathbf{B}$, where $d \cdot \mathbf{B}$ is the homothetic image of $\mathbf{B}$ with the factor d. As a consequence of this we get that

$$w_{\mathbf{L}}(\mathbf{B}) \leq w_{\mathbf{L}}(\mathbf{K}) \leq w_{\mathbf{L}}(d \cdot \mathbf{B}) = d \cdot w_{\mathbf{L}}(\mathbf{B}). \tag{1}$$

As any affinity does not change neither the lattice width nor the fact that certain lattice hyperplanes cover some lattice points we may assume that the concentric ellipsoids $\mathbf{B}$ and $d \cdot \mathbf{B}$ are concentric $d-$dimensional (closed) balls.

The lattice hyperplanes $H_1, H_2, \dots, H_n$ generate a tiling $\mathcal{T}$ of the ball $\mathbf{B}$. More precisely, a tile T of $\mathcal{T}$ is the closure of an open connected component of $(\mathbf{E}^d \setminus \cup_{j=1}^n H_j) \cap \text{int } \mathbf{B}$, where int $\mathbf{B}$ denotes the interior of $\mathbf{B}$. Obviously, T is a convex body. Assume that there is a point t of some T the distances of which from the lattice hyperplanes $H_1, H_2, \dots, H_n$ are larger than $c_0 \cdot d^2 \cdot m_{\mathbf{L}}$, where $c_0 = \frac{4}{c_1}$ is an absolute constant and $m_{\mathbf{L}}$ denotes the largest distance between two consequtive parallel lattice hyperplanes of $\mathbf{L}$. Then the closed $d-$dimensional ball $\mathbf{B}_\epsilon$ centered at t with radius $c_0 \cdot d^2 \cdot m_{\mathbf{L}} + \epsilon$ for some $\epsilon > 0$ is disjoint from $\cup_{j=1}^n H_j$. Obviously, $\mathbf{B} \cap \mathbf{B}_\epsilon \subset T$. Then either $\mathbf{B} \setminus \mathbf{B}_\epsilon \neq \emptyset$ or $\mathbf{B} \setminus \mathbf{B}_\epsilon = \emptyset$.

First assume that $\mathbf{B} \setminus \mathbf{B}_\epsilon \neq \emptyset$. As $\mathbf{B}$ as well as $\mathbf{B}_\epsilon$ are balls it is easy to see that there exists a closed $d-$dimensional ball $\mathbf{B}_\delta$ with diameter $c_0 \cdot d^2 \cdot m_{\mathbf{L}} + \delta$ for some $0 < \delta < \epsilon$ such that $\mathbf{B}_\delta \subset \text{int } (\mathbf{B} \cap \mathbf{B}_\epsilon) \subset \text{int } T$. As the union of the lattice hyperplanes $H_1, H_2, \dots, H_n$ covers the lattice points in $\mathbf{K} \cap \mathbf{L}$ and so in $\mathbf{B} \cap \mathbf{L}$ therefore int $T \cap \mathbf{L} = \emptyset$. Hence, $\mathbf{B}_\delta \cap \mathbf{L} = \emptyset$. Now recall the following theorem of Kannan and Lovász [8]. If $\mathbf{L}$ is a $d-$dimensional lattice of $\mathbf{E}^d$ and $\mathbf{M}$ is a centrally symmetric convex body of $\mathbf{E}^d$ with $\mathbf{M} \cap \mathbf{L} = \emptyset$, then $w_{\mathbf{L}}(\mathbf{M}) \leq c_0 \cdot d^2$. Thus, $w_{\mathbf{L}}(\mathbf{B}_\delta) \leq c_0 \cdot d^2$. However, $\mathbf{B}_\delta$ is a closed ball of diameter $c_0 \cdot d^2 \cdot m_{\mathbf{L}} + \delta$ yielding $w_{\mathbf{L}}(\mathbf{B}_\delta) > c_0 \cdot d^2$, a contradiction.

Second we assume that $\mathbf{B} \setminus \mathbf{B}_\epsilon = \emptyset$. Then $\mathbf{B} \subset \mathbf{B}_\epsilon$ and so int $\mathbf{B} = \text{int } (\mathbf{B} \cap \mathbf{B}_\epsilon) = \text{int } T$. Thus, $(\text{int } \mathbf{B}) \cap \mathbf{L} = \emptyset$. Hence, the $d-$dimensional closed ball which is concentric to $\mathbf{B}$ and the radius of which is say, half of the radius of $\mathbf{B}$ is disjoint from $\mathbf{L}$. Thus, the mentioned theorem of Kannan and Lovász [8] immediately yields that $\frac{1}{2} \cdot w_{\mathbf{L}}(\mathbf{B}) \leq c_0 \cdot d^2$ that is

$$c \cdot d^{-2} \cdot w_{\mathbf{L}}(\mathbf{B}) \leq 1. \tag{2}$$

From (1) and (2) we get that $c \cdot d^{-3} \cdot w_{\mathbf{L}}(\mathbf{K}) \leq 1 \leq n$ which then proves Theorem 1.

Thus, without loss of generality we may assume that in each tile T of $\mathcal{T}$ there is no point the distances of which from the lattice hyperplanes $H_1, H_2, \ldots, H_n$ are larger than $c_0 \cdot d^2 \cdot m_{\mathbf{L}}$. In other words, the closed strips of width $2 \cdot c_0 \cdot d^2 \cdot m_{\mathbf{L}}$ symmetric about the lattice hyperplanes $H_1, H_2, \ldots, H_n$ cover each tile of $\mathcal{T}$ that is the union of the strips covers $\mathbf{B}$. Thus, via the theorem of Bang [1] the sum $n \cdot (2 \cdot c_0 \cdot d^2 \cdot m_{\mathbf{L}})$ of the widths of the strips is not smaller than the diameter diam $\mathbf{B}$ of $\mathbf{B}$. So we have $n \cdot (2 \cdot c_0 \cdot d^2 \cdot m_{\mathbf{L}}) \geq$ diam $\mathbf{B}$. Consequently, $n \cdot (2 \cdot c_0 \cdot d^2) \geq w_{\mathbf{L}}(\mathbf{B})$ and so

$$n \geq c \cdot d^{-2} \cdot w_{\mathbf{L}}(\mathbf{B}). \tag{3}$$

From (1) and (3) we get that $n \geq c \cdot d^{-3} \cdot w_{\mathbf{L}}(\mathbf{K})$. This completes the proof of Theorem 1.

References

1. T. Bang, *A solution of the "plank problem"*, Proc. Amer. Math. Soc. **2** (1951), 990–993.
2. W. Blaschke, *Über Affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid*, Leipziger Ber. **69** (1917), 306–318.
3. M. Bognár, *On Fenchel's solution of the plank problem*, Acta Math. Acad. Sci. Hung. **12** (1961), 269–270.
4. J. Bourgain and V. D. Milman, *Sections euclidiennes et volume des corps symetriques convexes dans R^n*, C. R. Acad. Sc. Paris **I/13** (1985), 435–438.
5. W. Fenchel, *On Th. Bang's solution of the plank problem*, Mat. Tidsskr. **B 1951** (1951), 49–51.
6. F. John, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniv. Volume (1948), 187–204.
7. F. Juhnke, *Volumenminimale Ellipsoidüberdeckungen*, Beiträge zur Alg. und Geom. **30** (1990), 143–153.
8. R. Kannan and L. Lovász, *Covering minima and lattice-point-free convex bodies*, Annals of Mathematics **128** (1988), 577–602.
9. L. A. Santaló, *Un invariente afin para los courpos convexos de espacios de n dimensiones*, Portugal Math. **8** (1949), 155–161.

K. Bezdek and T. Hausel,
Dept. of Geometry,

52A20 CONVEX SETS AND RELATED GEOMETRIC TOPICS; Covering

Eötvös L. University,
1088 Budapest, Rákóczi út 5,
Hungary.

of lattice points by lattice
hyperplanes.