

HODGE COHOMOLOGY OF GRAVITATIONAL INSTANTONS

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Abstract

We study the space of L^2 harmonic forms on complete manifolds with metrics of fibred boundary or fibred cusp type. These metrics generalize the geometric structures at infinity of several different well-known classes of metrics, including asymptotically locally Euclidean manifolds, the (known types of) gravitational instantons, and also Poincaré metrics on \mathbb{Q} -rank 1 ends of locally symmetric spaces and on the complements of smooth divisors in Kähler manifolds. The answer in all cases is given in terms of intersection cohomology of a stratified compactification of the manifold. The L^2 signature formula implied by our result is closely related to the one proved by Dai [25] and more generally by Vaillant [67], and identifies Dai's τ -invariant directly in terms of intersection cohomology of differing perversities. This work is also closely related to a recent paper of Carron [12] and the forthcoming paper of Cheeger and Dai [17]. We apply our results to a number of examples, gravitational instantons among them, arising in predictions about L^2 harmonic forms in duality theories in string theory.

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1. Introduction

The Hodge theorem for a compact Riemannian manifold (M, g) identifies the space $L^2 \mathcal{H}^*(M, g)$ of L^2 harmonic forms on M with the de Rham cohomology of this space. When M is no longer compact, $L^2 \mathcal{H}^*(M, g)$ is still of considerable interest, but no general theorem identifies it with a topologically defined group. In a number of special noncompact geometric situations, there are topological interpretations of this *Hodge cohomology* space. These include the Hodge theorem for manifolds with cylindrical ends in Atiyah, Patodi, and Singer [2], Cheeger's seminal work [16], [15], [18] on Hodge theory on spaces with conic and iterated conic singularities and its relationship with intersection theory, the considerable literature on Hodge cohomology on locally symmetric spaces (cf., in particular, [68], [62]), and Mazzeo's work [52], [55] concerning (asymptotically) geometrically finite hyperbolic quotients.

The aim of this paper is to prove a Hodge-type theorem for two different classes of Riemannian manifolds, special cases of which arise frequently in many interesting problems in geometry and mathematical physics. These are fibred boundary and fibred cusp metrics. Manifolds with fibred boundary metrics include all identified classes of gravitational instantons, the name coined by Hawking for complete hyperkähler four-manifolds. Special cases of fibred cusp metrics include the familiar *Poincaré* metrics in the theory of locally symmetric spaces. Slightly more specifically, a product of a compact manifold with an asymptotically locally Euclidean (ALE) manifold is an example of a general fibred boundary metric, and a product of a compact manifold with a finite volume hyperbolic cusp is an example of a fibred cusp metric. The most general case incorporates twisted versions of these examples and also only requires that the fibration structure exist at the boundary. In particular, there are two special

and very familiar subclasses of metrics amongst these: the ALE manifolds, also called scattering metrics, and manifolds with asymptotically cylindrical ends, also called b -metrics, which are fibred boundary and fibred cusp metrics, respectively, with trivial fibre. We describe these rigorously and in more detail below.

Let \overline{M} be a smooth compact manifold with boundary, and suppose that x is a boundary-defining function. (Thus x vanishes on $\partial\overline{M}$ and $dx \neq 0$ there.) We recall four classes of metrics in terms of their behaviour in some neighbourhood \mathcal{U} of $\partial\overline{M}$. In the first two of these, \overline{M} is arbitrary, but in the latter two we assume that $Y \equiv \partial\overline{M}$ is the total space of a fibration $\phi : Y \rightarrow B$ with fibre F .

- The metric g is called a b -metric on the interior M of \overline{M} if in \mathcal{U} it takes the form

$$g = \frac{dx^2}{x^2} + h,$$

where h is a smooth metric on $\partial\overline{M}$ (i.e., nondegenerate up to the boundary).

- The metric g is called a fibred cusp metric if in \mathcal{U} it takes the form

$$g = \frac{dx^2}{x^2} + \tilde{h} + x^2k,$$

where \tilde{h} is a smooth extension to \mathcal{U} of ϕ^*h , where h is an arbitrary metric on B , and k is a symmetric two-tensor on ∂M which restricts to a metric on each fibre F .

- The metric g is called a scattering metric if in \mathcal{U} it takes the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},$$

where h is a smooth metric on $\partial\overline{M}$;

- The metric g is called a fibred boundary metric if in \mathcal{U} it takes the form

$$g = \frac{dx^2}{x^4} + \frac{\tilde{h}}{x^2} + k,$$

where \tilde{h} and k are as above.

We have made a simplification here in not allowing cross-terms in these metrics, and members of these restricted classes are usually called exact b -metrics, and so on. This is not serious because, as discussed in §2, Hodge cohomology is invariant under quasi-isometries, and so these cross-terms can always be deformed away without changing the Hodge cohomology. In fact, we henceforth assume that a product structure $[0, 1)_x \times \partial Y$ is fixed on \mathcal{U} and that the metrics h and k in each of the four cases are independent of x with respect to this decomposition. We simply write h instead of \tilde{h} . This multiwarped product structure simplifies computations, and general

fibred boundary and fibred cusp metrics may be deformed to ones of this type without affecting the Hodge cohomology.

These metrics, or at least special cases of them, are all familiar, albeit in different coordinate systems. Thus if we set $x = e^{-t}$, then a b -metric becomes $dt^2 + h$ on $\mathbb{R}^+ \times \partial M$, so it has cylindrical ends, while the same change of coordinates transforms a fibred cusp metric to $dt^2 + h + e^{-2t}k$, which is a standard form for a \mathbb{Q} -rank 1 cusp when ∂M is a torus bundle over a torus. Similarly, if we set $x = 1/r$, then a scattering metric becomes $dr^2 + r^2h$ with $r \rightarrow \infty$, which is the standard form of the infinite end of a cone, and corresponds to the ALE class of gravitational instantons, such as the Eguchi-Hanson metric. Finally, a fibred boundary metric transforms under this coordinate change to $dr^2 + r^2h + k$, which is a common form for metrics in the asymptotically locally flat (ALF) and ALG classes of gravitational instantons, such as the Taub-NUT metric and reduced 2-monopole moduli space metric.

The obvious compactification of M as the manifold with boundary \overline{M} is useful for many purposes, but to state the Hodge theorems here, we define a new compactification X by collapsing the fibres F of $\partial\overline{M}$. When F is a sphere, X is a manifold, but in general X is a stratified space with one singular stratum, which we denote by B (hopefully this should cause no confusion), and principal stratum $M = X \setminus B$. A neighbourhood of B is a cone bundle with link F over B . In particular, when B is trivial, X is the one-point compactification of M , whereas when F is trivial, $X = \overline{M}$. In any case, we set $b = \dim B$ and $f = \dim F$ throughout this paper. X is called a *Witt space* if $H^{f/2}(F) = 0$, and as we explain below, the analysis is much simpler in this case.

Our main theorems relate the Hodge cohomology of M , with either a fibred boundary or fibred cusp metric, to the intersection cohomology of X . We refer to §2 for a review of these latter spaces and an explanation of the notation in the following.

THEOREM 1

Let (M, g) be a manifold of dimension n with fibred boundary metric. Then for any degree $0 \leq k \leq n$, there are natural isomorphisms

$$L^2 \mathcal{H}^k(M, g) \longrightarrow \begin{cases} \text{Im} (IH_{f+(b+1)/2-k}^k(X, B) \longrightarrow IH_{f+(b-1)/2-k}^k(X, B)), & b \text{ odd,} \\ IH_{f+b/2-k}^k(X, B), & b \text{ even,} \end{cases}$$

where the notation $IH_j^k(X, B)$ is explained in §2.2.2, equation (9).

THEOREM 2

Let (M, g) be a manifold of dimension n with fibred cusp metric. Then for $0 \leq k \leq n$,

there is a natural isomorphism

$$L^2 \mathcal{H}^k(M, g) \longrightarrow \text{Im} \left(IH_{\underline{m}}^k(X, B) \longrightarrow IH_{\bar{m}}^k(X, B) \right),$$

where \underline{m} and \bar{m} are the lower-middle and upper-middle perversities. These give the same cohomology when X is a Witt space, in which case we write simply

$$L^2 \mathcal{H}^k(M, g) \cong IH_{\underline{m}}^k(X, B).$$

The perversity functions that arise in Theorem 1 are somewhat nonstandard, but they appear naturally in this problem. We shall return in another paper to a closer examination of the relationships between perversity functions and weighted L^2 -cohomologies in these and other related geometric settings. However, for now, note that an interesting special case occurs when F is the sphere S^f , in which case X is a manifold and intersection cohomology reduces to ordinary cohomology. Then Theorem 1 becomes the following.

COROLLARY 1

Let (M, g) be a manifold of dimension n with a fibred boundary metric where the fibre of $Y = \partial M$ is a sphere; thus M is identified with the complement of the submanifold B in the compact manifold X . Then for any degree $0 \leq k \leq n$, there are natural isomorphisms

$$L^2 \mathcal{H}^k(M, g) \cong \begin{cases} H^k(X, B), & k \leq \frac{b}{2}, \\ H^k(X), & \frac{b}{2} < k < n - \frac{b}{2}, \\ H^k(X \setminus B), & k \geq n - \frac{b}{2} \end{cases} \quad (1)$$

if b is even, and

$$L^2 \mathcal{H}^k(M, g) \cong \begin{cases} H^k(X, B), & k \leq \frac{b-1}{2}, \\ \text{Im} \left(H^k(X, B) \longrightarrow H^k(X) \right), & k = \frac{b-1}{2} + 1, \\ H^k(X), & \frac{b+1}{2} < k < n - \frac{b+1}{2}, \\ \text{Im} \left(H^k(X) \longrightarrow H^k(X \setminus B) \right), & k = n - \frac{b+1}{2}, \\ H^k(X \setminus B), & k \geq n - \frac{b-1}{2} \end{cases} \quad (2)$$

if b is odd.

The specialization of Theorem 2 is even simpler.

COROLLARY 2

Let (M, g) be a manifold of dimension n with a fibred cusp metric, where $F = S^f$

as in the previous corollary. Then the compactification X is a manifold, and for any degree $0 \leq k \leq n$,

$$L^2 \mathcal{H}^k(M, g) = H^k(X).$$

Two degenerate cases of Theorems 1 and 2 are fairly well known.

THEOREM 1A

Let (M, g) be a manifold of dimension n with scattering metric. Then there are natural isomorphisms

$$L^2 \mathcal{H}^k(M, g) \longrightarrow \begin{cases} H^k(M, \partial M), & k < n/2, \\ \text{Im}(H^k(M, \partial M) \rightarrow H^k(M)), & k = n/2, \\ H^k(M), & k > n/2. \end{cases}$$

THEOREM 2A

Let (M, g) be a manifold of dimension n with b -metric. Then for any degree $0 \leq k \leq n$, there is a natural isomorphism

$$L^2 \mathcal{H}^k(M, g) \longrightarrow \text{Im}(H^k(X, B) \rightarrow H^k(X - B)) \cong \text{Im}(H^k(M, \partial M) \rightarrow H^k(M)).$$

Theorem 2A is proved in [2], while Theorem 1A is stated in [57], but the proof does not seem to be readily available in the literature. We prove these first as a warm-up to the more general cases because the proofs are structured similarly but present fewer analytic and geometric demands.

In all these results, but particularly in the latter two where the notation is more familiar, it is apparent that the topological expressions on the right depend on the stratification (X, B) and not just on X . The traditional hypotheses about perversities were designed to make the corresponding intersection cohomology spaces independent of stratification, but as explained in §2, this independence is lost in certain degrees because of our use of slightly more general perversity functions.

As already indicated, there is a simpler proof of Theorem 2 when X is a Witt space. The reason is that with this hypothesis the range of d is closed in all degrees, and so the space of L^2 harmonic forms is isomorphic to the L^2 -cohomology. One can then directly apply techniques of [68] which are mainly sheaf-theoretic and topological. We discuss this further in §5.5. Note that since $L^2 \mathcal{H}^{n/2}(M, g)$ depends only on the conformal class of g , we can also compute the middle degree Hodge cohomology for fibred boundary metrics when X is Witt. In fact, this case shows a trick that is used to prove Theorem 1 in certain cases: if $k < n/2$ and $\omega \in L^2 \mathcal{H}^k(M, \hat{g})$, then $\omega \wedge \eta \in L^2 \mathcal{H}^{n-k}(M \times S^{n-2k}, \hat{g})$, in particular, is a middle-degree class, where \hat{g} is the product of a fibred boundary metric on M and the standard metric on the sphere,

and where η is the volume form on S^{n-2k} . The easier analytic argument now works provided the compactification $X \times S^{n-2k}$ is a Witt space, which requires that both $H^{(f+n)/2-k}(F) = H^{k-(f+n)/2}(F) = 0$ (one of which is of course always true). This can always be used, for example, to reduce Theorem 1A to a simple special case of Corollary 2, which follows easily from Theorem 2A. In the end, however, this would express the Hodge cohomology for a fibred boundary metric in terms of the homology of a different space altogether and hence is certainly less preferable.

In any case, when X is not a Witt space, one needs to do something to confront the main issue that the range of d is not closed. The analytic machinery we introduce in §§4 and 5 provides one avenue for doing this. Another possible approach involves Carron’s notion in [12] of nonparabolicity at infinity. In fact, Carron has used this method to characterize the L^2 -cohomology of arbitrary complete Riemannian manifolds with flat ends. There is a substantial, but not complete, overlap of his results with ours; we comment on this further in §6.

Other work very closely related to ours is a forthcoming paper by Cheeger and Dai [17] concerning the L^2 -cohomology of cone bundles. It is likely that we could deduce some of their results using the methods here and using parametrices in the edge calculus (see [53]). Their methods should certainly give some of our results too.

Hodge theorems are of course closely related to index theorems, and Theorems 1 and 2 imply a signature formula.

COROLLARY 3

Let (M, g) be a fibred boundary or fibred cusp metric. Then

$$\text{sgn}_{L^2}(M, g) = \text{sgn}(\text{Im}(IH_{\underline{m}}^*(X, B) \rightarrow IH_{\underline{m}}^*(X, B))).$$

This corollary is very closely related to the signature theorem for fibred cusp metrics proved by Dai [25, Theorem 3] in a special case (using Müller’s L^2 index theorem for manifolds with ends which are locally symmetric of \mathbb{Q} rank 1), and in more generality by Vaillant [67, page 103]. This theorem of Dai and Vaillant states that

$$\text{sgn}_{L^2}(M, g) = \text{sgn}(\text{Im}(H^*(M, \partial M) \rightarrow H^*(M))) + \tau, \tag{3}$$

where the final term is the τ -invariant of the fibration of ∂M defined by Dai [25]. Combining this with the above corollary gives the very interesting equality

$$\tau = \text{sgn}(\text{Im}(IH_{\underline{m}}^*(X, B) \rightarrow IH_{\underline{m}}^*(X, B))) - \text{sgn}(\text{Im}(H^*(M, \partial M) \rightarrow H^*(M))). \tag{4}$$

We discuss this further in §§6 and 7, and we shall explore this identity in another paper.

Our initial and primary motivation for this work came from predictions arising in duality theories in string theory, some of which we describe in §7. In particular,

physicists have predicted the dimensions of the spaces $L^2\mathcal{H}^*$ on the moduli space of magnetic monopoles on \mathbb{R}^3 (see [63]), multi-Taub-NUT gravitational instantons (see [64]), quiver varieties (see [66]), and certain G_2 - and $\text{Spin}(7)$ -manifolds (see [10]). In many of these cases, the metrics are of fibred boundary type, and our Theorem 1 confirms most of these predictions. A notable exception is the prediction for the G_2 -manifold in [10], which our results prove is false. Most of these results have been or could be proved by techniques available in the literature (see [43], [67]). In particular, as we explain in §6, taking [43] into account, Dai's signature theorem in [25] suffices to calculate $L^2\mathcal{H}^*$ for all hyperkähler metrics of fibred boundary type, that is, all known gravitational instantons. Our methods and results give a unified approach and have the advantage of using only basic asymptotic properties of the metric rather than any refined properties, for example, having a large symmetry group or special holonomy group. Moreover, the interpretation of Hodge cohomology in terms of the intersection cohomology of a compactification is very much in the spirit of the original Hodge theorem for compact manifolds. We hope (see [22]) that the results here, as well as those in [46], suggest the correct form for a general result that would encompass the remaining cases of these predictions.

This paper is organized as follows. In §2 we review L^2 -cohomology and the basics of intersection theory, focusing on spaces with only one singular stratum. We also define two different versions of weighted L^2 -cohomology. A review of the proof of the Hodge theorem for compact manifolds is presented in the brief §3; this provides the basic analytic structure for the proofs of our main theorems, and we emphasize here the main analytic points for which replacements are needed. The Hodge theorems for b - and scattering metrics are proved in §4; this is accompanied by a review of the requisite analysis of b -pseudodifferential operators. The more general Hodge theorems are proved in §5, first by identifying the Hodge cohomology with weighted cohomology, and then by relating weighted cohomology to intersection cohomology. We briefly explain the relationship of our results to those of Dai, Vaillant, Cheeger, Hitchin, and Carron in §6. Finally, in §7, we discuss the special cases of these theorems which provided our original motivation, where M is one of the *gravitational instantons* of currency in physics.

2. Cohomologies

We discuss various cohomology theories (in a loosely construed sense) that play significant rôles in this paper.

As a general word about notation, if \mathcal{F} is some function space on the Riemannian manifold (M, g) , then $\mathcal{F}\Omega^*(M)$ denotes the space of sections of the exterior bundle $\bigwedge^*(M)$ with this regularity. When $\mathcal{F} = L^2$ or a Sobolev space, then we indicate the dependence on the metric by writing $\mathcal{F}\Omega^*(M, g)$.

2.1. L^2 - and Hodge cohomology

We start with a review of some facts about L^2 -cohomology and its relationship to the space of L^2 harmonic forms.

The absolute cohomology $H^k(M)$ of a general (open) manifold M is identified with the de Rham complex of smooth forms with unrestricted growth at infinity:

$$\dots \longrightarrow \mathcal{C}^\infty \Omega^{k-1}(M) \longrightarrow \mathcal{C}^\infty \Omega^k(M) \longrightarrow \mathcal{C}^\infty \Omega^{k+1}(M) \longrightarrow \dots$$

Similarly, its compactly supported cohomology $H_c^k(M)$ is computed by the de Rham complex of smooth compactly supported forms:

$$\dots \longrightarrow \mathcal{C}_0^\infty \Omega^{k-1}(M) \longrightarrow \mathcal{C}_0^\infty \Omega^k(M) \longrightarrow \mathcal{C}_0^\infty \Omega^{k+1}(M) \longrightarrow \dots \tag{5}$$

It is well known (see [26]) that these same cohomologies can also be computed using the complexes of distributional forms $(\mathcal{C}^{-\infty} \Omega^*(M), d)$ and $(\mathcal{C}_0^{-\infty} \Omega^*(M), d)$. However, there are many interesting complexes incorporating restrictions on regularity and growth at infinity between these extremes. The most popular of these (for good reason) is L^2 -cohomology in the presence of a Riemannian metric. To define it, complete the differential complex (5) with respect to the norms on the exterior bundles associated to g and the volume form dV_g so as to obtain the Hilbert complex

$$\dots \longrightarrow L^2 \Omega_g^{k-1}(M) \longrightarrow L^2 \Omega_g^k(M) \longrightarrow L^2 \Omega_g^{k+1}(M) \longrightarrow \dots \tag{6}$$

Strictly speaking, this is not a complex since the differential d is defined at each stage only on a dense subspace. Thus the space of degree k should be defined as $\{\omega \in L^2 \Omega^k(M, g) : d\omega \in L^2 \Omega^{k+1}(M, g)\} \subseteq H^1 \Omega^k(M, g)$. The cohomology of (6) is called the L^2 -cohomology of M and denoted $H_{(2)}^k(M, g)$. In other words,

$$H_{(2)}^k(M, g) = \{\omega \in L^2 \Omega^k(M, g) : d\omega = 0\} / \{d\eta : \eta \in L^2 \Omega_g^{k-1}(M), d\eta \in L^2 \Omega_g^k(M)\}.$$

To set this into context, recall the Kodaira decomposition theorem, which states that for arbitrary manifolds, there is an orthogonal decomposition

$$L^2 \Omega^k(M, g) = L^2 \mathcal{H}^k(M, g) \oplus \overline{d \mathcal{C}_0^\infty \Omega^{k-1}} \oplus \overline{\delta \mathcal{C}_0^\infty \Omega^{k+1}}, \tag{7}$$

where the first summand consists of forms $\omega \in L^2 \Omega^k(M, g)$ such that both $d\omega = \delta\omega = 0$. This is the space of L^2 harmonic fields, or Hodge cohomology, and is our main object of study. The proof of the Kodaira decomposition is closely related to the essential self-adjointness of $d + \delta$ on $L^2 \Omega^*(M, g)$, which in turn follows from Gaffney's L^2 Stokes theorem (cf. [26]). It follows from this that the subspace of closed

forms is precisely the sum of the first two summands here, and hence

$$H_{(2)}^k(M, g) = L^2\mathcal{H}^k(M, g) \oplus \{d\eta \in L^2\Omega^k(M, g) : \eta \in L^2\Omega_g^{k-1}(M)\} / \overline{\{d\eta \in L^2\Omega_g^k(M), \eta \in L^2\Omega_g^{k-1}(M)\}}.$$

In particular, when the range of d from $L^2\Omega^{k-1}$ to $L^2\Omega^k$ is not closed, then $H_{(2)}^k$ is infinite-dimensional. This behaviour occurs in many instances, for example, on Euclidean space, and indeed is the reason for some of the difficulties in understanding L^2 -cohomology. However, we can define the reduced L^2 -cohomology

$$\overline{H}_{(2)}^k(M, g) = \{\omega \in L^2\Omega^k(M, g) : d\omega = 0\} / \overline{\{d\eta \in L^2\Omega^k(M, g), \eta \in L^2\Omega^{k-1}(M, g)\}}.$$

Combined with (7), this gives the useful isomorphism

$$\overline{H}_{(2)}^k(M, g) \cong L^2\mathcal{H}^k(M, g),$$

which reveals the surprising fact—certainly not apparent from the basic definition—that Hodge cohomology is invariant under quasi-isometric changes of the metric. In other words, if two metrics are comparable, $g' \leq cg, g \leq c'g'$ for constants $c, c' > 0$, then $\overline{H}_{(2)}^*(M, *)$ is the same when computed with respect to either metric, and hence the same is true for $L^2\mathcal{H}^*(M, *)$. Moreover, if $H_{(2)}^k(M, g)$ is finite-dimensional, then it is naturally isomorphic to $L^2\mathcal{H}^k(M, g)$.

Reduced L^2 -cohomology is not quite as tractable as it might appear. For example, it is quite important in calculations that there is a Mayer-Vietoris sequence for unreduced L^2 -cohomology, but this is true only in special cases for reduced L^2 -cohomology.

2.2. Intersection cohomology

We now review some definitions and facts about the intersection cohomology of stratified spaces.

2.2.1. Generalities

Let X be a stratified space of real dimension n with no codimension one singularities. We always assume, without further comment, that this space satisfies some extra hypotheses: if a point $q \in X$ is contained in the stratum of codimension ℓ , then it has a neighbourhood \mathcal{U} diffeomorphic to $\mathcal{V} \times C(L)$, where \mathcal{V} is diffeomorphic to a Euclidean ball and is contained in that stratum and $C(L)$ is the cone over a link L , which itself is a stratified space (of dimension $\ell - 1$).

A perversity \mathfrak{p} is an n -tuple of natural numbers $(p(1), p(2), \dots, p(n))$ satisfying $p(1) = p(2) = 0$ and $p(\ell - 1) \leq p(\ell) \leq p(\ell - 1) + 1$ for all $\ell \leq n$. Associated to such

a space X and perversity \mathfrak{p} is the intersection complex $IC_{\mathfrak{p}}^*(X)$, where, roughly speaking, the integer $p(\ell)$ regulates the dimension of the intersection of generic chains with the stratum of codimension ℓ . The homology of this complex is the intersection homology $IH_{\mathfrak{p}}^p(X)$. The dual intersection cohomology $IH_{\mathfrak{p}}^*(X)$ is more relevant to our purposes.

The following result is fundamental.

PROPOSITION 1 (see [37])

Let X be a stratified space, and let (\mathcal{L}^*, d) be a complex of fine sheaves on X with cohomology $H^*(X, \mathcal{L})$. Suppose that if \mathcal{U} is a neighbourhood in the principal (smooth) stratum of X , then $H^*(\mathcal{U}, \mathcal{L}) = H^*(\mathcal{U}, \mathbb{C})$, while if q lies in a stratum of codimension ℓ and $\mathcal{U} = \mathcal{V} \times C(L)$ as above, then

$$H^k(\mathcal{U}, \mathcal{L}) \cong IH_{\mathfrak{p}}^k(\mathcal{U}) = \begin{cases} IH_{\mathfrak{p}}^k(L), & k \leq \ell - 2 - p(\ell), \\ 0, & k \geq \ell - 1 - p(\ell). \end{cases} \tag{8}$$

Then there is a natural isomorphism between the hypercohomology $\mathbb{H}^*(X, \mathcal{L}^*)$ associated to this complex of sheaves and $IH_{\mathfrak{p}}^*(X)$, the intersection cohomology of perversity \mathfrak{p} .

Thus intersection cohomology with perversity \mathfrak{p} may be calculated using any fine sheaf so long as its local cohomology satisfies (8), which we refer to as the *local computation* (see also [18] and [7] for more on this).

This proposition is modified later in Section 2 to provide a link between weighted cohomology and intersection cohomology.

2.2.2. Intersection cohomology for spaces with only two strata

Suppose now that X has only two strata: the principal smooth stratum and the stratum of codimension ℓ , which we denote B . For convenience we assume that B is connected, although all results here generalize easily to allow B to have many components (even of different dimensions, so long as their closures are disjoint). We denote by F the link associated to any point $q \in B$. This is a smooth compact manifold of dimension $\ell - 1$ with trivial stratification, and $IH_{\mathfrak{p}}^*(F) = H^*(F)$ no matter the perversity \mathfrak{p} . We associate to X the manifold with boundary \overline{M} which is obtained by blowing up B , that is, replacing B by its spherical normal bundle. (This may be visualized as the complement of a tubular neighbourhood of B in X .) Notice that ∂M fibres over B with fibre F .

The only part of the perversity which affects $IC_{\mathfrak{p}}^*(X)$, and hence $IH_{\mathfrak{p}}^*(X)$, is the value $p(\ell)$. The basic hypothesis on \mathfrak{p} implies that $0 \leq p(\ell) \leq \ell - 2$, and by (8), only the spaces $H^k(F)$, $0 \leq k \leq \ell - 2 - p(\ell)$, are relevant for the calculation

of these intersection spaces. We now introduce an extension of these definitions by allowing $p(\ell)$ to take on any integer value. This does not give anything dramatically new; when $p(\ell) \leq -1$, the local calculations (8) agree with those for the computation of $H^*(X - B) = H^*(\overline{M})$, whereas when $p(\ell) \geq \ell - 1$, then they agree with those for the computation of $H^*(X, B) \cong H^*(M, \partial M)$. Thus for any $j \in \mathbb{Z}$ we fix the notation

$$IH_j^*(X, B) = \begin{cases} H^*(X - B), & j \leq -1, \\ IH_p^*(X), & 0 \leq j \leq \ell - 2, \\ H^*(X, B), & j \geq \ell - 1, \end{cases} \tag{9}$$

where in the middle case, p is any perversity with $p(\ell) = j$.

We note some properties of these extended groups. First,

$$IH_j^k(X, B) \cong IH_{\ell-2-j}^{n-k}(X, B),$$

just as with the standard intersection cohomology groups. Next, suppose that X is smooth and endowed with the stratification $(X \setminus B, B)$, where B is just a distinguished smooth $(n - \ell)$ -dimensional submanifold. The link at any point $q \in B$ is $S^{\ell-1}$, and so if $\mathcal{U} = \mathcal{V} \times C(S^{\ell-1})$ is a neighbourhood of a point $q \in B$, then

$$IH_p^k(\mathcal{U}) = \begin{cases} IH_p^k(S^{\ell-1}) = H^k(S^{\ell-1}), & k \leq \ell - 2 - p(\ell), \\ 0, & k \geq \ell - 1 - p(\ell). \end{cases}$$

If $0 \leq p(\ell) \leq \ell - 2$, this equals \mathbb{C} for $k = 0$ and 0 for $k > 0$, which is the same local calculation as for the ordinary cohomology of a smooth manifold; hence $IH_j^*(X, B) = H^*(X)$ in this case. As expected, this is independent of the submanifold B , and hence of the choice of stratification of X , because the perversity p is a *traditional* one. However, in the other cases, when $j \leq -1$ or $j \geq \ell - 1$, $IH_j^*(X, B)$ depends strongly on B . We also remark that this extension allows us to consider spaces with a codimension one stratum, that is, a boundary. In this case, the link of a point on the boundary is any point, so the local calculations corresponding to $j \leq -1$ and $j \geq 0$ give absolute and relative cohomologies, respectively.

This use of nonstandard perversities is now common in intersection theory; for example, they enter into calculations of weighted cohomology on locally symmetric spaces (see [60]).

Now we return to the class of manifolds of interest in this paper, where M is the interior of a compact n -dimensional manifold with boundary \overline{M} , such that ∂M is the total space of a fibration with base B and fibre F , $\dim B = b$, and $\dim F = f$. M has two natural compactifications: the first, \overline{M} , is obtained simply by adding ∂M , while the second, X , is the result of collapsing the fibres of ∂M in \overline{M} . We write the image of ∂M in X as B . Thus X is a stratified space with a single singular stratum, B , of codimension $\ell = n - b$.

Let us calculate the extended intersection groups $IH_j^*(\overline{M}, B)$. The first step is to localize the calculation around B . Let $N(B)$ denote a normal neighbourhood of B , so that $X = M \sqcup N(B)$. The overlap $M \cap N(B)$ retracts onto $\partial M \cong \partial N(B)$. For each j there is a Mayer-Vietoris sequence

$$\longrightarrow IH_j^k(\overline{M}, B) \longrightarrow H^k(M) \oplus IH_j^k(N(B), B) \longrightarrow H^k(\partial M) \longrightarrow .$$

This is elementary since $M \cap N(B)$ retracts onto a compact subset of $X \setminus B$. In any case, it suffices to calculate the groups $IH_j^k(N(B), B)$.

Assume that (\mathcal{L}_j^*, d) is a complex of fine sheaves, the hypercohomology of which is isomorphic to $IH_j^*(N(B), B)$. Choose an open cover $\{\mathcal{U}_\alpha\}$ of B in X such that the bundle $\partial M \rightarrow B$ is trivial over each \mathcal{U}_α ; this lifts to the cover $\underline{\mathcal{U}} = \{\phi^{-1}(\mathcal{U}_\alpha)\}$ of $N(B)$. The bigraded complex of Čech cochains with coefficients in \mathcal{L}_j^* ,

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & \\ & \uparrow d & & \uparrow d & & \uparrow d & \\ C^0(\underline{\mathcal{U}}, \mathcal{L}_j^2) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, \mathcal{L}_j^2) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, \mathcal{L}_j^2) & \xrightarrow{\delta} & \dots \\ & \uparrow d & & \uparrow d & & \uparrow d & \\ C^0(\underline{\mathcal{U}}, \mathcal{L}_j^1) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, \mathcal{L}_j^1) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, \mathcal{L}_j^1) & \xrightarrow{\delta} & \dots \\ & \uparrow d & & \uparrow d & & \uparrow d & \\ C^0(\underline{\mathcal{U}}, \mathcal{L}_j^0) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, \mathcal{L}_j^0) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, \mathcal{L}_j^0) & \xrightarrow{\delta} & \dots \end{array}$$

has hypercohomology that can be calculated using either of the two associated spectral sequences (cf. [8]). Consider first the spectral sequence that starts with with the vertical differential d . Any intersection of neighbourhoods $\phi^{-1}(\mathcal{U}_\alpha)$ is of the form $(0, s) \times F \times \mathcal{U}'$, where \mathcal{U}' is an intersection of the neighbourhoods \mathcal{U}_α in B . The

local calculation (8) gives that the E_1 -term of the spectral sequence is

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \\
 0 & \xrightarrow{\delta} & 0 & \xrightarrow{\delta} & 0 & \xrightarrow{\delta} & \dots \\
 \\
 C^0(\underline{\mathcal{U}}, H^{\ell-2-j}(F)) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, H^{\ell-2-j}(F)) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, H^{\ell-2-j}(F)) & \xrightarrow{\delta} & \dots \\
 \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 C^0(\underline{\mathcal{U}}, H^1(F)) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, H^1(F)) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, H^1(F)) & \xrightarrow{\delta} & \dots \\
 \\
 C^0(\underline{\mathcal{U}}, H^0(F)) & \xrightarrow{\delta} & C^1(\underline{\mathcal{U}}, H^0(F)) & \xrightarrow{\delta} & C^2(\underline{\mathcal{U}}, H^0(F)) & \xrightarrow{\delta} & \dots
 \end{array}$$

In all the rows below level $\ell - 1 - j$, this is the same as the E_1 -term of the Leray-Serre spectral sequence for the bundle $\partial M \rightarrow B$, but all rows at level $\ell - 1 - j$ and above are set to zero. The next differential, d_1 , is the horizontal Čech differential δ . Using it to calculate the E_2 -term gives a bigraded diagram that agrees below level j with the E_2 -term of the same Leray-Serre spectral sequence. The higher terms E_k of this truncated Leray-Serre spectral sequence converge to the extended intersection cohomology $IH_j^*(N(B), B)$.

One can, for example, see by examining the further terms of the resulting spectral sequence that this truncation does not change the limit $E^{p,q}$ for $p + q < \ell - 1 - j$. Thus for $k < \ell - 1 - j$, $\sum_{p+q=k} E^{p,q} = IH_j^k(N(B), B) \cong H^k(\partial M)$. Using this in the Mayer-Vietoris sequence, we find that for $k < \ell - 1 - j$, $IH_j^k(\overline{M}, B) \cong H^k(M)$.

2.3. Weighted cohomology and intersection cohomology

As we have already explained, one difficulty with L^2 -cohomology is that in many cases the range of d is not closed; this leads to the (somehow spurious) infinite dimensionality of the quotient spaces. There are many ways to circumvent this, each of which involves a perturbation of the Hilbert spaces $L^2\Omega^*(M)$. One possibility is to use an L^p -completion, $p \neq 2$ (cf. [69]), but the loss of the Hilbert space structure is unfortunate and unnecessary. An alternate and preferable method involves the use of weighted L^2 -norms. The associated theory is called weighted cohomology.

We do not attempt a general definition of weighted cohomology, but specialize

directly to the cases of interest here. Thus let \overline{M} be a compact smooth manifold with boundary, with boundary-defining function x . If $a \in \mathbb{R}$, then $x^a L^2(X)$ is the space of all functions (or forms) $u = x^a v$, where $v \in L^2(X)$. In the following, fix a fibred boundary metric g_{fb} and a fibred cusp metric g_{fc} on X ; we may as well assume that $g_{fc} = x^2 g_{fb}$. We also use the notation Ω_{fb}^* and Ω_{fc}^* . This is explained in §5, but for now we say only that this denotes the normalizations of the exterior bundle corresponding to g_{fb} and g_{fc} which are best suited for computations.

Definition 1

For $a \in \mathbb{R}$, define the Hilbert complexes

$$\dots \rightarrow x^a L^2 \Omega_{fc}^{k-1}(M, g_{fc}) \rightarrow x^a L^2 \Omega_{fc}^k(M, g_{fc}) \rightarrow x^a L^2 \Omega_{fc}^{k+1}(M, g_{fc}) \rightarrow \dots \quad (10)$$

and

$$\begin{aligned} \dots \rightarrow x^{a-1} L^2 \Omega_{fb}^{k-1}(M, g_{fb}) \rightarrow x^a L^2 \Omega_{fb}^k(M, g_{fb}) \\ \rightarrow x^{a+1} L^2 \Omega_{fb}^{k+1}(M, g_{fb}) \rightarrow \dots \end{aligned} \quad (11)$$

as completions of the de Rham complex of smooth compactly supported forms with respect to the stated norms at each degree. We then set $WH^k(M, g_{fc}, a)$ and $\mathscr{W}H^k(M, g_{fb}, a)$, respectively, to be the cohomology of these two complexes at degree k . Thus

$$WH^k(M, g_{fc}, a) = \frac{\{\omega \in x^a L^2 \Omega_{fc}^k(M, g_{fc}) : d\omega = 0\}}{\{d\eta : \eta \in x^a L^2 \Omega_{fc}^{k-1}(M, g_{fc}), d\eta \in x^a L^2 \Omega_{fc}^k(M, g_{fc})\}} \quad (12)$$

and

$$\mathscr{W}H^k(M, g_{fb}, a) = \frac{\{\omega \in x^a L^2 \Omega_{fb}^k(M, g_{fb}) : d\omega = 0\}}{\{d\eta : \eta \in x^{a-1} L^2 \Omega_{fb}^{k-1}(M, g_{fb}), d\eta \in x^a L^2 \Omega_{fb}^k(M, g_{fb})\}}. \quad (13)$$

We suppress the metric in this notation when it is unambiguous. Since fibred boundary and fibred cusp metrics are conformally related, these two cohomologies are essentially the same. More precisely,

$$\mathscr{W}H^k(M, g_{fb}, a) = WH^k(M, g_{fc}, (n/2) - k + a). \quad (14)$$

Thus for the remainder of this section we discuss only WH^k .

Our main goal now is to relate the weighted cohomology for fibred cusp metrics to intersection cohomology.

PROPOSITION 2

Suppose that $k - 1 + a - f/2 \neq 0$ for $0 \leq k \leq f$. Then

$$WH^*(M, g_{fc}, a) \cong IH^*_{[a+f/2]}(X, B),$$

where $[a + f/2]$ is the greatest integer less than or equal to $a + f/2$.

Proof

We prove this by considering the complex of sheaves associated to $x^a L^2$, so that its hypercohomology equals $WH^*(M, a)$, and show that its entries satisfy the appropriate local calculation (8). In order to apply Proposition 1, however, we must first show that this sheaf is fine.

For each k , define the presheaf

$$\mathcal{L}_a^k(\mathcal{U}) = \begin{cases} L^2\Omega^k(\mathcal{U}), & \mathcal{U} \cap B = \emptyset, \\ x^a L^2\Omega_{fc}^k(\mathcal{U} \setminus (\mathcal{U} \cap B)), & \mathcal{U} \cap B \neq \emptyset, \end{cases}$$

where the notation in this last line should be self-explanatory. The associated sheaf is denoted \mathcal{L}_a^k .

In general, the sheaf of (weighted) L^2 -forms on the compactification of a manifold is not fine unless one has a *good* partition of unity, that is, such that the cutoff functions χ_α have gradients that are bounded uniformly in α . However, such partitions of unity are easy to construct for fibred cusp metrics (cf. the essentially identical discussion in [68, proof of Proposition 4.4]). We construct a good cover and partition of unity on \overline{M} as follows. First, choose a finite cover $\{\mathcal{U}_\alpha\}$ of (the interior of) M such that all j -fold intersections of these sets are contractible. Choose a similar cover $\{\mathcal{V}_\beta\}$ of B , and let $\mathcal{U}'_\beta = \phi^{-1}(\mathcal{V}_\beta) \times (0, \epsilon)$, where $\phi : \partial M \rightarrow B$. Then for ϵ sufficiently small, $\{\mathcal{U}_\alpha, \mathcal{U}'_\beta\} = \{\mathcal{U}''_\gamma\}$ is a good cover for \overline{M} . Now choose a partition of unity $\{\chi''_\gamma\}$, where the elements satisfy no additional extra requirements over sets not intersecting \overline{M} but which have the form $\phi^* \tilde{\chi}_\beta(y) \hat{\chi}(x)$ on neighbourhoods intersecting the boundary; then it is easy to see that $|d\chi''_\alpha| \leq C$ uniformly in α , as required.

Now turn to the local cohomology computation, following the discussion in [68, §2]. Over neighbourhoods not intersecting B in X , we apply the standard Poincaré lemma. On the other hand, suppose that $\mathcal{U} = \mathcal{V} \times F \times (0, \epsilon)$, where $\mathcal{V} \subset B$. First, by an adapted form of the Künneth theorem, the (weighted) L^2 -cohomology of the product neighbourhood \mathcal{U} is the same as the weighted L^2 -cohomology of $F \times (0, \epsilon)$; this is valid since the weight function x^a does not depend on $y \in \mathcal{V}$. This reduces us to computing $WH^k(F \times (0, \epsilon), dx^2/x^2 + x^2k_F, a)$, where for the moment we write the metric on F as k_F to distinguish it from the form degree. Setting $r = -\log x$ to accord with the notation of [68], and regarding the weight as a norm on the trivial local coefficient system, then the conclusion of [68, Corollary 2.34] is the following.

- (i) $WH^k(X, a)$ is finite-dimensional; that is, the denominator in its definition is closed if and only if the same is true for $WH^j((0, \epsilon), dx^2/x^2, k - j + a - (f/2))$, and simultaneously $H^{k-j}(F) \neq 0$, $j = 0, 1$.
- (ii) If this condition is satisfied, then

$$WH^k(X, a) \cong WH^0((0, \epsilon), dx^2/x^2, k + a - f/2) \otimes H^k(F) \oplus WH^1((0, \epsilon), dx^2/x^2, k - 1 + a - f/2) \otimes H^{k-1}(F).$$

In fact, this follows once again from the Künneth theorem in [68]. We have

$$WH^0((0, \epsilon), dx^2/x^2, b) = \begin{cases} \mathbb{C}, & b < 0, \\ 0, & \text{otherwise,} \end{cases}$$

whereas $WH^1((0, \epsilon), dx^2/x^2, b) = 0$ if $b \neq 0$ and is infinite-dimensional when $b = 0$ (the range of d is not closed at weight zero).

Returning to the local calculation, we deduce that

$$WH^k(\mathcal{U}, a) = (WH^0((0, \epsilon), dx^2/x^2, k + a - f/2) \otimes H^k(F)) \oplus (WH^1((0, s), dx^2/x^2, k - 1 + a - f/2) \otimes H^{k-1}(F)).$$

Since we are assuming that $k - 1 + a - f/2 \neq 0$ when $0 \leq k \leq f$, we obtain finally

$$WH^k(\mathcal{U}, a) \cong WH^0((0, s), dx^2/x^2, k + a - f/2) \otimes H^k(F) \cong \begin{cases} H^k(F), & k < f/2 - a, \\ 0, & k \geq f/2 - a. \end{cases}$$

Since the codimension of B is $f + 1$, this is the same as the local calculation for $IH_p^*(X)$ when $p(f + 1) = [a + (f/2)]$. □

A closer reading of this proof, which we leave to the reader, gives the following.

COROLLARY 4

When a is sufficiently large, then

$$WH^k(M, g_{fc}, a) = \mathcal{W}H^k(M, g_{fb}, a) = H^k(M)$$

and

$$WH^k(M, g_{fc}, -a) = \mathcal{W}H^k(M, g_{fb}, -a) = H^k(M, \partial M)$$

for every $k = 0, \dots, n$. If $F = \emptyset$, then these equalities are true for any $a > 0$.

2.4. Representing intersection cohomology with conormal forms

It is quite useful later to be able to represent classes in intersection cohomology with forms having some better regularity, especially near B (or, in the other compactification of M , near $\partial\overline{M}$). On a manifold with boundary, a natural and useful replacement for smoothness at the boundary is *conormality*. This is closely associated with b -geometry, which is discussed in §4.1, and we refer ahead to that section for the definition of the space of b -vector fields $\mathcal{V}_b(M)$. For now, we say less formally that $V \in \mathcal{V}_b$ if it is a smooth vector field on M and is tangent to $\partial\overline{M}$. Let $\gamma \in \mathbb{R}$, and define the space of conormal functions of order γ by

$$\mathcal{A}^\gamma(M) = \{u : |V_1 \cdots V_\ell u| \leq Cx^\gamma, \forall \ell \text{ and } V_j \in \mathcal{V}_b\}.$$

Clearly, any conormal function is \mathcal{C}^∞ in the interior of M , and it has full tangential regularity at the boundary. This definition extends directly to sections of vector bundles.

We now define a complex of conormal forms. As we discuss later (cf. §5.1), the operator d acting on $\Omega_{fc}^*(M)$ involves differentiations with respect to elements of \mathcal{V}_b , but also involves the nonsmooth term $x^{-1}d_F$. Hence we set

$$\mathcal{A}^a \Omega_{fc,0}^k(M) = \{\alpha \in \mathcal{A}^a \Omega_{fc}^k(M) : d\alpha \in \mathcal{A}^a \Omega_{fc}^{k+1}(M)\},$$

so that $(\mathcal{A}^a \Omega_{fc,0}^*(M), d)$ is a complex. In essence, forms in this complex have a decomposition $\eta = \eta_0 + \eta'$, where $\eta' \in \mathcal{A}^{a+1} \Omega_{fc}^*(M)$ and $\eta_0 \in \mathcal{A}^a \Omega_{fc}^*(M)$ is fibre-harmonic as defined in §5.

It is well known (cf. [56, Proposition 6.13]) that the relative and absolute cohomology of a manifold with boundary can be calculated using complexes of conormal forms. Generalizing this, we have the following.

PROPOSITION 3

The cohomology of the conormal complex $(\mathcal{A}^{a-f/2} \Omega_{fc,0}^(M), d)$ is isomorphic to the weighted cohomology $WH^*(M, g_{fc}, a)$. Provided $k - 1 + a - f/2 \neq 0$ for $0 \leq k \leq f$, it is also isomorphic to $IH_{[a+f/2]}^*(X)$.*

The argument to prove this is nearly identical to that for Proposition 2. The point is simply that $\mathcal{A}^a \Omega_{fc,0}^*(M)$ is the space of global sections of a free sheaf, the local cohomology of which satisfies the same local calculation as the sheaf of appropriately weighted L^2 -forms. We omit the details.

3. Review of the compact Hodge theorem

Despite some trade-off in work, we mainly use the Hodge–de Rham operator $D_g = d + \delta_g$ rather than its square $D_g^2 = \Delta_g$, the Hodge Laplacian. We now review one

proof of the standard Hodge theorem on compact manifolds which is phrased in terms of D_g ; this is intended as a guide for the analogous arguments in the various noncompact settings considered below and is also meant to draw attention to certain analytic aspects of the argument which are standard when M is compact, but not so straightforward in these other settings.

Recall the two most important components of the argument when M is compact. First, the ellipticity of the self-adjoint operator $D = d + \delta$ (we drop the subscript g from now on) shows that it has a generalized inverse G , which is a pseudodifferential operator of order -1 . Write $L^2 \mathcal{H}^*(M) = \ker(D)$, and let Π denote the orthogonal projection $L^2 \Omega^*(M) \rightarrow L^2 \mathcal{H}^*(M)$, so that $GD = DG = I - \Pi$. Implicit in this equation is the fact that the kernel and cokernel of D are both identified with $L^2 \mathcal{H}^*(M)$. We have

$$\begin{aligned} G : H^s \Omega^*(M) &\longrightarrow H^{s+1} \Omega^*(M), \\ \Pi : H^s \Omega^*(M) &\longrightarrow \mathcal{C}^\infty \Omega^*(M), \quad \text{for all } s \in \mathbb{R}, \end{aligned} \tag{15}$$

and of course Π is finite rank. Also, d and δ commute with G . It follows directly that D is Fredholm, for example, on $L^2 \Omega^*(M)$. Furthermore, (15) also shows that the de Rham cohomology $H^k(M)$ can be calculated using any one of the complexes $\mathcal{C}^\infty \Omega^*(M)$, $L^2 \Omega^*(M)$, or $\mathcal{C}^{-\infty} \Omega^*(M)$, that is, the complexes of smooth, L^2 - or distributional (current) forms.

Now to the argument. First, let $\omega \in L^2 \mathcal{H}^k(M)$. Then $D^2 \omega = 0$, and since ω is smooth, there is no problem in the integration by parts,

$$\langle D^2 \omega, \omega \rangle = \langle D\omega, D\omega \rangle = \|d\omega\|^2 + \|\delta\omega\|^2,$$

so that $d\omega = \delta\omega = 0$. In particular, ω is closed and $[\omega] \in H^k(M)$ is well defined. This defines a map

$$\Phi : \mathcal{H}^k(M) \longrightarrow H^k(M).$$

We must show that Φ is both injective and surjective.

Suppose $\Phi(\omega) = [\omega] = 0$; that is, suppose $\omega = d\zeta$ for some $(k - 1)$ -form ζ . We may assume that we are calculating using the complex of smooth forms, so we can choose ζ to be smooth. Since there are no boundary terms to worry about, we can integrate by parts to obtain

$$\|\omega\|^2 = \langle \omega, d\zeta \rangle = \langle \delta\omega, \zeta \rangle = 0, \tag{16}$$

and so Φ is injective.

Next, let $[\eta] \in H^k(M)$, and choose a smooth representative η . Applying $GD = I - \Pi$ to it yields

$$\eta = D\zeta + \gamma, \quad \text{where } \zeta = G\eta, \gamma = \Pi\eta. \tag{17}$$

By (15) again, $\zeta \in \mathcal{C}^\infty \Omega^*$, and of course the same is true for $\gamma \in L^2 \mathcal{H}^*$. Because D and G act on forms of all degrees together, we do not know yet whether ζ is of pure degree $k - 1$ or γ of pure degree k , so we argue as follows. Write

$$\delta\zeta = \eta - d\zeta - \gamma;$$

then

$$\|\delta\zeta\|^2 = \langle \delta\zeta, \eta - d\zeta - \gamma \rangle = \langle \zeta, d\eta - d^2\zeta - d\gamma \rangle = 0.$$

Hence $\eta = d\zeta + \gamma$, where $\gamma \in L^2 \mathcal{H}^*(M)$. Now, clearly, neither ζ nor γ has terms of degree other than $k - 1$ and k , respectively. This establishes surjectivity of Φ and completes the proof.

When (M, g) is noncompact, each of these steps may fail in a variety of ways, and our main task is to show that they can be justified for fibred boundary and fibred cusp metrics. Most fundamentally, D may no longer be Fredholm on $L^2 \Omega^*$, and so we must find some other function space on which it does have closed range. In fact, in our cases it is Fredholm on a scale of weighted L^2 -spaces, and we must study the action of D on these spaces. In particular, we wish to find function spaces X and Y such that $D : X \rightarrow Y$ is Fredholm with cokernel identified with $L^2 \mathcal{H}^*(M)$. This serves as the replacement for (17). To justify the various integrations by parts, we must also establish that elements of $L^2 \mathcal{H}^*(M)$ decay at some definite rate at infinity and also show similar decay and regularity properties for the forms ζ .

4. Nonfibred ends

Although the L^2 Hodge theorems for b - (cylindrical) and scattering (asymptotically Euclidean) metrics, Theorems 1A and 2A, are already known, we nevertheless present here proofs of these results which address some (but not all) of the difficulties encountered in the general fibred boundary and fibred cusp cases.

We sometimes denote the Hodge–de Rham operator for a b or scattering metric by D_b or D_{sc} , respectively. Recall from the end of §3 that we need to find function spaces on which these operators have closed range, and we must also establish various decay and regularity properties for the L^2 harmonic forms, as well as for the other auxiliary forms that enter into the proof. To obtain these properties, we use the machinery of the b -calculus (see [56]; cf. also [53]). In other words, we adopt the point of view that in either case D is an elliptic element in an appropriate ring of degenerate differential operators on the manifold \overline{M} . Mapping and regularity properties of these operators can be investigated using a parametrix for D constructed in an associated calculus of degenerate pseudodifferential operators.

4.1. b -metrics and operators

Let g be an exact b -metric. Associated to it is the space of b vector fields \mathcal{V}_b , which by definition is the Lie algebra of all smooth vector fields on \overline{M} which are tangent

to ∂M . In a coordinate chart (x, y_1, \dots, y_{n-1}) near ∂M , where (y_1, \dots, y_{n-1}) are coordinates on ∂M extended to the collar neighbourhood \mathcal{U} and x is a boundary-defining function, any $Z \in \mathcal{V}_b$ can be written as

$$Z = a(x, y)x\partial_x + \sum_{j=1}^{n-1} b_j(x, y)\partial_{y_j}, \quad a, b_j \in \mathcal{C}^\infty(\overline{M}).$$

Notice that \mathcal{V}_b contains precisely those smooth vector fields on \overline{M} which have point-wise bounded norms with respect to any b -metric. The vector fields $x\partial_x, \partial_{y_j}$ form a local spanning set of a vector bundle over M called the b -tangent bundle, ${}^bT M$. This bundle is canonically isomorphic to the ordinary tangent bundle $T M$ only over the interior, M , of \overline{M} , but the canonical map ${}^bT M \rightarrow T M$ given by evaluating sections at a point extends to a map that is neither injective nor surjective over ∂M ; its null space is one-dimensional and is spanned by $x\partial_x$. The dual of ${}^bT M$ is the b -cotangent bundle, ${}^bT^* M$, which is locally spanned by the one-forms $dx/x, dy_j$. We write ${}^b\bigwedge^* M$ and $\mathcal{C}^\infty\Omega_b^*(M)$ for the exterior powers of this bundle and its space of smooth sections, respectively.

A differential operator P on M is called a b -operator if it can be written locally as a sum of products of elements of \mathcal{V}_b . Thus in these coordinates,

$$P = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(x, y)(x\partial_x)^j \partial_y^\alpha$$

with all coefficients $a_{j,\alpha} \in \mathcal{C}^\infty(\overline{M})$. If P is an operator on a space of sections of a bundle over M , then the coefficients $a_{j,\alpha}$ are smooth endomorphisms of the bundle. The b -symbol

$${}^b\sigma_m(P)(x, y; \zeta, \eta) = i^{-m} \sum_{j+|\alpha|=m} a_{j,\alpha}(x, y)\zeta^j \eta^\alpha$$

is invariantly defined as a homogeneous function on ${}^bT^* M$, and P is elliptic in this setting if ${}^b\sigma_m(P)$ is nonvanishing (or invertible if P is a system) for $(\zeta, \eta) \neq 0$.

Our primary example of a b -differential operator is the Hodge–de Rham operator $D = d + \delta$ with respect to a b -metric g on M . To illustrate the definitions above, we determine its form now, assuming that the metric h which appears in the decomposition of g does not depend on x in the boundary neighbourhood \mathcal{U} .

Near ∂M any element of $\Omega_b^k(M)$ can be written as

$$\omega = \alpha + \frac{dx}{x} \wedge \beta,$$

where $\alpha(x, y)$ and $\beta(x, y)$ are families of k - and $(k - 1)$ -forms, respectively, on ∂M depending on x as a smooth parameter. The L^2 -norm is given by

$$\|\omega\|^2 = \int_M (|\alpha|_h^2 + |\beta|_h^2) \frac{dx dy}{x}.$$

Since b -metrics are special cases of fibred cusp metrics, where the fibration $\partial M \rightarrow B$ has trivial fibres, we cohere with the more general notation of this paper and identify ∂M with B . The induced differential is written d_B , and the codifferential, induced by the metric h on B , is written δ_B . We have

$$d\omega = d_B\alpha + \frac{dx}{x} \wedge (x\partial_x\alpha - d_B\beta), \tag{18}$$

$$\delta\omega = \delta_B\alpha - x\partial_x\beta - \frac{dx}{x} \wedge \delta_B\beta. \tag{19}$$

Finally, the b symbol of D is computed just as in the standard case, so that if $\zeta = (\zeta, \eta) \in {}^bT^*M$, then ${}^b\sigma_1(D) = i(\zeta \wedge \cdot + \iota_{\zeta}\cdot)$. This gives the following.

PROPOSITION 4

The operator

$$D = d + \delta : \mathcal{C}^\infty\Omega_b^*(M) \longrightarrow \mathcal{C}^\infty\Omega_b^*(M)$$

on (M, g) is an elliptic b -differential operator of order 1.

Remark. It is natural to write forms in terms of the b covector fields dy_j and dx/x since these have (essentially) unit length, but it is also important since, in a poorly chosen coframe, the expression of D might no longer be a b -operator. For example, this is the case if we use the standard basis dx and dy_j .

Unlike the usual interior calculus, symbol ellipticity alone is not enough to determine whether a b -differential operator P is Fredholm. For this, one must also use another model for P called the indicial operator I_P . This operator acts on functions on $B_y \times \mathbb{R}_s^+$ and is invariant with respect to dilations in s ; for a general P written as above,

$$I_P = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(y)(s\partial_s)^j \partial_y^\alpha.$$

To analyze this operator we use its dilation invariance. Thus, conjugating I_P by the Mellin transform in s ,

$$u(s, y) \longmapsto u_M(\gamma, y) = \int_0^\infty s^\gamma u(s, y) \frac{ds dy}{s}, \quad \gamma \in \mathbb{C},$$

yields the indicial family, $I_P(\gamma)$, which is a holomorphic family of elliptic operators on B (when P is b -elliptic). By the analytic Fredholm theorem, this family is either never invertible for any γ or else is invertible for all $\gamma \in \mathbb{C} \setminus \Lambda$, where Λ is a discrete set of complex numbers called the indicial set, the elements of which are called the indicial roots of P . It is not hard to see that the first possibility never holds. We use an

alternate (equivalent) characterization of this indicial set which is more intuitive and certainly easier to calculate:

$$\gamma \in \Lambda \iff \exists \phi \in \mathcal{C}^\infty(Y) \text{ such that } P(x^\gamma \phi(y)) = \mathcal{O}(x^{\gamma+1}), \quad \text{where } Y = \partial M.$$

Notice that $P(x^\gamma \phi(y)) = \mathcal{O}(x^\gamma)$ for all γ and ϕ , and so $\gamma \in \Lambda$ if and only if there is some additional cancellation, which arises precisely when there is an element $s^\gamma \phi(y)$ in the null space of I_P .

Again we illustrate this through the operator D . Since we are assuming that h does not depend on x in \mathcal{U} , we can identify I_D with D near ∂M , and so all approximate solutions of $D\omega = 0$ in the sense above are exact solutions in this boundary neighbourhood. Now write $\omega = \omega' x^\gamma$, where $\omega' = \alpha' + dx/x \wedge \beta'$ and neither α' nor β' depend on x . Then by (18) and (19), in \mathcal{U} ,

$$D(\omega' x^\gamma) \equiv x^\gamma I_D(\gamma)(\omega') = x^\gamma \left(D_B \alpha' - \gamma \beta' + \frac{dx}{x} \wedge (\gamma \alpha' - D_B \beta') \right). \quad (20)$$

Hence γ is an indicial root if and only if there is a solution ω' of the equations

$$D_B \alpha' = \gamma \beta', \quad D_B \beta' = \gamma \alpha', \quad (21)$$

which implies

$$\Delta_B \alpha' = \gamma^2 \alpha', \quad \Delta_B \beta' = \gamma^2 \beta'. \quad (22)$$

Thus γ is an indicial root of D if and only if $\gamma^2 \in \text{spec}(\Delta_B)$ on $\Omega^*(B)$. Note that the operators in (22) preserve the form degree and so are easier to analyze than the operators in (21). However, arbitrary solutions of (22) do not necessarily satisfy (21); in other words, we must be cautious not to introduce spurious indicial roots by all solutions of the decoupled equations. From the Kodaira decomposition on $\Omega^* B$, the only coupling in (21) is between closed k -forms and coclosed $(k-1)$ -forms for each k . Thus let ϕ_j and ψ_j be a complete set of eigenforms for Δ_B on coclosed $(k-1)$ -forms and closed k -forms, with eigenvalue λ_j^2 and such that $d_B \phi_j = \lambda_j \psi_j$, $\delta_B \psi_j = \lambda_j \phi_j$ for $\lambda_j \neq 0$. If we write

$$\alpha' = \sum \alpha_j(x) \psi_j, \quad \beta' = \sum \beta_j(x) \phi_j,$$

then (21) gives

$$\gamma \alpha_j = \lambda_j \beta_j, \quad \gamma \beta_j = \lambda_j \alpha_j,$$

which implies $\gamma^2 = \lambda_j^2$, as expected. We see finally that

$$\omega' = \sum_j \left\{ \left(\alpha_j^+ + \frac{dx}{x} \wedge \beta_j^+ \right) x^{\lambda_j} + \left(\alpha_j^- + \frac{dx}{x} \wedge \beta_j^- \right) x^{-\lambda_j} \right\}, \quad (23)$$

where $\alpha_j^\pm, \beta_j^\pm$ are both eigenforms of Δ_B with eigenvalue λ_j^2 . We have proved the following.

PROPOSITION 5

The indicial set Λ for the operator D with respect to a b -metric g consists of the values $\pm\lambda$, where $\lambda^2 \in \text{spec}(\Delta_B)$ acting on ${}^b\Omega^*(M)|_B$.

Note here that these calculations seem to leave open the possibility that zero is a double root, which would allow for the possibility of solutions of the indicial equation of the form $\omega = \omega' \log x + \omega'' x^0$. However, (20) admits no solutions of this form, and so we see that the double root is spurious and arises merely from the algebraic calculations above.

We conclude this section by discussing some general mapping properties of b -operators on weighted L^2 -spaces as well as regularity results for their solutions. Proofs of these theorems may be found in [56].

Let $L_b^2(M) = L^2(M, dx dy/x)$; this is the same as $L^2(M, dV_g)$ if g is any b -metric. We also define

$$H_b^\ell(M) = \{u \in L_b^2(M) : V_1 \cdots V_j u \in L_b^2(M), \forall j \leq \ell \text{ and } V_i \in \mathcal{V}_b\}$$

and

$$x^\gamma H_b^\ell(M) = \{u = x^\gamma v : v \in H_b^\ell(M)\},$$

whenever $\ell \in \mathbb{N}$ and $\gamma \in \mathbb{R}$.

PROPOSITION 6

Let P be an elliptic differential b -operator of order m , acting between sections of the vector bundles E and F over M , with indicial set Λ . Then the mapping

$$P : x^\gamma H_b^{\ell+m}(M; E) \longrightarrow x^\gamma H_b^\ell(M; F)$$

is Fredholm if and only if $\gamma \notin \text{Re}(\zeta) : \zeta \in \Lambda$.

To state the final proposition, we introduce the important subspace of polyhomogeneous distributions, sitting in the space of conormal distributions:

$$\mathcal{A}_{\text{phg}}^*(M) = \left\{ u \in \mathcal{A}^*(M) : u \sim \sum_{\text{Re } \gamma_j \rightarrow \infty} \sum_{k=0}^{N_j} u_{jk}(y) x^{\gamma_j} (\log x)^k u_{jk} \in \mathcal{C}^\infty(\partial M) \right\}.$$

These expansions are meant in the standard asymptotic sense as $x \rightarrow 0$ and hold along with all derivatives. The superscript $*$ here may be replaced by an index set I containing all pairs (γ_j, k) which are allowed to appear in this expansion.

PROPOSITION 7

If $u \in x^\gamma L_b^2(M; E)$ and $Pu = 0$, then $u \in \mathcal{A}_{\text{phg}}^I(M; E)$, where I is an index set

derived from the index set Λ for P truncated below the weight γ . If $Pu = f$, where $u \in x^\gamma L_b^2(M; E)$ and $f \in \mathcal{A}^{\gamma'}(M; F)$ for some $\gamma' > \gamma$, $\gamma' \notin \text{Re } \Lambda$, then $u = v + w$, where $v \in \mathcal{A}_{\text{phg}}^I(M; E)$ and $w \in \mathcal{A}^{\gamma'}(M; F)$.

The powers γ appearing in the polyhomogeneous expansion in this proposition are of the form $\gamma_j + \ell$, where each γ_j is an element of the index set for P and $\ell \in \mathbb{N}_0$. Logarithms can arise either from indicial roots with multiplicity greater than one or else (as in classical ordinary differential equation theory) when two indicial roots differ by an integer (for more details on this, see [56]). All the roots we encounter in this paper are of multiplicity one (although this fact does not really affect the arguments much), and we justify this in the various cases below, as we did following Proposition 5.

These results about b -operators may be proved in a variety of ways, some fairly elementary (e.g., see [2] for the analysis of Δ_g on cylinders using separation of variables). We refer, however, to [56] and [53] for proofs based on the calculus of b -pseudodifferential operators. This general theory is quite flexible and is ideally suited for the proofs of more general index theorems in the b -category. A thorough treatment of this calculus, along with many applications, is given in [56].

We do not need to know much about these operators beyond their mapping properties, but for the sake of completeness, we say a few words about them. The calculus ${}^b\Psi^*(M)$ is designed in part to contain parametrices for elliptic b -operators. Elements $A \in \Psi_b^*(M)$ are characterized in terms of the structure of their Schwartz kernels κ_A . Each such κ_A is a distribution on $M^2 = M \times M$ with singularities along the diagonal and side faces of this double space; kernels of elements in ${}^b\Psi^*(M)$ are characterized by the fact that they lift to distributions on a resolution M_b^2 of M^2 with only polyhomogeneous singularities. This resolution is the normal blowup of M^2 along its corner and is obtained by replacing the corner $(\partial M)^2$ by its interior normal spherical bundle.

4.2. Analysis for scattering metrics and operators

We next consider scattering metrics on M . The analysis of general elliptic operators in the scattering calculus is considerably more subtle than for operators in the b -calculus, but because we only consider the Hodge–de Rham operator, various simplifications permit us to reduce directly to the b -calculus. (Later in the paper, however, we need to use the calculus of fibred boundary pseudodifferential operators, which is much closer in spirit to the scattering calculus than to the b -calculus.)

Recall that a scattering metric g has the form $g = g'/x^2$, where g' is a b -metric. We define the Lie algebra \mathcal{V}_{sc} of scattering vector fields to consist of all smooth vector fields on \overline{M} which have bounded length with respect to any scattering metric g . Clearly,

$$\mathcal{V}_{\text{sc}} = x\mathcal{V}_b = \{V : V = xW, W \in \mathcal{V}_b\};$$

alternately, in local coordinates (x, y_1, \dots, y_{n-1}) near ∂M , \mathcal{V}_{sc} is spanned by the vector fields $x^2\partial_x$ and $x\partial_{y_j}$. By definition, these form the full set of sections of the scattering tangent bundle ${}^{sc}TM$; its dual, ${}^{sc}T^*M$, is locally smoothly trivialized by the sections

$$\frac{dx}{x^2}, \frac{dy_1}{x}, \dots, \frac{dy_{n-1}}{x}.$$

The space of smooth sections of the exterior powers of this bundle is $\mathcal{C}^\infty\Omega_{sc}^*(M)$. Thus any $\omega \in \mathcal{C}^\infty\Omega_{sc}^*(M)$ can be written as

$$\omega = \sum_k \omega_k = \sum_k \left(\frac{\alpha_k}{x^k} + \frac{dx}{x^2} \wedge \frac{\beta_{k-1}}{x^{k-1}} \right), \quad \alpha_k, \beta_{k-1} \in \mathcal{C}^\infty. \tag{24}$$

An advantage of this normalization is that

$$\|\omega\|^2 = \int_M \sum_k (|\alpha_k|^2 + |\beta_{k-1}|^2) \frac{dx dy}{x^{n+1}}.$$

An operator P is a scattering differential operator if it can be locally written as a finite sum of multiples of elements of \mathcal{V}_{sc} :

$$P = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(x, y)(x^2\partial_x)^j(x\partial_y)^\alpha, \quad a_{j,\alpha} \in \mathcal{C}^\infty(\overline{M}).$$

Its scattering symbol is defined as

$${}^{sc}\sigma_m(P)(x, y; \zeta, \eta) = i^{-m} \sum_{j+|\alpha|=m} a_{j,\alpha}(x, y)\zeta^j\eta^\alpha.$$

P is elliptic in this calculus if this symbol is invertible for $(\zeta, \eta) \neq 0$.

The analysis of $(\Delta - \lambda)u = 0$ is quite different depending on whether λ is negative or positive; for example, in the former case, solutions decay rapidly, while in the latter they oscillate with slow decay as $x \rightarrow 0$. Accordingly, the nature of the resolvent changes dramatically when $\lambda \in \text{spec}(\Delta)$ (cf. [57], [38]). Because of this, the general theory of parametrices, mapping properties, and regularity theory for elliptic scattering operators is fairly complicated. Fortunately, we can sidestep this calculus by virtue of the following.

PROPOSITION 8

If g is a scattering metric on M , then

$$D = d + \delta : \mathcal{C}^\infty\Omega_{sc}^*(M) \longrightarrow x\mathcal{C}^\infty\Omega_{sc}^*(M)$$

is an elliptic first-order scattering operator of the form $D = xD'$, where D' is an elliptic first-order b -operator.

Remark. It seems initially somewhat confusing that D' is a b -operator when acting between sections of the scattering form bundles (normalized as above) but not when acting between sections of the b -form bundles. We can understand why this is true, however, when we consider that the endomorphism $dx \wedge$ has the same operator norm on forms as the does the form dx . This norm depends upon the metric on M . Thus dx/x is a unit norm endomorphism on the bundle of forms when M has a b -metric, whereas dx/x^2 is the unit endomorphism on the bundle of forms when M has a scattering metric. There is a similar shift in the power of x in the other coordinates, so in the scattering case, an extra power of x is absorbed into the denominator of the endomorphism part of the Laplacian. This makes D' a b -operator on the bundle of scattering forms, although it is not an operator on the bundle of b -forms.

Proof

Write $\omega \in \mathcal{C}^\infty \Omega_{sc}^*(M)$ as in (24), and set $\alpha = \sum \alpha_k, \beta = \sum \beta_k$. Then a brief calculation gives

$$D\omega = \sum_k \left(\frac{x(D_B\alpha)_k - x^2\partial_x\beta_k + (n-k-1)x\beta_k}{x^k} + \frac{dx}{x^2} \wedge \frac{x^2\partial_x\alpha_k - kx\alpha_k - x(D_B\beta)_k}{x^k} \right), \tag{25}$$

where $(D_B\zeta)_k$ is the component of degree k of $D_B\zeta$ for $\zeta = \alpha$ or β . This shows immediately that $D' \equiv x^{-1}D$ is a b -operator; it differs from $D_{g'}$, where $g' = x^2g$ is the associated b -metric, only in terms of order zero. Of course, these affect the indicial set Λ markedly. □

The mapping properties of D and the regularity properties of its solutions may be deduced directly from the corresponding properties for D' in Propositions 6 and 7. Note, however, that the extra factor of x causes a shift in the weight of the function spaces.

PROPOSITION 9

Suppose that g is a scattering metric, so that $D = xD'$ as above. Let Λ denote the indicial set for D' . Then

$$D : x^\gamma H_b^{\ell+1} \Omega_{sc}^*(M) \longrightarrow x^{\gamma+1} H_b^\ell \Omega_{sc}^*(M)$$

is Fredholm for any $\ell \in \mathbb{N}_0$ and $\gamma \notin \{\Re(\lambda) : \lambda \in \Lambda\}$.

PROPOSITION 10

If $\omega \in x^\gamma L^2 \Omega_{sc}^(M)$ for any $\gamma \in \mathbb{R}$ and $D\omega = 0$, then $\omega \in \mathcal{A}_{phg}^1 \Omega_{sc}^*(M)$, where*

I is some augmented index set determined by the indicial set Λ of D' and the cutoff weight γ . If, on the other hand, $D\omega = \eta$, where $\eta \in x^{\gamma'+1}\mathcal{A}^*\Omega_{sc}^*(M)$ for some $\gamma' > \gamma$, then $\omega = \omega' + \omega''$ with $\omega' \in \mathcal{A}_{phg}^*\Omega_{sc}^*(M)$ and $\omega'' \in \mathcal{A}^{a+1}\Omega_{sc}^*(M)$.

We conclude this section with a computation of the relevant part of the indicial set Λ for D' . As in the b -case, this set is determined by the spectrum of Δ_B , but the computation is more intricate.

First, define the numerical operators N_1 and N_2 :

$$N_1\beta_k = (n - k - 1)\beta_k, \quad N_2\alpha_k = -k\alpha_k$$

(i.e., N_1 and N_2 are diagonal on $\Omega^*(B)$ with respect to the decomposition by degree). Let

$$\omega = \sum \omega_k, \quad \omega_k = \frac{\alpha_k}{x^k} + \frac{dx}{x^2} \wedge \frac{\beta_{k-1}}{x^{k-1}},$$

where all α_j and β_j are independent of dx . Then $D(x^\gamma \omega) = x^{\gamma+1}I_{D'}(\gamma)(\omega)$, where

$$I_{D'}(\gamma)(\omega) = \sum_k \left(\frac{(D_B\alpha + (N_1 - \gamma)\beta)_k}{x^k} + \frac{dx}{x^2} \wedge \frac{(-D_B\beta + (N_2 + \gamma)\alpha)_{k-1}}{x^{k-1}} \right).$$

Writing I for $I_{D'}$, this vanishes when

$$I(\gamma) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv \begin{pmatrix} D_B & N_1 - \gamma \\ N_2 + \gamma & -D_B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{26}$$

Although this equation seems strongly coupled and hence difficult to analyze, computations can be simplified using the special structure that $\Delta = D^2$ preserves degree. On the indicial level this gives

$$\begin{aligned} I(\gamma + 1)I(\gamma) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ = \begin{pmatrix} D_B & N_1 - \gamma - 1 \\ N_2 + \gamma + 1 & -D_B \end{pmatrix} \begin{pmatrix} D_B & N_1 - \gamma \\ N_2 + \gamma & -D_B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Multiplying out this matrix of operators and using the easily verified fact that

$$[D_B, N_j] = d_B - \delta_B,$$

we have

$$\begin{aligned} \left(\begin{array}{cc} \Delta_B + (N_1 - \gamma - 1)(N_2 + \gamma) & 2d_B \\ 2\delta_B & \Delta_B + (N_2 + \gamma + 1)(N_1 - \gamma) \end{array} \right) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

The coupling here occurs only between closed k -forms and coclosed $(k - 1)$ -forms.

We do not need to calculate all the indicial roots of D' , although this can be done readily from these formulae. Instead, we focus on the special value $\gamma = n/2 - 1$. This is a critical value in our calculations because $x^{n/2}$ is just on the border of lying in $L^2(dV_g) = L^2(x^{-n-1}dx dy)$ and we need to analyze the map $D : x^{\gamma-1}L^2 \rightarrow x^\gamma L^2$ for γ near this borderline value. Thus setting $\gamma = n/2 - 1$ gives

$$\begin{aligned} \left(N_1 - \frac{n}{2}\right)\left(N_2 + \frac{n}{2} - 1\right)\alpha_k &= \left(\frac{n}{2} - k - 1\right)^2 \alpha_k, \\ \left(N_2 + \frac{n}{2}\right)\left(N_2 - \frac{n}{2} + 1\right)\beta_{k-1} &= \left(\frac{n}{2} - k + 1\right)^2 \beta_{k-1}. \end{aligned}$$

Hence if ω lies in the null space of $I_D(n/2 - 1)$, then for all k we have

$$\begin{aligned} \left(\Delta_B + \left(\frac{n}{2} - 1 - k\right)^2\right)\alpha_k + 2d_B\beta_{k-1} &= 0, \\ \left(\Delta_B + \left(\frac{n}{2} + 1 - k\right)^2\right)\beta_{k-1} + 2\delta_B\alpha_k &= 0. \end{aligned}$$

Decompose these equations using an eigendecomposition for Δ_B such that $\alpha_k = a\psi_k$, $\beta_{k-1} = b\phi_{k-1}$, where both ψ_k and ϕ_{k-1} are eigenforms with eigenvalue $\lambda^2 \geq 0$ and $d\phi_{k-1} = \lambda\psi_k$, $\delta\psi_k = \lambda\phi_{k-1}$. Then

$$\begin{pmatrix} \lambda^2 + (n/2 - k - 1)^2 & 2\lambda \\ 2\lambda & \lambda^2 + (n/2 - k + 1)^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so there are nontrivial solutions only if this matrix is singular. Its determinant equals $(\lambda^2 + (n/2 - k)^2 - 1)^2$; hence there are no solutions unless $|k - n/2| \leq 1$. First, if $\lambda = 0$, then $k = n/2 \pm 1$, and the null space consists of harmonic forms $\alpha_{n/2-1}$ and $\beta_{n/2+1}$. Next, if $\lambda^2 = 1$ is in the spectrum of Δ_B , then there are solutions for α_k and β_{k-1} only if $k = n/2$, and elements of the null space of $I(n/2)I(n/2 - 1)$ are obtained by taking $a + b = 0$. Finally, there are solutions of a similar type when $k = (n \pm 1)/2$ and $\lambda^2 = 3/4 \in \text{spec}(\Delta_B)$. One can then verify that only the solutions corresponding to $\lambda = 0$ also lie in the null space of $I(n/2 - 1)$; hence these are the only ones that appear in the polyhomogeneous expansions for solutions of $D\omega = 0$.

Note that $\gamma = n/2 - 1$ is not an indicial root of multiplicity two. As in the b -setting, this follows by checking that there are no solutions of (26) of the form $\omega' \log x + \omega''$.

We conclude these computations by noting that if $\omega \in L^2(dV_g)$ satisfies $D\omega = 0$, then $D^2\omega = 0$ and the usual integration by parts, which is justified in L^2 , gives $d\omega = \delta\omega = 0$ individually. To relate this to the preceding calculations, this implies that if $\gamma > n/2$ is an indicial root for $x^{-1}D$, then it is also one for both $x^{-1}d$ and

$x^{-1}\delta$ (and conversely), and these are much simpler to compute. In fact,

$$I_{x^{-1}d}(\gamma)(\omega) = \sum_k \left(\frac{d_B \alpha_k}{x^{k+1}} + \frac{dx}{x} \wedge \frac{(\gamma - k)\alpha_k - d_B \beta_{k-1}}{x^k} \right)$$

and

$$I_{x^{-1}\delta}(\gamma)(\omega) = \sum_k \left(\frac{(\delta_B \alpha_k + (n - k - \gamma)\beta_{k-1})}{x^{k-1}} + \frac{dx}{x} \wedge \frac{-\delta_B \beta_{k-1}}{x^{k-2}} \right).$$

Hence ω is in the null space of both these operators provided $d_B \alpha = \delta_B \beta = 0$ and also

$$\delta_B \alpha_k = -(n - k - \gamma)\beta_{k-1}, \quad d_B \beta_{k-1} = (\gamma - k)\alpha_k.$$

Thus α_k is closed, β_{k-1} is coclosed, and both are in the null space of $\Delta_B + (\gamma - k)(n - k - \gamma)$. On an eigenspace with eigenvalue λ^2 for Δ_B , we must have

$$\gamma^2 - n\gamma + (k(n - k) - \lambda^2) = 0,$$

and by assumption, above we must choose the root that is greater than $n/2$. (Of course, solutions of these equations, no matter the value of γ , give indicial roots of $x^{-1}D$, corresponding to non- L^2 -solutions. The point of the earlier calculations is that there are other indicial roots corresponding to solutions which are not individually closed and coclosed.) In summary, these comprise the subset

$$\Lambda' = \{ \gamma_j^\pm : \text{roots of } \gamma^2 - n\gamma + k(n - k) - \lambda_j^2 = 0, \lambda_j^2 \in \text{spec}(\Delta_B) \} \quad (27)$$

inside the possibly larger set of all indicial roots of $x^{-1}D$. Note, in particular, that when $\lambda_j^2 = 0$, $\gamma_j^\pm = k, n - k$.

4.3. Hodge theorems for b - and scattering metrics

Having assembled these analytic facts and calculations, we now complete the proofs of the Hodge theorems for b - and scattering metrics following the outline from the compact case. We invert the usual order of presentation and discuss first the b -case, which specializes Theorem 2, and afterwards the scattering case, which specializes Theorem 1. These results equate Hodge cohomology with weighted cohomology only; Corollary 4 shows that the results are indeed the same as stated in Theorems 2A and 1A, respectively.

THEOREM 2B

Let g be an exact b -metric on the manifold M . Then for sufficiently small $\epsilon > 0$ and for every $k = 0, \dots, n$, there is a canonical isomorphism

$$\Phi : L^2 \mathcal{H}^k(M) \longrightarrow \text{Im} (WH^k(M, g, \epsilon) \longrightarrow WH^k(M, g, -\epsilon)). \quad (28)$$

Proof

As in the compact case, if $\omega \in L^2 \mathcal{H}^k(M)$, then $d\omega = 0$. Further, by Proposition 7 and (23), ω is polyhomogeneous with an expansion of the form $\sum \omega_j^\pm(y)x^{\pm\lambda_j}$, where the tangential and normal parts of ω_j^\pm are eigenforms on ∂M with eigenvalue λ_j^2 . Since $\omega \in L^2$, we see that all coefficient forms ω_j^- vanish, as do those $\omega_j^+ = 0$ corresponding to values of j with $\lambda_j = 0$. Hence $\omega = \alpha + dx/x \wedge \beta$, where $\alpha, \beta = \mathcal{O}(x^{\underline{\lambda}})$, where $\underline{\lambda} = \inf\{|\lambda_j| \neq 0 : \lambda_j^2 \in \text{spec}(\Delta_B)\}$. Thus $[\omega] \in WH^k(M, g, \epsilon)$ is well defined provided $\epsilon < \underline{\lambda}$.

If $\omega \in L^2 \mathcal{H}^k(M)$ and $\Phi(\omega) = 0$, then $\omega = d\zeta$ for some $\zeta \in x^{-\eta}L^2\Omega_b^{k-1}(M)$. Computing cohomology with the complex of conormal forms as explained in §2, we can take $\zeta = \mu + dx/x \wedge \nu$, where $\mu, \nu \in \mathcal{A}^{-\eta}([0, 1)_x \times B; \wedge^*(B))$. In the integration by parts $\|\omega\|^2 = \langle \omega, d\zeta \rangle = \langle \delta\omega, \zeta \rangle = 0$, the boundary term equals $\lim_{x \rightarrow 0} \langle \alpha, \nu \rangle_B$, and this vanishes since $\epsilon < \underline{\lambda}$. Hence $\omega = 0$, and so Φ is injective.

Next, by Proposition 6, the map

$$D : x^{-\eta}H_b^1\Omega^*(M) \longrightarrow x^{-\eta}L^2\Omega^*(M) \tag{29}$$

is Fredholm when $\epsilon \in (0, \underline{\lambda})$. This gives the decomposition

$$x^{-\eta}L^2\Omega^*(M, dV_g) = (\text{ran } D|_{x^{-\eta}H_b^1\Omega^*}) \oplus (\text{ran } D|_{x^{-\eta}H_b^1\Omega^*})^\perp.$$

The second summand on the right is finite-dimensional and could be replaced with any other finite-dimensional subspace of $x^{-\eta}L^2\Omega^*$ which is complementary to the range of D since the orthogonality of this decomposition does not play any role. In particular, we claim that we can replace this term by $L^2 \mathcal{H}^*(M)$. To see this, note simply that the natural pairing between $x^{-\eta}L^2\Omega^*$ and $x^\eta L^2\Omega^*$ identifies the orthogonal complement of the range of D on $x^{-\eta}L^2\Omega^*$ with the null space of D on $x^\eta L^2\Omega^*$, which equals $L^2 \mathcal{H}^*(M)$. In any case, we have shown that for any $\eta \in x^{-\eta}L^2\Omega^*$, there exist elements $\zeta \in x^{-\eta}H_b^1\Omega^*$ and $\gamma \in L^2 \mathcal{H}^*$ such that

$$\eta = D\zeta + \gamma. \tag{30}$$

Now we prove surjectivity of Φ . Fix any $[\eta]$ in the space on the right in (28), and choose a conormal representative $\eta \in \mathcal{A}^{-\eta}\Omega^*$ for it. Decompose η as $D\zeta + \gamma$ as above, in the space $x^{-\eta}L^2\Omega^*$. Proposition 7 shows that ζ is partially polyhomogeneous; that is, it is a sum of a finite number of terms of the form $\zeta_{j,\ell} x^{\sigma_j} (\log x)^\ell$ and a term $\zeta' \in \mathcal{A}^\epsilon\Omega^*$. All exponents σ_j lie in the interval $(-\epsilon, \epsilon)$, and because we can choose ϵ as small as desired, we may assume that the only terms that appear have $\sigma_j = 0$. The remaining terms correspond to solutions of the indicial operator $I_D(0)$, and the analysis in §4.1 shows that zero is an indicial root of multiplicity one, so no log terms occur. Thus $\zeta = \zeta_0 + \zeta'$, where $I_D(0)\zeta_0 = 0$; if we write $\zeta_0 = \mu_0 + (dx/x) \wedge \nu_0$, then both μ_0 and ν_0 are harmonic on B .

As in §3, to conclude that $\delta\zeta = 0$ we must check that all three terms, $\langle \delta\zeta, \eta \rangle$, $\langle \delta\zeta, \gamma \rangle$, and $\langle \delta\zeta, d\zeta \rangle$, vanish. This is true formally, that is, integrating by parts and neglecting the boundary terms, so it suffices to check that these boundary terms also vanish. For the first two this is straightforward since $|\zeta|$ is bounded and both η and γ vanish at $x = 0$. For the final term the boundary contribution is

$$\int_M d(\zeta \wedge *\delta\zeta) = \int_B \mu_0 \wedge d_B *_B \nu_0 = \langle \mu_0, \delta_B \nu_0 \rangle_B,$$

and this vanishes since $\nu_0 \in L^2 \mathcal{H}^*(B)$. Hence $\delta\zeta = 0$, and so $\eta = d\zeta + \gamma$, as required. As in the compact case, there are only forms of degree k here. This finishes the proof. □

Now suppose that

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}$$

is a scattering metric on M .

THEOREM 1B

Let g be a scattering metric on M . Then for any $\epsilon > 0$ sufficiently small, there is a canonical isomorphism

$$\Phi : L^2 \mathcal{H}^k(M) \longrightarrow \text{Im}(\mathcal{W} H^k(M, g, \epsilon) \rightarrow \mathcal{W} H^k(M, g, -\epsilon)). \tag{31}$$

Proof

If $\omega \in L^2 \mathcal{H}^k(M, g)$, then $d\omega = 0$. By Proposition (10), $\omega \in \mathcal{A}^{\underline{\lambda}} \Omega_{\text{sc}}^k(M)$ for some $\underline{\lambda} > 0$. Hence $[\omega] \in \mathcal{W} H^k(M, g, \epsilon)$ is well defined provided $0 < \epsilon < \underline{\lambda}$. Thus $\Phi(\omega)$ is well defined.

Suppose $\Phi(\omega) = 0$, so that $\omega = d\zeta$ for some $\zeta \in x^{-\epsilon-1} L^2 \Omega_{\text{sc}}^{k-1}$. Write

$$\omega = \frac{\alpha}{x^k} + \frac{dx}{x^2} \wedge \frac{\beta}{x^{k-1}} \quad \text{and} \quad \zeta = \frac{\mu}{x^{k-1}} + \frac{dx}{x^2} \wedge \frac{\nu}{x^{k-2}}.$$

We may assume that ζ is conormal and hence that $|\mu|, |\nu| \in \mathcal{O}(x^{n/2+\epsilon'-1})$ for some $\epsilon' > \epsilon$. This implies that $\lim_{x \rightarrow 0} \langle x^{-k+1} \mu, x^{-n+k} \beta \rangle_B = 0$, whence $\|\omega\|^2 = \langle d\zeta, \omega \rangle = \langle \zeta, \delta\omega \rangle = 0$. This shows that $\omega = 0$ and thus that Φ is injective.

The surjectivity argument proceeds as before. Since

$$D : x^{-\epsilon-1} H_b^1 \Omega^*(M) \longrightarrow x^{-\eta} L^2 \Omega^*(M)$$

is Fredholm, we have

$$x^{-\eta} L^2 \Omega^*(M) = (\text{ran } D|_{x^{-\epsilon-1} H_b^1}) \oplus (\text{ran } D|_{x^{-\epsilon-1} H_b^1})^\perp.$$

The same argument as in the b -case identifies this orthocomplement with $L^2 \mathcal{H}^*$. Now let $\eta \in \mathcal{A}^\epsilon \Omega^*$ represent a nontrivial class $[\eta]$ in the space on the right in (31). Write $\eta = D\zeta + \gamma$, $\gamma \in L^2 \mathcal{H}^*$. By Proposition 10, ζ is partially polyhomogeneous and is a finite sum of terms $\zeta_{j,\ell} x^{\sigma_j} (\log x)^\ell$ and some $\zeta' \in \mathcal{A}^\epsilon \Omega_{\text{sc}}^*$. By taking ϵ small enough, we can eliminate all but the term of weight $n/2 - 1$, and by the computations in §4.3, this indicial root occurs with multiplicity one, so there are no log terms. In fact, those computations give

$$\zeta = \zeta_0 + \zeta', \quad \zeta_0 = \left(\frac{\alpha_{n/2-1}}{x^{n/2-1}} + \frac{dx}{x^2} \wedge \frac{\beta_{n/2+1}}{x^{n/2+1}} \right) x^{n/2-1} + \zeta', \quad |\zeta'| = \mathcal{O}(x^{\epsilon'}),$$

where $\alpha_{n/2-1}$ and $\beta_{n/2+1}$ are harmonic on B . Using the same reasoning as in the proof of Theorem 2B, we see that the boundary terms in the integrations by parts $\langle \delta\zeta, \eta \rangle = \langle \zeta, d\eta \rangle$ and $\langle \delta\zeta, \gamma \rangle = \langle \zeta, d\gamma \rangle$ both vanish. Since $d\eta = d\gamma = 0$, these terms vanish altogether. Finally, $\langle \delta\zeta, d\zeta \rangle$ equals the sum of the (vanishing) interior term, $\langle \zeta, d^2\zeta \rangle = 0$, and a boundary term. Since d and δ are both x times b -operators, $d\zeta'$ and $\delta\zeta'$ both decay, so only ζ_0 contributes. This boundary term equals

$$\int_M d\zeta_0 \wedge * \delta\zeta_0 = \pm \int_M d(\zeta_0 \wedge d * \zeta_0) = \pm \langle \alpha_{n/2-1}, \delta_B \beta_{n/2+1} \rangle = 0.$$

Hence $\delta\zeta = 0$, and finally, $\eta = d\zeta + \gamma$, where $\gamma \in x^{-\epsilon-1} L^2 \Omega^{k-1}$ and $\gamma \in L^2 \mathcal{H}^k$. \square

5. Fibred ends

We now turn to the general case, where both the base and fibre in the fibration of ∂M are nontrivial. As in the b - and scattering cases, we must determine the explicit structure of D , calculate its indicial roots, and understand its mapping properties and the regularity (polyhomogeneity) of elements of $L^2 \mathcal{H}^*$. For the construction of a parametrix for D , we invoke the fibred boundary calculus of pseudodifferential operators, as developed in [54] and extended in [67]. This serves as a replacement for the b -calculus in this context but is more intricate. To help mitigate the analytic requisites, we include a discussion of this parametrix construction in the very special case where M is a global product and the fibred boundary or fibred cusp metric respects this decomposition. Although the Hodge theorems in these cases follow directly via a Künneth theorem from those for b - and scattering metrics, we sketch an explicit parametrix construction for D in hopes that this gives some insight into the more general case.

We begin with a general discussion of the fibred boundary calculus and then proceed immediately to a discussion of D and its parametrix in the product case. This is followed by a review of the geometry of fibrations and the structure of D in the general case. The identifications of Hodge cohomology with weighted cohomology are then proved, as usual following the general line of argument from §3. Section 5.5 relates the weighted cohomology to intersection cohomology.

5.1. The fibred boundary calculus

Suppose that $\phi : Y = \partial M \rightarrow B$ is a fibration with fibre F , $\dim B = b$, and $\dim F = f$. Fixing an extension of this fibration to a collar neighbourhood \mathcal{U} of ∂M in M , we choose a fibred boundary metric

$$g_{\text{fb}} = \frac{dx^2}{x^4} + \frac{\phi^*(h)}{x^2} + k_F,$$

where h is a metric lifted from B and k_F is a symmetric 2-tensor that restricts to a metric on each fibre. There is an associated fibred cusp metric $g_{\text{fc}} = x^2 g_{\text{fb}}$. These metrics stand in the same relationship to one another as do scattering and b -metrics. As we saw in those cases, only the b -calculus (but not the scattering calculus) is required to analyze the operator D in both cases. Similarly, the fibred boundary calculus is enough to analyze the Hodge–de Rham operators for both g_{fb} and g_{fc} . (Indeed, there is no calculus directly associated to g_{fc} , for reasons indicated below.)

The fibred boundary calculus relies on the choice of a 1-jet of the defining function x along the fibres at ∂M . Making such a choice, define the Lie algebra of fibred boundary vector fields

$$\mathcal{V}_{\text{fb}} = \{V \in \mathcal{V}_b(M) : V \text{ tangent to fibres } F \text{ at } \partial M, Vx = \mathcal{O}(x^2)\}.$$

To understand this more clearly, choose local coordinates (x, y, z) , where y are coordinates on B , pulled back to Y via ϕ and then extended into the manifold, z are functions on Y that restrict to coordinates on the fibres, similarly extended inward, and x is in the given equivalence class of defining functions. Then \mathcal{V}_{fb} is spanned locally over \mathcal{C}^∞ by the vector fields $x^2 \partial_x, x \partial_{y_j}, \partial_{z_\ell}$. If $(\tilde{x}, \tilde{y}, \tilde{z})$ is a new choice of coordinates adapted to the fibration, then $\partial_{\tilde{z}}$ transforms into a vector field with one component equal to $(\partial x / \partial \tilde{z}) \partial_x$, and this explains why we need to fix the differential of x along each fibre in order that the coefficient here vanish to second order.

In contrast, the vector fields associated to a fibred cusp metric are $x \partial_x, \partial_{y_j}, x^{-1} \partial_{z_\ell}$; these are singular, but much more seriously, their span is not closed under Lie bracket. Involutivity is a basic requirement in the microlocalization procedure leading to the construction of the associated pseudodifferential calculus, and this explains why there is no separate fibred cusp calculus. The elements of \mathcal{V}_{fb} constitute the full set of sections of the fb tangent bundle ${}^{\text{fb}}TM$. We use its dual, the fb cotangent bundle, and the bundle of fb exterior forms, $\bigwedge_{\text{fb}}^*(M)$. Thus in the coordinates above,

$$\mathcal{C}^\infty \Omega_{\text{fb}}^k(M) \ni \omega = \sum_{i=0}^k \frac{\alpha_i}{x^i} + \frac{dx}{x^2} \wedge \sum_{j=0}^{k-1} \frac{\beta_j}{x^j},$$

where α_i is a sum of wedge products of i -forms in y and $(k - i)$ -forms in z , and β_j is a sum of wedge products of j -forms in y and $(k - j - 1)$ -forms in z , all of which are

smooth in the ordinary sense on \overline{M} . (This decomposition is recast more invariantly later.)

We now define the space of fb differential operators on M , the associated ϕ -symbol, and finally, the corresponding notion of symbol ellipticity. This leads to the following.

PROPOSITION 11

For an exact fibred boundary metric, the Hodge–de Rham operator $D = d + \delta$ is an elliptic first-order fibred boundary differential operator. For an exact fibred cusp metric, the operator D is of the form $x^{-1}D'$, where D' is an elliptic first-order fibred boundary operator.

Elliptic fibred boundary operators may be analyzed using the calculus of fibred boundary pseudodifferential operators from [54]. We use the elaboration of this theory developed by Vaillant [67]. He constructs parametrices for any Dirac-type operator associated to a fibred boundary or fibred cusp metric and in particular proves the following.

PROPOSITION 12 ([67, Proposition 3.28])

Let \mathcal{D} be a Dirac-type operator associated to a fibred boundary metric (e.g., either D or D' above). Suppose that $\omega \in x^\gamma L^2\Omega_{\text{fb}}^(M)$ satisfies $\mathcal{D}\omega = 0$. Then $\omega \in \mathcal{A}_{\text{phg}}^*\Omega_{\text{fb}}^*(M)$.*

We also require a replacement for the other parts of Propositions 7 and 10 as well as replacements for the basic mapping properties, as in Propositions 6 and 9. The precise forms of these results in the fibred boundary setting are somewhat different, and as explained in the preamble to this section, to motivate these results we take a detour and investigate the mapping and regularity properties for product metrics. This involves little more than rephrasing the corresponding results for b - and scattering metrics but is included to help orient the reader. We also include a discussion of the indicial root structure for D in these two cases; the computations are more transparent in the product cases, but the general results are qualitatively the same.

5.2. The product case

5.2.1. Fibred boundary metrics

Suppose that $M = N \times F$, where $\partial N = B$, and fix a fibred boundary metric g on M which is of the form $g' + k$, where g' is a scattering metric on N and k is a metric on the compact manifold F .

We write $Y = \partial M = B \times F$. Since TY splits canonically as $TB \oplus TF$, we have

$$\bigwedge^k T^*Y = \bigoplus_{p+q=k} \bigwedge^{p,q} Y, \quad \bigwedge^{p,q} Y = \bigwedge^p T^*B \otimes \bigwedge^q T^*F.$$

Thus any $\omega \in \bigwedge^k M$ can be written as

$$\omega = \frac{\alpha}{x^k} + \frac{dx}{x^2} \wedge \frac{\beta}{x^{k-1}}, \quad \text{where } \alpha \in \bigoplus_j \bigwedge^{k-j,j} Y, \quad \beta \in \bigoplus_j \bigwedge^{k-1-j,j} Y$$

depend parametrically on x .

The Hodge–de Rham operator $D = D_M$ acts on ω , regarded as a column vector $(\alpha, \beta)^t$, as

$$\begin{pmatrix} 0 & -x^2 \partial_x + (b - k + 1)x \\ x^2 \partial_x - kx & 0 \end{pmatrix} + \begin{pmatrix} xD_B + D_F & 0 \\ 0 & -xD_B - D_F \end{pmatrix}. \tag{32}$$

Here D_F acts on a (p, q) -form $\eta \wedge \nu$ as $(-1)^p \eta \wedge (D_F \nu)$. In the more general (non-product) case, D has a similar decomposition, but the second matrix has extra terms coming from the nontrivial geometry of the bundle.

The space of harmonic forms on the compact manifold F is finite-dimensional.

Let

$$\Pi_0 : L^2 \Omega^*(F) \longrightarrow L^2 \mathcal{H}^*(F), \quad \Pi_{\perp} = I - \Pi_0$$

be the natural orthogonal projectors. These extend naturally to $L^2 \Omega_{\text{fb}}^*(M)$, and we have

$$D_M = \Pi_0 D_M \Pi_0 + \Pi_{\perp} D_M \Pi_0 + \Pi_0 D_M \Pi_{\perp} + \Pi_{\perp} D_M \Pi_{\perp}.$$

Since $[D_M, \Pi_0] = 0$ in the product case, the second and third terms vanish and this reduces to

$$D_M = \Pi_0 D_M \Pi_0 \oplus \Pi_{\perp} D_M \Pi_{\perp}.$$

We use this decomposition to construct a parametrix for D_M . First,

$$\Pi_0 D_M \Pi_0 = D_N \otimes \text{Id}_{\mathcal{H}^*(F)},$$

and so by the theory from §§4.1 and 4.2, if $a \in \mathbb{R}$ is not an indicial root for $x^{-1} D_N$, this operator is Fredholm as a mapping from $x^a L^2 \Omega^* \rightarrow x^{a+1} L^2 \Omega^*$. We write the generalized inverse as

$$G_0^a : x^{a+1} L^2 \Omega_{\text{fb}}^*(N) \otimes \mathcal{H}^*(F) \longrightarrow x^a H^1 \Omega_{\text{fb}}^*(N) \otimes \mathcal{H}^*(F).$$

The second term in the decomposition of D_M has square $\Delta_N + \Pi_{\perp} \Delta_F \Pi_{\perp}$. Since $\Delta_N \geq 0$ and $\Pi_{\perp} \Delta_F \Pi_{\perp} \geq c > 0$, we have that $\Pi_{\perp} D_M \Pi_{\perp} : x^a H^1 \Omega_{\text{fb}}^* \rightarrow x^a L^2 \Omega_{\text{fb}}^*$ is an isomorphism for any a . Thus for any a we get

$$G_{\perp}^a \equiv (\Pi_{\perp} D_M \Pi_{\perp})^{-1} : x^a \Pi_{\perp} L^2 \Omega_{\text{fb}}^*(M) \longrightarrow x^a \Pi_{\perp} H^1 \Omega_{\text{fb}}^*(M).$$

Altogether, we have proved that

$$G_0^a \oplus G_\perp^a \equiv G^a : x^{a+1} \Pi_0 L^2 \Omega_{\text{fb}}^*(M) \oplus x^a \Pi_\perp L^2 \Omega_{\text{fb}}^*(M) \longrightarrow x^a H^1 \Omega_{\text{fb}}^*(M)$$

is bounded. Clearly, $I - G^a D_M = \Pi_M^a$ is the projector onto the null space of D_M in $x^a L^2 \Omega_{\text{fb}}^*$, which is the same as the null space of $\Pi_0 D_M \Pi_0$, that is, $L^2 \mathcal{H}^*(N) \otimes \mathcal{H}^*(F)$. By Proposition 10, Π_M^a maps $x^a L^2$ into the space of polyhomogeneous fibre harmonic forms. We emphasize that the indicial roots for D_M are exactly the same as for D_N . In particular, the *critical* root $(1/2) \dim N - 1 = (b - 1)/2$ has multiplicity one!

In summary, we have proved that the mappings

$$D_M : x^a H^1 \Omega_{\text{fb}}^*(M) \longrightarrow x^{a+1} \Pi_0 L^2 \Omega_{\text{fb}}^*(M) \oplus x^a \Pi_\perp L^2 \Omega_{\text{fb}}^*(M) \quad (33)$$

and

$$D_M : x^{a-1} \Pi_0 H^1 \Omega_{\text{fb}}^*(M) \oplus x^a \Pi_\perp H^1 \Omega_{\text{fb}}^*(M) \longrightarrow x^a L^2 \Omega_{\text{fb}}^*(M) \quad (34)$$

are Fredholm when a , respectively, $a - 1$, is not an indicial root of D_N .

The generalized inverse G^a has other mapping properties. Suppose $\eta = D_M \zeta$, where $\eta \in \mathcal{A}^a \Omega_{\text{fb}}^*(M)$ and $\zeta \in x^{c-1} \Pi_0 H^1 \Omega_{\text{fb}}^*(M) \oplus x^c \Pi_\perp H^1 \Omega_{\text{fb}}^*(M)$ for some $c < a$. Then, in fact, $\zeta \in \Pi_0 \mathcal{A}_{\text{phg}}^* \Omega_{\text{fb}}^*(M) + \mathcal{A}^a \Omega_{\text{fb}}^*(M)$.

5.2.2. Fibred cusp metrics

Now suppose that $M = N \times F$ has a fibred cusp metric g_{fc} ; notice that this is a warped product (since k_F is multiplied by x^2). We obtain a parametrix for the associated Hodge–de Rham operator D as above. Write all forms in terms of the (essentially orthonormal) coframe, dx/x , dy , $x dz$, and denote the space of forms with this normalization as \bigwedge_{fc}^* . Thus

$$\bigwedge_{\text{fc}}^*(M) \ni \omega = x^k \alpha + \frac{dx}{x} \wedge x^k \beta, \quad \text{where } \alpha, \beta \in \bigoplus_j \bigwedge^{j,k} Y.$$

Write $\omega \in \Omega_{\text{fc}}^{*,k}(M)$ if it decomposes into terms all with fibre degree k . D_M acts on the pair (α, β) as the matrix of operators

$$\begin{pmatrix} 0 & -x \partial_x - (f - k) \\ x \partial_x + k & 0 \end{pmatrix} + \begin{pmatrix} D_B + x^{-1} D_F & 0 \\ 0 & -D_B - x^{-1} D_F \end{pmatrix}. \quad (35)$$

As before, this splits as $D_M = \Pi_0 D_M \Pi_0 \oplus \Pi_\perp D_M \Pi_\perp$. If α and β are (j, k) - and $(j - 1, k)$ -forms, respectively, then

$$\Pi_\perp D_M \Pi_\perp = x^{-1} \tilde{D},$$

where

$$\tilde{D} = D_{N,sc} + \begin{pmatrix} \Pi_{\perp} D_F \Pi_{\perp} & (b - j + 1 - f + k)x \\ (k - j)x & -\Pi_{\perp} D_F \Pi_{\perp} \end{pmatrix}.$$

The diagonal terms in this final matrix are constant in x and invertible on $\Pi_{\perp} L^2 \Omega_{fc}^*$, and reasoning as before, for any $a \in \mathbb{R}$, the mapping

$$\Pi_{\perp} D_M \Pi_{\perp} : x^a \Pi_{\perp} L^2 \Omega_{fc}^*(M) \longrightarrow x^{a-1} \Pi_{\perp} L^2 \Omega_{fc}^*(M)$$

has bounded inverse, G_{\perp}^a .

On the other hand, $\Pi_0 D_M \Pi_0 \in \text{Diff}_b^1(N; \Omega_{fc}^{*,k} \mathcal{H}^*(F))$ is a b -operator (it has no $x^{-1} d_F$ or $x^{-1} \delta_F$ terms!) and equals

$$\begin{pmatrix} D_B & -x \partial_x - (f - k) \\ x \partial_x + k & -D_B \end{pmatrix}. \tag{36}$$

This operator preserves fibre degrees, so we can reduce to any fixed $\Omega_{fc}^{*,k}(M)$, for example, when computing indicial roots. We have

$$I_{(\Pi_0 D_M \Pi_0)^2}(\gamma) = \begin{pmatrix} D_B^2 - (\gamma + f - k)(\gamma + k) & 0 \\ 0 & D_B^2 - (\gamma + f - k)(\gamma + k) \end{pmatrix}.$$

The critical exponent in the surjectivity calculation is $\gamma = -f/2$, and inserting this into the expression above gives $D_B^2 + (k - f/2)^2$ in both diagonal components. Hence elements in the null space are in $L^2 \mathcal{H}^*(B)$ and are of fibre degree $k = f/2$. As before, at this point one also checks that $-f/2$ is not an indicial root of multiplicity two, which simply involves showing as usual that (36) has no solutions of the form $\omega' x^{-f/2} \log x + \omega'' x^{-f/2}$.

In any case, so long as a is not in the indicial set of $\Pi_0 D_M \Pi_0$, then

$$\Pi_0 D_M \Pi_0 : x^a \Pi_0 L^2 \Omega_{fc}^*(M) \longrightarrow x^a \Pi_0 L^2 \Omega_{fc}^*(M)$$

is Fredholm with generalized inverse G_0^a .

Altogether, this gives the generalized inverse

$$G^a = G_0^a \oplus G_{\perp}^a : x^a \Pi_0 L^2 \Omega_{fc}^*(M) \oplus x^{a-1} \Pi_{\perp} L^2 \Omega_{fc}^*(M) \longrightarrow x^a H^1 \Omega_{fc}^*(M),$$

and $I - G^a D_M = \Pi_M^a$ is the projection onto the null space of $\Pi_0 D_M \Pi_0$ at weight a , all elements of which are polyhomogeneous.

In summary, the mappings

$$D_M : x^a H^1 \Omega_{fc}^*(M) \longrightarrow x^a \Pi_0 L^2 \Omega_{fc}^*(M) \oplus x^{a-1} \Pi_{\perp} L^2 \Omega_{fc}^*(M) \tag{37}$$

and

$$D_M : x^a \Pi_0 H^1 \Omega_{fc}^*(M) \oplus x^{a+1} \Pi_{\perp} H^1 \Omega_{fc}^*(M) \longrightarrow x^a L^2 \Omega_{fc}^*(M) \tag{38}$$

are Fredholm when a is not an indicial root of D_N .

As in the fibred boundary case, if $\eta = D_M\zeta$, where $\eta \in \mathcal{A}^a\Omega_{\text{fc}}^*(M)$ and $\zeta \in x^c\Pi_0 H^1\Omega_{\text{fc}}^*(M) \oplus x^{c+1}\Pi_{\perp} H_{\text{fc}}^1(M)$ for some $c < a$, then $\zeta \in \Pi_0\mathcal{A}_{\text{phg}}^*\Omega_{\text{fc}}^*(M) + \mathcal{A}^a\Omega_{\text{fc}}^*(M)$.

5.3. Manifolds with nonproduct fibre bundle ends

5.3.1. Geometry of fibrations

In this section we review some of the geometry associated to a Riemannian fibration and use it to describe the precise structure of D_Y . The exposition here is drawn from [3, §10.1], [25], [67], and [4], but since the notation in these sources varies considerably, it has seemed worthwhile to develop this material in detail.

Suppose that $G = \phi^*(h) + k$ is a metric on the total space of a fibration Y , where $\phi : Y \rightarrow B$ and $\phi^{-1}(b) = F_b$. As before, we assume that k annihilates the horizontal subbundle $T^H Y$, which is the orthogonal complement of the vertical tangent bundle T^V , and we let $P^V : TY \rightarrow T^V Y$, $P^H : TY \rightarrow T^H Y$ denote the orthogonal projections. The tangent bundle TB is naturally identified via ϕ_* with $T^H Y$, and we denote the lift of a section $X \in \mathcal{C}^\infty(B; TB)$ by \tilde{X} . In the following, we denote sections of $T^V Y$ and $T^H Y$ by U_1, U_2, \dots and $\tilde{X}_1, \tilde{X}_2, \dots$, respectively. Finally, let ∇^L denote the Levi-Civita connection of G .

The extent to which these subbundles fail to be parallel with respect to ∇^L is measured in terms of two tensor fields, the second fundamental form of the fibres, and the curvature of the horizontal distribution. The second fundamental form is the symmetric bilinear form on $T^V Y$ defined by

$$\mathbb{I}\tilde{X}(U_1, U_2) = \langle \nabla_{U_1}^L U_2, \tilde{X} \rangle. \tag{39}$$

We let $\mathbb{I}\mathbb{I}(U_1, U_2)$ be the horizontal vector given by

$$\langle \mathbb{I}\mathbb{I}(U_1, U_2), \tilde{X} \rangle = \mathbb{I}\tilde{X}(U_1, U_2),$$

and we let $\mathbb{I}\tilde{X}(U_1)$ denote the vertical vector determined by

$$\langle \mathbb{I}\tilde{X}(U_1), U_2 \rangle = \mathbb{I}\tilde{X}(U_1, U_2).$$

The nonintegrability of the horizontal distribution is measured by its curvature

$$\mathcal{R}(\tilde{X}_1, \tilde{X}_2) = P^V([\tilde{X}_1, \tilde{X}_2]), \tag{40}$$

which is tensorial and vertical. We define the horizontal vector $\hat{\mathcal{R}}_U(\tilde{X}_1)$ by

$$\langle \hat{\mathcal{R}}_U(\tilde{X}_1), \tilde{X}_2 \rangle = \langle \mathcal{R}(\tilde{X}_1, \tilde{X}_2), U \rangle = \langle [\tilde{X}_1, \tilde{X}_2], U \rangle. \tag{41}$$

Four additional facts are used repeatedly. First, the bracket of a vertical vector field with the horizontal lift of a vector field from B is again vertical; that is,

$$[\tilde{X}, U] \in \mathcal{C}^\infty(Y, T^V Y).$$

This is proved by noting that vertical vector fields are characterized by the fact that they annihilate functions of the form $\phi^* f$, $f \in \mathcal{C}^\infty(B)$. Second, the Koszul formula determines the Levi-Civita connection in terms of the metric and Lie brackets:

$$\begin{aligned} \langle \nabla_{V_1}^L V_2, V_3 \rangle &= \frac{1}{2} \{ \langle [V_1, V_2], V_3 \rangle - \langle [V_2, V_3], V_1 \rangle + \langle [V_3, V_1], V_2 \rangle \\ &\quad + V_1 \langle V_2, V_3 \rangle + V_2 \langle V_1, V_3 \rangle - V_3 \langle V_1, V_2 \rangle \}, \end{aligned}$$

for any $V_1, V_2, V_3 \in \mathcal{C}^\infty(Y, TY)$.

Third, by definition of the induced Levi-Civita connection ∇^F on the fibres,

$$\langle \nabla_{U_1}^L U_2, U_3 \rangle = \langle \nabla_{U_1}^F U_2, U_3 \rangle.$$

Finally, since vertical and horizontal vector fields are perpendicular and because the vertical distribution is integrable,

$$\langle [U_1, U_2], \tilde{X} \rangle = U_1 \langle \tilde{X}, U_2 \rangle = U_2 \langle \tilde{X}, U_1 \rangle = 0.$$

We now determine the vertical and horizontal components of $\nabla_{V_1}^L V_2$ when the V_j are, successively, vertical and horizontal fields. First, by definition, the horizontal part of $\nabla_{U_1}^L U_2$ is

$$\langle \nabla_{U_1}^L U_2, \tilde{X} \rangle = \langle \mathbb{I}\mathbb{I}\mathbb{I}(U_1, U_2), \tilde{X} \rangle,$$

and also by definition, the vertical part is $\nabla_{U_1}^F U_2$.

From the Koszul formula and the expansion of $\tilde{X} \langle U_1, U_2 \rangle$ using (39),

$$\langle \nabla_{\tilde{X}}^L U_1, U_2 \rangle = \langle [\tilde{X}, U_1], U_2 \rangle - \langle \mathbb{I}\mathbb{I}\mathbb{I}_{\tilde{X}}(U_1), U_2 \rangle,$$

or in other words,

$$P^V \nabla_{\tilde{X}}^L U = [\tilde{X}, U] - \mathbb{I}\mathbb{I}\mathbb{I}_{\tilde{X}}(U).$$

As for the horizontal component of $\nabla_{\tilde{X}}^L U$, most of the terms in the Koszul formula vanish, leaving only

$$\langle \nabla_{\tilde{X}_1}^L U, \tilde{X}_2 \rangle = -\frac{1}{2} \langle [\tilde{X}_1, \tilde{X}_2], U \rangle = -\frac{1}{2} \langle \hat{\mathcal{R}}_U(\tilde{X}_1), \tilde{X}_2 \rangle.$$

Next,

$$\langle \nabla_{U_1}^L \tilde{X}, U_2 \rangle = -\langle \tilde{X}, \nabla_{U_1}^L U_2 \rangle = -\langle \mathbb{I}\mathbb{I}\mathbb{I}_{\tilde{X}}(U_1), U_2 \rangle$$

is the vertical part of $\nabla_{U_1}^L \tilde{X}$, and the horizontal part is

$$\langle \nabla_U^L \tilde{X}_1, \tilde{X}_2 \rangle = \langle \nabla_{\tilde{X}_1}^L U + [U, \tilde{X}_1], \tilde{X}_2 \rangle = \langle \nabla_{\tilde{X}_1}^L U, \tilde{X}_2 \rangle = -\frac{1}{2} \langle \hat{\mathcal{R}}_U(\tilde{X}_1), \tilde{X}_2 \rangle,$$

where the Koszul formula is used for the final equality.

Finally, putting the covariant derivative on the other side of the inner product and using the last equality of the previous displayed formula,

$$\langle \nabla_{\tilde{X}_1}^L \tilde{X}_2, U \rangle = \frac{1}{2} \langle \hat{\mathcal{R}}(\tilde{X}_1, \tilde{X}_2), U \rangle,$$

and at last,

$$\langle \nabla_{\tilde{X}_1}^L \tilde{X}_2, \tilde{X}_3 \rangle = \langle \nabla_{X_1}^B X_2, X_3 \rangle,$$

where ∇^B is the Levi-Civita connection on (B, h) . This last formula holds because all the terms in the Koszul formula expansion depend only on h .

In summary, we have proved the following.

PROPOSITION 13

The Levi-Civita connection decomposes into vertical and horizontal components as

$$\begin{aligned} \nabla_{U_1}^L U_2 &= \nabla_{U_1}^F U_2 + \text{III}(U_1, U_2), \\ \nabla_{\tilde{X}}^L U &= ([\tilde{X}, U] - \text{III}_{\tilde{X}}(U)) - \frac{1}{2} \hat{\mathcal{R}}_U(\tilde{X}), \\ \nabla_U^L \tilde{X} &= -\text{III}_{\tilde{X}}(U) - \frac{1}{2} \hat{\mathcal{R}}_U(\tilde{X}), \\ \nabla_{\tilde{X}_1}^L \tilde{X}_2 &= \frac{1}{2} \hat{\mathcal{R}}(\tilde{X}_1, \tilde{X}_2) + (\nabla_{X_1}^B X_2)^\sim. \end{aligned} \tag{42}$$

We wish to define a new connection that preserves the splitting of TY . As a first guess, one might do this by projecting ∇^L onto the vertical and horizontal subspaces, that is, by defining $\nabla_V U = P^V(\nabla_V^L U)$, $\nabla_V(\tilde{X}) = P^H(\nabla_V^L(\tilde{X}))$, where V is any vector (either horizontal or vertical). The formulae above indicate which terms should be subtracted from ∇^L to accomplish this. However, there is another natural choice, which turns out to be more convenient for many computational purposes, given by using the projected connection on the vertical bundle and lifting the connection on the horizontal bundle from the Levi-Civita connection on B . In other words, we define

$$\nabla := (P^V \nabla^L) \oplus \nabla^B$$

or, even more specifically,

$$\begin{aligned} \nabla_{U_1} U_2 &= P^V(\nabla_{U_1}^L U_2), & \nabla_{\tilde{X}} U &= P^V(\nabla_{\tilde{X}}^L U) = [\tilde{X}, U] - \text{III}_{\tilde{X}}(U), \\ \nabla_U \tilde{X} &= 0, & \nabla_{\tilde{X}_1} \tilde{X}_2 &= (\nabla_{X_1}^B X_2). \end{aligned}$$

We use this connection henceforth. Notice that it differs from the projected connection only in the removal of the terms $(1/2)\widehat{\mathcal{R}}_U(\tilde{X})$. One important feature of ∇ vis a vis computations related to the families index theorem is that it is in *upper triangular form* with respect to the vertical/horizontal splitting (cf. [3]).

The difference tensor $Q = \nabla^L - \nabla$ is given by

$$\begin{aligned}
 Q_{U_1}(U_2) &= \text{III}(U_1, U_2), & Q_{\tilde{X}}(U) &= -\frac{1}{2}\widehat{\mathcal{R}}_U(\tilde{X}), \\
 Q_U(\tilde{X}) &= -\text{III}_{\tilde{X}}(U) - \frac{1}{2}\widehat{\mathcal{R}}_U(\tilde{X}), & Q_{\tilde{X}_1}(\tilde{X}_2) &= \frac{1}{2}\mathcal{R}(\tilde{X}_1, \tilde{X}_2).
 \end{aligned}$$

We note also that the torsion tensor of ∇ is the negative of the skew-symmetrization of Q .

We now express the de Rham differential d_Y and its adjoint in terms of ∇ , III , and \mathcal{R} . Because the connections ∇^L and ∇ are both metric connections, they act on 1-forms by duality. That is, if ϕ is the 1-form given by $\langle w, \cdot \rangle$, then $\nabla_Z \phi$ is the 1-form given by $\langle \nabla_Z w, \cdot \rangle$. The action extends to forms of higher degree as a derivation.

Let $e_i, i = 1, \dots, f$, and $\eta_\mu, \mu = 1, \dots, b$, be orthonormal frame fields for F and B , respectively, and let $\{e^i\}, \{\eta^\mu\}$ be the dual coframe fields. It is standard that

$$d_Y = \sum_{i=1}^f e^i \wedge \nabla_{e_i}^L + \sum_{\mu=1}^b \eta^\mu \wedge \nabla_{\eta_\mu}^L \tag{43}$$

with analogous formulae for d_F and d_B . Now substitute $\nabla_L = \nabla + Q$ into (43) to get first

$$\begin{aligned}
 d_Y e^j &= d_F e^j + \sum \eta^\mu \wedge \nabla_{\eta_\mu} e^j \\
 &\quad - \sum \left((\text{III}_{\eta_\mu}(e_i), e_j) \eta^\mu \wedge e^i + \frac{1}{2} \langle \mathcal{R}(\eta_\mu, \eta_\nu), e_j \rangle \eta^\mu \wedge \eta^\nu \right), \tag{44}
 \end{aligned}$$

and then

$$d_Y \eta^\mu = d_B \eta^\mu. \tag{45}$$

The last formula initially has many terms, all of which cancel, but the result is no surprise since $d_Y \phi^* = \phi^* d_B$.

Now extend to forms of higher degrees. First, the splitting of TY induces a decomposition

$$\Lambda^k(T^*Y) = \bigoplus_{p+q=k} \Lambda^{p,q}(T^*Y),$$

where

$$\Lambda^{p,q}(T^*Y) = \Lambda^p((T^V Y)^*) \otimes \Lambda^q((T^H Y)^*).$$

We regard the space of sections $\Omega^{p,q}(Y)$ as the completed tensor product $\Omega^p(B) \hat{\otimes} \Omega^q(Y, T^V Y)$. By construction, ∇ preserves this splitting. Thus for $\omega \in \Omega^{p,q}(Y)$ with $\omega = \phi^*(\alpha) \wedge \beta$, $\alpha \in \Omega^p(B)$, and $\beta \in \mathcal{C}^\infty(Y, \Lambda^q(T^V Y)^*)$,

$$d_F(\phi^*(\alpha) \wedge \beta) = (-1)^p \phi^*(\alpha) \wedge d_F \beta,$$

and we also define

$$\tilde{d}_B \phi^*(\alpha) \wedge \beta = \phi^*(d_B \alpha) \wedge \beta + (-1)^p \phi^*(\alpha) \wedge \left(\sum_{\mu} \eta^\mu \wedge \nabla_{\eta_\mu} \beta \right).$$

Rewrite (44) as

$$d_Y e^j = d_F e^j + \tilde{d}_B e^j - \mathbb{I}\mathbb{I}(e^j) - \frac{1}{2} \mathcal{R}(e^j),$$

where

$$\mathbb{I}\mathbb{I}(e^j) = \mathbb{I}\mathbb{I}_{\mu ij} \eta^\mu \wedge e^i, \quad \mathcal{R}(e^j) = \mathcal{R}_{\mu\nu j} \eta^\mu \wedge \eta^\nu.$$

To simplify notation, let $R = -(1/2)\mathcal{R}$. Then we have proved the first part of the following.

PROPOSITION 14

We have $d_Y = d_F + \tilde{d}_B - \mathbb{I}\mathbb{I} + R$, $\delta_Y = \delta_F + (\tilde{d}_B)^* - \mathbb{I}\mathbb{I}^* + R^*$.

The second part is tautologous. Notice that

$$\begin{aligned} d_F : \Omega^{p,q}(Y) &\rightarrow \Omega^{p,q+1}(Y), & \tilde{d}_B : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+1,q}(Y), \\ \mathbb{I}\mathbb{I} : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+1,q}(Y), & R : \Omega^{p,q}(Y) &\rightarrow \Omega^{p+2,q-1}(Y). \end{aligned}$$

We can deduce some useful information from the fact that both d_Y and d_F are legitimate differentials; that is, their squares are zero. First, there is a Kodaira decomposition on the fibres, so any smooth form α on Y can be decomposed uniquely and orthogonally as $\alpha = d_F \eta + \delta_F \mu + \gamma$, where γ is fibre harmonic. Thus

$$\Pi_0 d_F = \Pi_0 \delta_F = d_F \Pi_0 = \delta_F \Pi_0 = 0. \tag{46}$$

Second, applying $d_Y^2 = 0$ to a form of pure bidegree and decomposing into bidegrees gives

$$\begin{aligned} R^2 &= 0, & d_F^2 &= 0, \\ d_F(\tilde{d}_B - \mathbb{I}\mathbb{I}) + (\tilde{d}_B - \mathbb{I}\mathbb{I})d_F &= 0, \\ R(\tilde{d}_B - \mathbb{I}\mathbb{I}) + (\tilde{d}_B - \mathbb{I}\mathbb{I})R &= 0, \\ d_F R + R d_F &= -(\tilde{d}_B - \mathbb{I}\mathbb{I})^2 \end{aligned} \tag{47}$$

with analogous relationships between the adjoints of these operators.

Now define the operator

$$\mathfrak{d} = \Pi_0(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0;$$

this acts on the space of fibre-harmonic forms.

PROPOSITION 15

The operator \mathfrak{d} and its adjoint \mathfrak{d}^* are differentials; that is, $\mathfrak{d}^2 = (\mathfrak{d}^*)^2 = 0$.

Proof

It suffices to prove only one of these. Recalling that $\Pi_0 = I - \Pi_\perp$, we have

$$\mathfrak{d}^2 = \Pi_0(\tilde{d}_B - \mathbb{I}\mathbb{I})^2\Pi_0 - \Pi_0(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_\perp(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0;$$

substituting from (47) and using (46), this equals

$$-\Pi_0(d_F R + R d_F + (\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_\perp(\tilde{d}_B - \mathbb{I}\mathbb{I}))\Pi_0 = -\Pi_0(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_\perp(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0.$$

Now, $d_F(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0 = -(\tilde{d}_B - \mathbb{I}\mathbb{I})d_F\Pi_0 = 0$, so for any form α , $(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0\alpha = d_F\eta + \gamma$ with γ fibre-harmonic, and hence $\Pi_\perp(\tilde{d}_B - \mathbb{I}\mathbb{I})\Pi_0\alpha = d_F\eta$. Finally,

$$\mathfrak{d}^2\alpha = -\Pi_0(\tilde{d}_B - \mathbb{I}\mathbb{I})d_F\eta = \Pi_0d_F(\tilde{d}_B - \mathbb{I}\mathbb{I})\eta = 0. \quad \square$$

COROLLARY 5

Let $\mathbb{D} = \mathfrak{d} + \mathfrak{d}^*$, and suppose that $\mathbb{D}\alpha = 0$ for some fibre-harmonic form α . Then $\mathfrak{d}\alpha = \mathfrak{d}^*\alpha = 0$, and so the terms $\alpha_{p,q}$ of pure bidegree also satisfy $\mathbb{D}\alpha_{p,q} = 0$.

This follows just as for the usual Hodge Laplacian, for $\mathbb{D}^2 = \mathfrak{d}^*\mathfrak{d} + \mathfrak{d}\mathfrak{d}^*$ preserves bidegree, and so

$$0 = \langle \mathbb{D}^2\alpha, \alpha \rangle = \|\mathfrak{d}\alpha\|^2 + \|\mathfrak{d}^*\alpha\|^2;$$

in addition, we have that both \mathfrak{d} and \mathfrak{d}^* commute with \mathbb{D} .

5.3.2. Hodge–de Rham operators in general

The structure of the Hodge–de Rham operators for general exact fibred boundary and fibred cusp metrics is obtained by substituting the expression for d_Y from Proposition 14 into (32) and (35). To distinguish them, we write D_{fb} for the operator D_M associated to the fibred boundary metric g_{fb} and write D_{fc} for this operator associated to the fibred cusp metric g_{fc} . The action of D_{fb} on $\omega = \alpha/x^k + (dx/x^2) \wedge \beta/x^{k-1} \in \Omega_{\text{fb}}^{k,*}(M)$ is given by replacing the second matrix in (32) with

$$\begin{pmatrix} D_F + xD_B - x(II + II^*) + x^2(R + R^*) & 0 \\ 0 & -D_F - xD_B + x(II + II^*) - x^2(R + R^*) \end{pmatrix}.$$

Similarly, the action of D_{fc} on $\omega = x^k\alpha + (dx/x) \wedge x^k\beta \in \Omega_{fc}^{*,k}(M)$ is obtained by substituting

$$\begin{pmatrix} x^{-1}D_F + D_B - (II + II^*) + x(R + R^*) & 0 \\ 0 & -x^{-1}D_F - D_B + (II + II^*) - x(R + R^*) \end{pmatrix}$$

for the second matrix in (35).

As explained in the beginning of §5, the construction of parametrices for D_{fb} and D_{fc} requires the machinery of fibred boundary pseudodifferential operators. The basic strategy is the same in that one inverts $\Pi_0 D \Pi_0$ and $\Pi_{\perp} D \Pi_{\perp}$ separately but now must also show that the off-diagonal terms $\Pi_0 D \Pi_{\perp}$ and $\Pi_{\perp} D \Pi_0$, which no longer vanish, play only an insignificant role. This is all carried out by Vaillant [67] (cf. especially [67, Proposition 3.27], although beware that the Fredholm result is misstated in the special case $\lambda_0 = 0$), and we simply quote the two results we need, looking back to the product case for motivation. Before stating these, we remark that the operators Π_0, Π_{\perp} are only defined right at the boundary. However, the fibred boundary structure requires that we have fixed the one-jet of a defining function x along the fibres, and this implies that the spaces $x^c \Pi_0 L^2 \oplus x^{c \pm 1} \Pi_{\perp} L^2$ are well defined for any $c \in \mathbb{R}$ (because the weights differ only by 1).

PROPOSITION 16

Suppose that a is not an indicial root for $\Pi_0 D_{fb} \Pi_0$. Then

$$D_{fb} : x^a H_{fb}^1(M) \longrightarrow x^{a+1} \Pi_0 L^2 \Omega_{fb}^*(M) \oplus x^a \Pi_{\perp} L^2 \Omega_{fb}^*(M) \tag{48}$$

and

$$D_{fb} : x^{a-1} \Pi_0 H^1 \Omega_{fb}^*(M) \oplus x^a \Pi_{\perp} H_{fb}^1(M) \longrightarrow x^a L^2 \Omega_{fb}^*(M) \tag{49}$$

are Fredholm. If $D_{fb}\omega = 0$, then ω is polyhomogeneous with exponents in its expansion determined by the indicial roots of $\Pi_0 x^{-1} D_{fb} \Pi_0$, while if $\eta \in \mathcal{A}^a \Omega_{fb}^*(M)$, $\zeta \in x^{c-1} \Pi_0 H^1 \Omega_{fb}^*(M) \oplus x^c \Pi_{\perp} H_{fb}^1(M)$ for $c < a$ and $\eta = D_{fb}\zeta$, then $\zeta \in \Pi_0 \mathcal{A}_{phg}^1 \Omega_{fb}^*(M) + \mathcal{A}^a \Omega_{fb}^*(M)$.

PROPOSITION 17

Suppose that a is not an indicial root for $\Pi_0 D_{fc} \Pi_0$. Then

$$D_{fc} : x^a H_{fc}^1(M) \longrightarrow x^a \Pi_0 L^2 \Omega_{fc}^*(M) \oplus x^{a-1} \Pi_{\perp} L^2 \Omega_{fc}^*(M) \tag{50}$$

is Fredholm. If $a + 1$ is not an indicial root, then

$$D_{fc} : x^a \Pi_0 H^1 \Omega_{fc}^*(M) \oplus x^{a+1} \Pi_{\perp} H^1 \Omega_{fc}^*(M) \longrightarrow x^a L^2 \Omega_{fc}^*(M) \tag{51}$$

is Fredholm. If $D_{fc}\omega = 0$, then ω is polyhomogeneous, with exponents in its expansion determined by the indicial roots of $\Pi_0 D_{fc} \Pi_0$, while if $\eta \in \mathcal{A}^a \Omega_{fc}^*(M)$,

$\zeta \in x^c \Pi_0 H^1 \Omega_{fc}^*(M) \oplus x^{c+1} \Pi_{\perp} H_{fc}^1(M)$ for $c < a$ and $\eta = D_{fc} \zeta$, then $\zeta \in \Pi_0 \mathcal{A}_{phg}^1 \Omega_{fc}^*(M) + \mathcal{A}^a \Omega_{fc}^*(M)$.

We remark that the indicial roots for the operators $\Pi_0 D \Pi_0$, $D = D_{fb}$ or D_{fc} , are different than in the product case because of the term $\text{III} + \text{III}^*$ and because of the action of \tilde{d}_B on the fibre part of forms; on the other hand, the term $\mathcal{R} + \mathcal{R}^*$ is lower order at $x = 0$ and does not affect the indicial roots.

5.4. Hodge theorems for fibred boundary and fibred cusp metrics

We now complete the proofs of the identifications of L^2 harmonic forms with weighted cohomology in the two cases.

THEOREM 1C

If (M, g) is a manifold with fibred boundary metric, then for every k there is a natural isomorphism

$$L^2 \mathcal{H}^k(M) \longrightarrow \text{Im} \left(\mathcal{W} H^k(M, g_{fb}, \epsilon) \longrightarrow \mathcal{W} H^k(M, g_{fb}, -\epsilon) \right). \tag{52}$$

Proof

If $\omega \in L^2 \mathcal{H}^k(M)$, then Proposition 16 shows that ω is polyhomogeneous and hence lies in $x^{\epsilon_0} L^2 \Omega_{fb}^k(M)$ for some $\epsilon_0 > 0$ (with polyhomogeneous coefficients). This gives the mapping

$$\begin{aligned} L^2 \mathcal{H}^k(M) &\longrightarrow \mathcal{W} H^k(M, g_{fb}, \epsilon) \\ &\longrightarrow \text{Im} \left(\mathcal{W} H^k(M, g_{fb}, \epsilon) \longrightarrow \mathcal{W} H^k(M, g_{fb}, -\epsilon) \right). \end{aligned}$$

If $[\omega] = 0$, then $\omega = d\zeta$ for some $\zeta \in x^{-\epsilon-1} L^2 \Omega_{fb}^{k-1}(M)$; by the discussion in §2.4, we can choose ζ to be conormal. Write

$$\omega = \sum_{p,q} \frac{\alpha_{p,q}}{x^p} + \frac{dx}{x^2} \wedge \frac{\beta_{p,q}}{x^p}, \quad \zeta = \sum_{p,q} \frac{\mu_{p,q}}{x^p} + \frac{dx}{x^2} \wedge \frac{\nu_{p,q}}{x^p},$$

where $|\alpha_{p,q}|, |\beta_{p,q}| = \mathcal{O}(x^{(b+1)/2+\epsilon_0})$ and $|\mu_{p,q}|, |\nu_{p,q}| = \mathcal{O}(x^{(b-1)/2+\epsilon})$. The usual integration by parts gives

$$\begin{aligned} \|\omega\|^2 &= \int_M d\zeta \wedge * \omega = \int_M d(\zeta \wedge * \omega) = \lim_{x \rightarrow 0} \int_{B \times F} \zeta \wedge * \omega \\ &= \lim_{x \rightarrow 0} \sum_{p,q} \int_Y \frac{\mu_{p,q}}{x^p} \wedge \frac{*_Y \beta_{p,q}}{x^{b-p}}, \end{aligned}$$

which vanishes by the decay properties of the $\mu_{p,q}$ and $\beta_{p,q}$. Thus $\omega = 0$, and this proves injectivity.

For surjectivity, we note that for sufficiently small $\epsilon > 0$, the space $L^2 \mathcal{H}^*(M)$ can be identified with the cokernel of the map

$$D_{\text{fb}} : x^{-\epsilon-1} \Pi_0 H^1 \Omega_{\text{fb}}^*(M) \oplus x^{-\epsilon} \Pi_{\perp} H^1 \Omega_{\text{fb}}^*(M) \longrightarrow x^{-\epsilon} L^2 \Omega_{\text{fb}}^*(M).$$

Thus we can write

$$x^{-\epsilon} L^2 \Omega_{\text{fb}}^*(M) = \text{Im}(D_{\text{fb}}|_{x^{-\epsilon-1} \Pi_0 H^1 \Omega_{\text{fb}}^*(M) + x^{-\epsilon} \Pi_{\perp} H^1 \Omega_{\text{fb}}^*(M)}) \oplus L^2 \mathcal{H}^*(M).$$

So suppose that $\eta \in x^{\eta} L^2 \Omega_{\text{fb}}^k(M)$ is a polyhomogeneous representative for a class in the space on the right in (52). Then $\eta = D_{\text{fb}} \zeta + \gamma$, where $\zeta \in \Pi_0 \mathcal{A}_{\text{phg}}^* \Omega_{\text{fb}}^*(M) \oplus \Pi_{\perp} \mathcal{A}^{\epsilon} \Omega_{\text{fb}}^*(M)$ and $\gamma \in L^2 \mathcal{H}^*(M)$. In fact, comparing orders of vanishing in x , we see that $\zeta = \zeta_0 + \zeta'$, $\zeta' \in \mathcal{A}^{\eta} \Omega_{\text{fb}}^*(M)$, and $\zeta_0 \in \ker I_{\Pi_0 x^{-1} D_M \Pi_0}((b-1)/2)$.

We must analyze the structure of ζ_0 more closely. Acting on pairs (α, β) , the indicial operator has the form

$$I_{\Pi_0 x^{-1} D_{\text{fb}} \Pi_0} \left(\frac{b-1}{2} \right) = \begin{pmatrix} \mathbb{D} & N_1 - (b-1)/2 \\ N_2 + (b-1)/2 & -\mathbb{D} \end{pmatrix},$$

where $\mathbb{D} = \mathfrak{d} + \mathfrak{d}^*$. The operators N_1 and N_2 are defined by $N_1 \beta_k = (b-k) \beta_k$ and $N_2 \alpha_k = -k \alpha_k$ (which agrees with the scattering case since $n = b+1$ there). Following the calculation and reasoning for the scattering case, we expand in terms of an eigenbasis for \mathbb{D}^2 and deduce that this indicial root has rank 1 and that an element of the null space of this indicial operator has the form

$$\zeta_0 = x^{(b-1)/2} \left(\frac{\alpha_{(b-1)/2}}{x^{(b-1)/2}} + \frac{dx}{x^2} \wedge \frac{\beta_{(b+1)/2}}{x^{(b+1)/2}} \right),$$

where $\alpha_{(b-1)/2}, \beta_{(b+1)/2} \in \ker \mathbb{D}$.

We now have

$$\begin{aligned} \|\delta \zeta\|^2 &= \langle \eta - d\zeta - \gamma, \delta \zeta \rangle = \langle d(\eta - d\zeta - \gamma), \zeta \rangle = \lim_{x \rightarrow 0} \int_Y \zeta_0 \wedge d * \zeta_0 \\ &= \lim_{x \rightarrow 0} \int_Y \alpha_{(b-1)/2} \wedge d_Y * \beta_{(b+1)/2} = \langle \alpha_{(b-1)/2}, \mathfrak{d}^* \beta_{(b+1)/2} + \mathbf{R}^* \beta_{(b+1)/2} \rangle_Y. \end{aligned}$$

But $\mathbf{R}^* \beta_{(b+1)/2}$ is a $((b-3)/2, *)$ -form, so it pairs trivially with $\alpha_{(b-1)/2}$. Hence $\delta \zeta = 0$.

The rest of the argument is as in the scattering case. \square

THEOREM 2C

If (M, g_{fc}) is a manifold with fibred cusp metric, then there is a natural isomorphism

$$L^2 \mathcal{H}^*(M) \longrightarrow \text{Im} (WH^*(M, \epsilon) \longrightarrow WH^*(M, -\epsilon)). \quad (53)$$

Proof

The proofs of the existence of this mapping and its injectivity are nearly identical to those in the fibred boundary case, so we omit them.

For the surjectivity argument, we decompose

$$x^{-\eta} L^2 \Omega^*(M) = (\text{ran } D_M|_{x^{-\eta} \Pi_0 H_b^1 \oplus x^{-\epsilon+1} \Pi_{\perp} H_b^1}) \oplus (\text{ran } D_M|_{x^{-\eta} \Pi_0 H_b^1 \oplus x^{-\epsilon+1} \Pi_{\perp} H_b^1})^{\perp}.$$

So we can write any $\eta \in x^{\eta} L^2 \Omega_{\text{fc}}^*(M)$ which represents a nontrivial class as $\eta = D\zeta + \gamma$, where $\gamma \in L^2 \mathcal{H}^*(M)$ and $\zeta \in x^{-\eta} L^2 \Omega_{\text{fc}}^*(M)$. Since the indicial root $\gamma = -f/2$ occurs with multiplicity one, we have $\zeta = \zeta_0 + \zeta'$, where $\zeta' \in \mathcal{A}^{\epsilon} \Omega_{\text{fc}}^*(M)$ and

$$\zeta_0 = \sum_k \left(x^k a_k + \frac{dx}{x} \wedge x^k \beta_k \right) x^{-f/2},$$

where a_k and β_k are independent of x and dx . Matching up powers of x in $\eta = D\zeta + \gamma$, we find that ζ_0 is in the null space of the operator $I_{\mathfrak{D}'}$, $\mathfrak{D}' = \Pi_0 D_{\text{fc}} \Pi_0$, which acts on $(*, k)$ -forms by

$$I_{\mathfrak{D}'} \left(-\frac{f}{2} \right) = \begin{pmatrix} \mathbb{D} & k - \frac{f}{2} \\ k - \frac{f}{2} & \mathbb{D} \end{pmatrix}.$$

This implies that α and β must both be forms on B with coefficients in $\mathcal{H}^{f/2}(F)$ and in the kernel of \mathbb{D} . Thus the boundary term in the integration by parts vanishes as in the fibred boundary case. □

5.5. *From weighted cohomology to intersection cohomology*

To prove our main theorems, we must relate the weighted cohomology groups appearing in the statements of Theorems 1C and 2C to intersection cohomology groups. Most of the work has already been done in §2.3, so it remains only to reinterpret the answers.

The statement for fibred cusp metrics is slightly simpler, so we consider that case first. We have proved that when (M, g) is a fibred cusp metric, then

$$L^2 \mathcal{H}^*(M) \cong \text{Im} \left(WH^*(M, \epsilon) \longrightarrow WH^*(M, -\epsilon) \right).$$

Using Proposition 2, this is equivalent to

$$L^2 \mathcal{H}^*(M) \cong \text{Im} \left(IH_{[\epsilon+(f/2)]}^*(X, B) \longrightarrow IH_{[-\epsilon+(f/2)]}^*(X, B) \right),$$

where X is the compactification of M defined in the introduction. The two spaces on the right correspond to intersection cohomology with the middle perversities

$$\underline{\mathfrak{m}}(f+1) = \begin{cases} \frac{f-1}{2}, & f \text{ odd,} \\ \frac{f}{2}, & f \text{ even,} \end{cases} \quad \overline{\mathfrak{m}}(f+1) = \begin{cases} \frac{f-1}{2}, & f \text{ odd,} \\ \frac{f}{2} - 1, & f \text{ even,} \end{cases}$$

respectively. This proves the main theorem (Theorem 2).

THEOREM 2'

Suppose that (M, g) is a manifold with fibred cusp metric. Then

$$L^2 \mathcal{H}^*(M) \cong \text{Im}(IH_{\mathfrak{m}}^*(X, B) \longrightarrow IH_{\mathfrak{m}}^*(X, B)).$$

We remark on a few special cases of this result.

If $f = 0$ (i.e., (M, g) has cylindrical ends), then

$$IH_{[\epsilon+(f/2)]}^*(X, B) = H^*(M, \partial M), \quad IH_{[-\epsilon+(f/2)]}^*(X, B) = H^*(M),$$

and so the Hodge cohomology is equated with the image of the relative in absolute cohomology, as already proved in §4.

If $\dim F = f > 0$, then the two spaces on the right coincide when f is odd, or if we have only $H^{f/2}(F) = 0$, that is, if (X, B) is a Witt space. In either case, $L^2 \mathcal{H}^*(M)$ equals the (unique) middle perversity intersection cohomology $IH_{\mathfrak{m}}^*(X, B)$.

We can see this simplification directly from the analysis in the last section. Recall the decomposition $\eta = d\zeta + \gamma$ for the closed form $\eta \in \mathcal{A}^\eta \Omega_{\text{fc}}^k(M)$. We have $\zeta = \zeta_0 + \zeta'$, where $\zeta' \in \mathcal{A}^\epsilon \Omega_{\text{fc}}^*(M)$ and ζ_0 is the sum of pullbacks of forms on B wedged with elements of $H^{f/2}(F)$. But the assumption that X is a Witt space gives $\zeta_0 = 0$, and hence $[\eta] = [\gamma]$ already in $WH(M, g_{\text{fc}}, \epsilon)$. Thus in this case,

$$WH(M, g_{\text{fc}}, -\epsilon) = WH(M, g_{\text{fc}}, \epsilon) = WH(M, g_{\text{fc}}, 0) = L^2 \mathcal{H}^*(M),$$

and all these spaces are finite-dimensional. This already follows from [68, Corollary 2.34]. Finally, the discussion in §2.3 shows how to interpret this in terms of intersection cohomology.

However, when $H^{f/2}(F) \neq 0$, the unweighted L^2 -cohomology is infinite-dimensional and the two middle perversity intersection cohomologies are different. In this case, some sort of more elaborate analysis, such as we have carried out in this paper, is needed.

We obtain the Hodge theorem for fibred boundary metrics by a translation from the fibred cusp case. To do this, first rewrite the isomorphism

$$L^2 \mathcal{H}^k(M) \cong \text{Im}(\mathcal{W}H^k(M, g_{\text{fb}}, \epsilon) \longrightarrow \mathcal{W}H^k(M, g_{\text{fb}}, -\epsilon))$$

in terms of weighted L^2 -cohomology for the associated fibred cusp metric $g_{\text{fc}} = x^2 g_{\text{fb}}$. This gives

$$L^2 \mathcal{H}^k(M) \cong \text{Im}((WH^k(M, g_{\text{fc}}, n/2 - k + \epsilon) \longrightarrow WH^k(M, g_{\text{fc}}, n/2 - k - \epsilon)),$$

and hence by Proposition 2, we get the following.

THEOREM 1'

If (M, g) is a fibred boundary metric, then

$$L^2 \mathcal{H}^k(M) \cong \text{Im} (IH_{[(n+f)/2-k+\epsilon]}^k(X, B) \longrightarrow IH_{[(n+f)/2-k-\epsilon]}^k(X, B)).$$

We list the various cases.

Suppose b is even. Since $n = b + f + 1$, this is the same as when $n + f$ is odd, and then the two groups are the same, so that

$$L^2 \mathcal{H}^k(M) \cong IH_{f+b/2-k}^k(X, B) \cong \begin{cases} H^k(X, B), & k \leq \frac{b}{2}, \\ IH_{f-1}^k(X, B), & k = \frac{b}{2} + 1, \\ \vdots \\ IH_0^k(X, B), & k = n - \frac{b}{2} + 1, \\ H^k(X \setminus B), & k \geq n - \frac{b}{2}. \end{cases}$$

Just as in the fibred cusp case, when b is even, the form ζ_0 which arises in the surjectivity argument must vanish since it lies in $\Omega^{(b\pm 1)/2,*} = \{0\}$ on the boundary. Hence the map Φ is now surjective onto $\mathcal{H}^*(M, g_{\text{fb}}, \epsilon)$. In this case the range of D is closed, and the theorem follows from the techniques of [68].

When b is odd,

$$L^2 \mathcal{H}^k(M) \cong \text{Im} (IH_{f+(b+1)/2-k}^k(X, B) \longrightarrow IH_{f+(b-1)/2-k}^k(X, B))$$

$$\cong \begin{cases} H^k(X, B), & k \leq \frac{b-1}{2}, \\ \text{Im}(H^k(X, B) \longrightarrow IH_{f-1}^k(X, B)), & k = \frac{b-1}{2} + 1, \\ \text{Im}(IH_{f-1}^k(X, B) \longrightarrow IH_{f-2}^k(X, B)), & k = \frac{b-1}{2} + 2, \\ \vdots \\ \text{Im}(IH_1^k(X, B) \longrightarrow IH_0^k(X, B)), & k = n - \frac{b-1}{2} - 2, \\ \text{Im}(IH_0^k(X, B) \longrightarrow H^k(X \setminus B)), & k = n - \frac{b-1}{2} - 1, \\ H^k(X \setminus B), & k \geq n - \frac{b-1}{2}. \end{cases}$$

Simpler corollaries of this theorem, for cases when F is a sphere and X a smooth manifold, were stated in Corollary 1 in the introduction.

6. Relationship to other works

We now briefly discuss some consequences of the Hodge theorems proved here and their relationship to other work in the field.

Carron's Hodge theorem for manifolds with flat ends

In a recent paper [12], Carron has calculated the Hodge cohomology for manifolds with finitely many ends, on all of which it is assumed that the curvature tensor vanishes identically. He uses two main tools: a precise geometric structure theorem for

flat ends (see [27]), and his theory of nonparabolicity at infinity in order to obtain new function spaces that are extensions of $H_0^1\Omega^*(M)$ and on which the range of D is closed. This work has substantial overlap with ours in the sense that many but not all fibred boundary and fibred cusp metrics are nonparabolic at infinity and satisfy the extra conditions implied by the flatness hypothesis.

The signature formula of Dai and Vaillant

As discussed in the introduction, an immediate corollary of Theorems 1 and 2 is that

$$\text{sgn}_{L^2}(M, g) = \text{sgn Im} (IH_{\underline{m}}(X, B) \longrightarrow IH_{\overline{m}}(X, B)). \tag{54}$$

This formula holds both for fibred boundary and fibred cusp metrics.

On the other hand, there is an L^2 signature theorem for manifolds with fibred cusp ends proved by Dai [25] and generalized by Vaillant [67]:

$$\text{sgn}_{L^2}(M, g) = \text{sgn Im} (H^*(M, \partial M) \longrightarrow H^*(M)) + \tau. \tag{55}$$

The final term here is the τ -invariant, originally defined by Dai, which is a sum of signatures coming from the higher terms in the Leray spectral sequence for the fibration of ∂M . Combining these two signature theorems now identifies $\tau = \tau(\partial M)$ with the difference of the two algebraic signatures in (54) and (55) (see (4) in the introduction). The original definition of τ involves algebraic signatures on the higher terms (i.e., the E_k -terms, $k \geq 3$) of the Leray spectral sequence of the fibration for ∂M . It seems very tempting to conjecture that the summands in this definition arise from signatures on the weighted cohomology for weights $\pm a$, where a varies from some small positive number to one sufficiently large so that the weighted cohomologies $WH(M, g, \pm a)$ equal the relative and absolute cohomologies, respectively. There should be finitely many jumps in this deformation, and the intermediate weighted cohomologies should correspond to intersection cohomologies with perversities varying from lower middle or upper middle to one of the extremes. We shall return to a precise exploration of these ideas elsewhere.

Hitchin's Hodge theorem

Section 7 contains an explanation of our Hodge and signature theorems in several interesting examples. Most of those examples are hyperkähler, and the Hodge cohomology of such manifolds has been recently studied by Hitchin [44]. Amongst his results is one particularly relevant to our paper.

THEOREM 3 (Hitchin)

Let M be a complete hyperkähler manifold of real dimension $4k$ such that one of the Kähler forms ω_i satisfies $\omega_i = d\beta$, where β has linear growth. Then any L^2

harmonic form on M is of degree $2k$ and is self-dual or antiself-dual provided that k is even (resp., odd).

This implies the following.

COROLLARY 6

If M is a hyperkähler manifold as above, then $\dim L^2\mathcal{H}^*(M, g) = |\operatorname{sgn}_{L^2}(M, g)|$.

Hence for the class of hyperkähler manifolds satisfying the hypothesis of Hitchin’s theorem (including most of the examples in §7), the Hodge cohomology can be computed from the L^2 -signature index theorem of Dai and Vaillant.

We obtain two consequences that follow from this result and the analysis developed for the proofs of our main theorems. The first gives an interesting topological obstruction to the existence of a fibred boundary or fibred cusp hyperkähler metric satisfying the linear growth hypothesis of Theorem 3.

COROLLARY 7

If M is a hyperkähler manifold as in Theorem 3 which is either of fibred cusp or fibred boundary type, then the intersection form on $H^*(M, \partial M)$ is semidefinite, so that $\operatorname{sgn}(M)$ is nonpositive if k is odd and nonnegative if k is even.

Proof

To be definite, suppose that g is a fibred cusp metric. We know by Theorem 3 that the intersection form on $L^2\mathcal{H}^{2k}(M, g)$ is semidefinite of the correct sign. On the other hand, the topological signature of a manifold with boundary is by definition the index of the intersection form on the image of (middle degree) relative cohomology in absolute. Thus we must show that this latter intersection form is also semidefinite.

Suppose that η and ν are smooth closed compactly supported $2k$ -forms that represent nontrivial classes in $\operatorname{Im}(H^{2k}(M, \partial M) \rightarrow H^{2k}(M))$. By Theorem 2 or, rather, by its proof in §5, we have $\eta = d\zeta + \gamma$, $\nu = d\zeta + \rho$, where $\gamma, \rho \in L^2\mathcal{H}^{2k}(M)$; we also have $\zeta = \zeta_0 + \zeta'$, where $\zeta' \in \mathcal{A}^\epsilon \Omega_{\text{fc}}^{2k}(M)$ and ζ_0 is polyhomogeneous with growth at just the critical value for square integrability and in addition is fibre-harmonic in form and in the kernel of \mathbb{D} . There is a similar decomposition for ζ .

We now compute that

$$\begin{aligned} \int_M \eta \wedge \nu &= \int_M (d\zeta + \gamma) \wedge (d\zeta + \rho) \\ &= \int_M d\zeta \wedge d\zeta + \int_M d\zeta \wedge \rho + \int_M \gamma \wedge d\zeta + \int_M \gamma \wedge \rho. \end{aligned}$$

Now integrate by parts in each of the first three terms on the right. Using the infor-

mation in the last paragraph, the boundary terms all vanish, and we are left with the equality of the pairing of η and ν with the pairing of γ and ρ , as desired. \square

Remark. This topological obstruction is investigated further in [41] for toric hyperkähler varieties.

The arguments in the proof above also yield the following.

COROLLARY 8

If M has a hyperkähler fibred boundary metric as above, then the τ -invariant of ∂M is nonpositive if k is odd and nonnegative if k is even.

7. Examples

A mathematically interesting theme in contemporary research in string theory involves the use of duality to predict the dimensions of spaces of L^2 harmonic forms on various classes of noncompact manifolds. Probably the most famous of these is the S -duality conjecture made by Sen in [63, page 220], which predicts the dimension of the Hodge cohomology on moduli spaces of monopoles on \mathbb{R}^3 ; these moduli spaces include the Atiyah-Hitchin manifold, the Taub-NUT space, and its higher-dimensional generalizations. A similar S -duality prediction in [66, page 57] concerns the Hodge cohomology of quiver varieties, while [39, Conjecture 1] contains a mathematical conjecture about the Hodge cohomology of moduli of Higgs bundles. Similarly to Sen's conjecture, these last predictions equate the Hodge cohomology of these moduli spaces with the image of compactly supported cohomology in absolute cohomology. We also mention the predictions about Hodge cohomology in [64] for multi-Taub-NUT spaces and in [10] for the G_2 -space constructed in that paper.

The justification of these predictions has been a key motivation for our work. In this final section we examine these conjectures in light of the results of this paper. The point is that, particularly in the low-dimensional cases, the moduli spaces in these conjectures carry natural fibred boundary metrics, and hence our Theorem 1 can be applied. We discuss several examples where we can confirm the predictions, but notably, we also show that the L^2 harmonic form predicted to exist on the ALF G_2 -space of [10] does not in fact exist. This is labeled as a $U(1)$ -puzzle in [10, §6] and awaits further explanation.

Many of the calculations below have been or could be done using techniques already in the literature. For example, Hitchin [44] has already settled Sen's S -duality conjecture for the Atiyah-Hitchin and Taub-NUT manifolds. Likewise, the computations for all hyperkähler ALE spaces follow from Theorem 3 above and the computation of Hodge cohomology in the b -case, which was previously known (see [2], [56]).

For spaces with hyperkähler metrics of fibred boundary type, the calculations follow from Theorem 3 again and the signature formula (55) of Dai and Vaillant. We hope the advantages of our more unified approach to these problems is apparent and that our results give new topological insight even in the previously understood cases. We state as a corollary those applications that we believe are new.

7.1. Gravitational instantons

A gravitational instanton is by definition (see [42]) a four-dimensional complete hyperkähler manifold. In all known topologically finite and noncompact examples, the metric is of fibred boundary type. These examples can be separated into three classes: ALE (short for asymptotically locally Euclidean), where F is a point; ALF (short for asymptotically locally flat), where $F = S^1$; and ALG (by induction), where $F = S^1 \times S^1$.

The space $L^2\mathcal{H}^2(M)$ of L^2 harmonic 2-forms for gravitational instantons is particularly interesting since it contains the curvatures of $U(1)$ Yang-Mills connections. Because of this, we also mention what is known about $SU(2)$ Yang-Mills connections on gravitational instantons and how these $U(1)$ Yang-Mills connections fit into that picture as subspaces of reducible connections.

7.1.1. ALE gravitational instantons

In his thesis, Kronheimer [49], [50] classified all ALE gravitational instantons. The underlying manifolds in this classification are (diffeomorphic to) minimal resolutions of \mathbb{C}^2/Γ , where Γ is a finite subgroup of $SU(2)$. These are of type A_k , D_k , E_6 , E_7 , or E_8 . If we denote the resolution of \mathbb{C}^2/Γ by M_Γ , the correspondence is given by the fact that the intersection form on $H_c^2(M_\Gamma)$ is isomorphic to the Cartan matrix of some simply laced Lie algebra of type ADE. Topologically, this means that M_Γ retracts to a configuration of Lagrangian 2-spheres forming the corresponding Dynkin diagram. The intersection form gives a pairing $H_c^2(M_\Gamma) \times H^2(M_\Gamma) \rightarrow \mathbb{Z}$, and since the Cartan matrix defining the form is always negative definite, we see that the forgetful map $H_c^2(M_\Gamma) \rightarrow H^2(M_\Gamma)$ is an isomorphism.

Now apply Theorem 1 to get the well-known result that $L^2\mathcal{H}^k(M)$ is nontrivial only in degree 2 and

$$L^2\mathcal{H}^2(M_\Gamma, g_{\text{ALE}}) \cong H^2(M_\Gamma).$$

In particular, if k is the number of conjugacy classes in Γ , then $\dim L^2\mathcal{H}^2(M_\Gamma, g_{\text{ALE}}) = k - 1$.

A nice explicit construction of $k - 1$ independent elements giving a basis of $L^2\mathcal{H}^2(M_\Gamma)$ in this case appears in [35]. The paper [51] combines this with [49] to construct all finite-energy $U(k)$ Yang-Mills instantons on M_Γ .

7.1.2. ALF gravitational instantons

There is no classification known for ALF gravitational instantons parallel to that of Kronheimer for the ALE case. However, Cherkis and Kapustin [20] have recently conjectured a classification scheme; using a physics argument, they predict that all ALF instantons are of the types A_k , D_k , so that D_0 stands for the Atiyah-Hitchin manifold.

Consider first the A_k - (for $k \geq 1$) and D_k - (for $k \geq 4$) families. The underlying manifolds of these gravitational instantons are the same as in the ALE case, although the metrics are of course now ALF. Thus now Γ is either a cyclic or dihedral subgroup of $SU(2)$ and M_Γ is the minimal resolution of \mathbb{C}^2/Γ . The A_k -family was constructed first in [42] (see below for the details), while the D_k -family appears in [20] and [19].

The following corollary confirms the prediction made in [64] concerning the Hodge cohomology of ALF gravitational instantons in the A_k -case but includes the D_k -case as well.

COROLLARY 9

Suppose that $\Gamma \subset SU(2)$ is a finite cyclic or dihedral subgroup, and let k be the number of conjugacy classes in Γ . If $(M_\Gamma, g_{\text{ALF}})$ is the associated ALF gravitational instanton, then $\dim L^2\mathcal{H}^2(M_\Gamma) = k$; $L^2\mathcal{H}^d(M_\Gamma)$ is trivial for $d \neq 2$.

Proof

In both the A_k and D_k settings, $\overline{X_\Gamma} = X_\Gamma \cup S^2$. The Mayer-Vietoris sequence gives $H^*(\overline{X_\Gamma}) \cong H^2(X_\Gamma) \oplus H^0(S^2)$. Therefore by (1), $\dim L^2\mathcal{H}^2(M, g_{\text{ALF}}) = \dim H^2(X_\Gamma) + 1 = k$.

Alternatively, apply Theorem 3 and (55). One calculates that the τ -invariant of the fibration at infinity is -1 ; hence $\text{sgn}_{L^2}(M_\Gamma, g_{\text{ALF}}) = \text{sgn}(M_\Gamma) - 1 = -k$. The result follows by applying Theorem 3 again. \square

A consequence of this result is that for an ALF gravitational instanton M_Γ there is, up to scaling, a unique L^2 harmonic form; this form is exact but not, of course, in the range of d on L^2 . In the A_k -case, the metric and all L^2 harmonic 2-forms are known explicitly. We now explain this in more detail and determine which L^2 -harmonic form is exact.

The explicit construction of the ALF gravitational instantons of type A_k uses the Gibbons-Hawking ansatz (see [33])

$$g_{\text{ALF}} = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(d\theta + \alpha)^2,$$

where α is a 1-form on \mathbb{R}^3 such that $d\alpha = *dV$. There is a metric g_{ALF}^k of this type that lives on a four-manifold M_k and admits an isometric circle action with k fixed

points. Away from these fixed points, M_k fibres over $\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$ with S^1 fibres, and it induces a degree -1 fibration around each $p_i \in \mathbb{R}^3$. Here (x_1, x_2, x_3) is the standard coordinate system on \mathbb{R}^3 and $\theta \in S^1$. Finally,

$$V = \sum_1^k \frac{2m}{|x - p_i|} + 1, \quad m > 0.$$

These are called Gibbons-Hawking or multi-Taub-NUT metrics, and g_{ALF}^1 is the famous Taub-NUT metric.

The paper [61] explicitly describes the k -dimensional space $L^2 \mathcal{H}^2(M_k)$ as

$$\Omega_i = d\xi_i, \quad i = 1, \dots, k,$$

where

$$\xi_i = \alpha_i - \frac{V_i}{V}(d\theta + \alpha), \quad \text{where } V_i = \frac{2m}{|x - p_i|} \text{ and } d\alpha_i = *dV_i.$$

This description is only local in the given coordinate chart, and indeed, ξ_i extends globally only as a connection on a $U(1)$ -bundle. Its curvature Ω_i is globally defined. There is one exception: the connection $\zeta = \sum \xi_i = V^{-1}(d\theta + \alpha) - d\theta$ is gauge equivalent to $V^{-1}(d\theta + \alpha)$, which extends globally as the metric dual of the Killing vector field $\partial/\partial\theta$ from the circle action. Its curvature is the L^2 harmonic 2-form $d(V^{-1}(d\theta + \alpha))$. For the Taub-NUT metric, that is, when $k = 1$, this 2-form was discovered by Gibbons [32] and exhibited as support for Sen’s S -duality conjecture. (As already noted, Hitchin [44] settled Sen’s conjecture in this case by proving that there are no other nontrivial L^2 harmonic forms.)

Our result explains the topological origin of Gibbons’s L^2 harmonic 2-form. For although M_1 is diffeomorphic to \mathbb{R}^4 , its compactification (as an ALF space) is $X_1 = \mathbb{C}P^2$. The nontrivial cohomology of $\mathbb{C}P^2$ in degree 2 is the topological source of Gibbons’s L^2 harmonic 2-form.

The other infinite family of ALF gravitational instantons, of type D_k , was constructed in [19] and [20] as moduli spaces of certain singular $SU(2)$ monopoles on \mathbb{R}^3 . The metrics are defined using twistor theory and so are not as explicit as the Gibbons-Hawking metrics above. However, for $k \geq 4$, Theorem 1 again gives a k -dimensional space of L^2 harmonic 2-forms, a one-dimensional subspace of which is exact. It would be interesting to find these harmonic forms explicitly.

We now come to the Atiyah-Hitchin manifold M (see [1]). As explained in [43], the compactification of this space is obtained by adding a copy of $\mathbb{R}P^2$, and in fact, $M \cup \mathbb{R}P^2 = S^4$. Hence (1) shows that $L^2 \mathcal{H}^*(M) = 0$. However, $\pi_1(M)$ is \mathbb{Z}_2 , and the universal cover \tilde{M} has compactification $\tilde{M} \cup \mathbb{R}P^2 = \mathbb{C}P^2$. Therefore $L^2 \mathcal{H}^2(\tilde{M})$ is one-dimensional. This 2-form was constructed by Sen in [63], and Hitchin [44] proved

its uniqueness. Our proof of Sen’s conjecture, through (1), explains the topological origin of this form since it comes from the one-dimensional $H^2(\mathbb{C}P^2)$.

In contrast to the ALE case, very little is known about Yang-Mills instantons on these ALF gravitational instantons (though, of course, the discussion above can be applied to understand the situation for $U(1)$ Yang-Mills instantons). Recently, new families of $SU(2)$ Yang-Mills instantons on multi-Taub-NUT spaces have been found (cf. [28], [30]). In particular, [30] contains an intrinsic construction of the L^2 harmonic forms Ω_i defined above as the curvatures of reducible $SU(2)$ Yang-Mills instantons.

We conclude this section with a final example, the well-known Euclidean Schwarzschild space M , which is a complete Ricci-flat 4-manifold but not hyperkähler. Its Hodge cohomology is calculated in [29] using techniques from [44], and it is shown there that $L^2 \mathcal{H}^k(M) = 0$ when $k \neq 2$ and $L^2 \mathcal{H}^2(M)$ is two-dimensional, with a one-dimensional subspace of (anti)-self-dual solutions. This is explained neatly by (1): namely, M is diffeomorphic to $\mathbb{R}^2 \times S^2$ and is ALF with $F = S^1 = \partial(\mathbb{R}^2)$; hence it compactifies as $X = S^2 \times S^2$. Applying (1), we see that the Hodge cohomology of M is concentrated in degree 2, and

$$\dim L^2 \mathcal{H}^2(M) = \dim H^2(X) = \dim H^2(S^2 \times S^2) = 2.$$

As explained in [29], the self-dual L^2 harmonic 2-forms on M had already appeared in the physics literature in the disguise of $SU(2)$ Yang-Mills instantons (see [14]).

7.1.3. ALG gravitational instantons

The ALG gravitational instantons are the most recent of these spaces to be studied, and examples have only recently been constructed (see [21]); they arise as moduli spaces of periodic monopoles on $\mathbb{R}^2 \times S^1$. In these examples the underlying manifold M is an elliptic fibration of type D_1, D_2, D_3, D_4 or E_6, E_7, E_8 (cf. [21] for the precise meaning of this). They have a fibred boundary metric with $F = T^2$, and hence their compactification $X = M \cup S^1$ is not a Witt space. Theorem 1 gives the following.

COROLLARY 10

Let (M, g_{ALG}) be an ALG gravitational instanton with a fibred boundary metric with $F = T^2$. Then

$$L^2 \mathcal{H}^2(M, g_{\text{ALG}}) \cong \text{Im} (H^2(M, \partial M) \rightarrow H^2(M))$$

is an isomorphism or, in other words, $\dim L^2 \mathcal{H}^2(M, g_{\text{ALG}})$ equals the rank of the intersection matrix on $H^2(M, \partial M)$.

Proof

The intersection cohomology of X can be calculated using Mayer-Vietoris, so that the

result follows from Theorem 1. However, another approach may be more transparent. By Theorem 3 and the signature formula (55), it is enough to show that the fibration $\partial(M) \rightarrow B$ has τ -invariant equal to zero. But this follows from [25, pages 316–319], where it is shown that $\tau = 0$ on any fibration that admits a flat connection. This applies in the present situation because over the one-dimensional base $B = S^1$ any connection is flat. \square

In the examples of type D_4 , the intersection matrix is the Cartan matrix of type \hat{D}_4 (see [21]). Hence in this case $L^2 \mathcal{H}^2(M, g_{\text{ALG}})$ is four-dimensional.

A parallel construction in [21] of certain moduli spaces of solutions to Hitchin's equations (or, equivalently, Higgs bundles) yields manifolds with hyperkähler metrics g_{Hit} which have the same complex structure and underlying topology as the moduli spaces of periodic monopoles discussed above. A conjecture in [21] states that the corresponding elements of these two classes of moduli spaces are in fact isometric. For example, it is known that the moduli space of rank 2 parabolic Higgs bundles on $\mathbb{C}P^1 \setminus \{p_1, p_2, p_3, p_4\}$ is an elliptic fibration (given by the Hitchin map) with one singular fibre of type \hat{D}_4 .

If this conjecture is valid in general, then Corollary 10 implies that for the four-dimensional moduli space of solutions to Hitchin's equations on a cylinder, $L^2 \mathcal{H}^2(M, g_{\text{Hit}}) \cong \text{Im}(H^2(M, \partial M) \rightarrow H^2(M))$. This would be the first evidence, albeit indirect, for [39, Conjecture 1].

7.2. ALE toric hyperkähler manifolds

Toric hyperkähler manifolds were defined and first studied in [6]. An algebraic geometric account of the underlying varieties, with some novel applications to combinatorics, is given in [40].

Let $U(1)^d$ act on \mathbb{H}^n , preserving the hyperkähler structure, and let $M_\xi = \mathbb{H}^n //_{\xi} U(1)^d$ be a smooth toric hyperkähler manifold of dimension $4n - 4d$. The notation $X //_{\xi} G$ here denotes a hyperkähler quotient (see [45]). This construction determines a family of metrics on M_ξ corresponding to the regular values of the hyperkähler moment map. For any such value, consider the family $M_{t\xi}$, $t > 0$. The asymptotics of the metrics in the family $M_{t\xi}$ are the same for $t \neq 0$ (i.e., these metrics are quasi-isometric, with increasing quasi-isometry constant as $t \rightarrow 0$). As $t \rightarrow 0$, $M_{t\xi}$ degenerates to the singular space $M_0 = \mathbb{H}^n //_{/0} U(1)^d$. If we suppose that M_0 has only one isolated singularity, then the metrics in this family maintain the same asymptotics at infinity even when $t = 0$. In this case, M_0 is the cone over a 3-Sasakian compact smooth manifold. This implies that M_ξ is ALE.

The question of when M_0 has only one isolated singularity is intimately related to 3-Sasakian geometry (see [9]), and we quote a result from [6, Theorem 4.1]: M_0 has

only one isolated singularity if and only if the action of $U(1)^d$ on \mathbb{H}^n is unimodular (this means that the generic quotient M_ξ is smooth) and generic (this means that the vector configuration described by the embedding $U(1)^d \subset U(1)^n$ is generic; see [5]). Now Theorem 1 and [41] give the following.

COROLLARY 11

Suppose that the toric hyperkähler manifold M_ξ is smooth and generic. Then

$$L^2 \mathcal{H}^{2n-2d}(M_\xi) \cong \text{Im}(H^{2n-2d}(M_\xi, \partial M_\xi) \rightarrow H^{2n-2d}(M_\xi)) \cong H^{2n-2d}(M_\xi),$$

and $L^2 \mathcal{H}^k(M_\xi) = 0$ in all other degrees.

The Hodge cohomology is concentrated in the middle degree because M_ξ has no cohomology above the middle dimension. It is proven in [41] that the intersection form on $H^{2n-2d}(M_\xi, \partial M_\xi)$ is definite, which in the case of a smooth and generic toric hyperkähler variety is consistent with Corollary 7. It follows that the forgetful map $H^{2n-2d}(M_\xi, \partial M_\xi) \rightarrow H^{2n-2d}(M_\xi)$ is an isomorphism for any smooth toric hyperkähler variety, proving the last isomorphism in Corollary 11.

There are two extreme cases for a smooth generic toric hyperkähler manifold M_ξ . One occurs when $d = n - 1$, and these are just the ALE gravitational instantons of type A_k , which we have discussed earlier. The other extreme is when $d = 1$, and then we obtain the Calabi metric on T^*CP^{n-1} . From the argument above, it has an ALE metric and its Hodge cohomology is supported in the middle degree $2n - 2$, where it is one-dimensional. An explicit generator for this space was found in [47].

A closely related example is the ALE Ricci-flat Kähler metric on T^*S^n , constructed by Stenzel in [65]. Theorem 1 shows that there is a one-dimensional space of L^2 harmonic n -forms on that manifold when n is even. For $n = 2$ this is just the Eguchi-Hanson metric. For general $n = 2k$, physicists have found explicit expressions for the L^2 harmonic k -form (see [24]).

7.3. Spin(7)- and G_2 -metrics

There has been recent interest amongst physicists in constructing new noncompact complete Spin(7)- and G_2 -metrics (cf. [10]), and there have been predictions about the L^2 harmonic forms on such spaces. All known examples have fibred boundary metrics, and so our results, Theorem 1, (1), and (2) can be used to check these predictions. We mention just two examples.

In fact, our Theorem 1 suggested that physicists look for an L^2 harmonic 3-form on a particular example, an ALE G_2 -metric on a rank 3 real vector bundle over S^4 , constructed first in [11]. We have the following as a simple corollary of Theorem 1.

COROLLARY 12

The G_2 -metric of [11] on a rank 3 real vector bundle over S^4 supports exactly a one-dimensional space of degree 3 and a one-dimensional space of degree 4 L^2 harmonic forms.

Armed with the knowledge that such forms existed, physicists (see [23]) were able to find their explicit forms (see [23, (2.18)]) and also [23, footnote 4].

There is another example of a G_2 -metric, constructed in [10], which lives on $\mathbb{R}^4 \times S^3$. It is ALF with $F = S^1$, and so our result (2) implies the following.

COROLLARY 13

There are no nontrivial L^2 harmonic forms on the G_2 -space of [10].

A prediction coming from duality arguments between M-theory and type IIA string theory suggested the existence on this space of an L^2 harmonic 2-form or, equivalently, a finite energy U(1) Yang-Mills field, whose counterpart exists in dual theory. Corollary 13 shows that this prediction fails; actually, the methods of [44] were already used in [10, §6] to establish the nonexistence of L^2 harmonic 2-forms on this G_2 -manifold. Those authors call this the U(1)-puzzle.

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