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Examples of mirror partners arising from integrable systems

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Abstract. In this note we present pairs of hyperkähler orbifolds which satisfy two different versions of mirror symmetry. On the one hand, we show that their Hodge numbers (or more precisely, stringy E-polynomials) are equal. On the other hand, we show that they satisfy the prescription of Strominger, Yau, and Zaslow: that a Calabi-Yau and its mirror should fiber over the same real manifold, with special Lagrangian fibers which are tori dual to each other. Our examples arise as moduli spaces of local systems on a curve with structure group SL(n); the mirror is the corresponding space with structure group PGL(n). The special Lagrangian tori come from an algebraically completely integrable Hamiltonian system: the Hitchin system. © Académie des Sciences/Elsevier, Paris

Partenaires miroirs provenants des systèmes intégrables

Résumé. *Nous présentons dans cette note des paires de V–variétés hyperkähleriennes satisfaisant deux formulations différentes de la symétrie miroir. Nous montrons d'une part que leurs nombres de Hodge (plus précisément, leurs E–polynômes de cordes) coincident. D'autre part, nous montrons qu'elles satisfont le critère de Strominger, Yau et Zaslow, c'est–à–dire qu'elles sont fibrées sur la même variété réelle, de sorte que les fibres soient des tores Lagrangiens spéciaux duaux entre eux. Nos exemples se présentent comme espaces de modules de systèmes locaux sur une courbe avec groupe de structure* $\mathrm{SL}(n)$*; le miroir est l'espace correspondant avec groupe de structure* $\mathrm{PGL}(n)$*. Les tores Lagrangiens spéciaux proviennent d'un système Hamiltonien complètement intégrable algébriquement, le système de Hitchin. © Académie des Sciences/Elsevier, Paris*

Version Française Abrégée

Il s'agit dans cette note de donner quelques exemples de variétés algébriques complexes où l'on peut vérifier directement les prédictions de symétrie miroir, au sens de Strominger-Yau-Zaslow [11].

Soient M et $\hat{M}$ des V-variétés Calabi-Yau de dimension complexe n. On appelle $\hat{M}$ un *partenaire miroir faible* de M s'il existe une V-variété N de dimension réelle n, et des applications lisses $\pi : M \to N$, $\hat{\pi} : \hat{M} \to N$ telles que, sur un ouvert dense de N, les fibres $L_x = \pi^{-1}(x)$ et $\hat{L}_x = \hat{\pi}^{-1}(x)$ sont des tores Lagrangiens spéciaux, et que les systémes locaux $\pi_1(L_x)$ et $\pi_1(\hat{L}_x)$ sont en dualité.

Considérons le cas où M est *hyperkähler*, c'est-à-dire Kähler par rapport à trois structures complexes $J_1, J_2, J_3 : TM \to TM$ satisfaisant les relations de commutation des quaternions imaginaires.

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LEMME. – *Dans ce cas, $L \subset M$ est Lagrangien spécial par rapport à J_1 s'il est Lagrangien holomorphe par rapport à J_2.*

Si de plus M est un système Hamiltonien complètement intégrable algébriquement (SHCIA), alors il existe sur un ouvert dense une fibration par tores Lagrangiens holomorphes, comme on s'y attend. L'exemple-clé est le *système de Hitchin*, que l'on peut décrire comme suit.

Soit C une courbe lisse complexe projective de genre g, $p \in C$ un point de base, et n un entier, le rang. Pour chaque $d \in \mathbf{Z}$, soit M^d_{Dol} l'espace de modules des fibrés de Higgs semistables (E, ϕ) sur C satisfaisant $\Lambda^n E \cong \mathcal{O}(d\,p)$ et $\operatorname{tr}\phi = 0$. De la meme façon, soit M^d_{DR} l'espace de modules des systèmes locaux sur C avec groupe de structure $\mathrm{SL}(n)$ sur $C\backslash\{p\}$, dont la monodromie autour de p est $e^{2\pi i d/n}$. Nous renvoyons à Simpson [7,8] pour les définitions exactes.

THÉORÈME (Hitchin, Simpson). – *Avec la notation ci-dessus,*

(1) *il existe un homéomorphisme $M^d_{\mathrm{DR}} \simeq M^d_{\mathrm{Dol}}$, et une structure hyperkählerienne sur la partie lisse dont les structures complexes J_1 et J_2 sont celles provenant de M^d_{DR} et M^d_{Dol};*
(2) *il existe une famille algébrique equisingulière $\mathcal{M}^d_{\mathrm{Hod}} \to \mathbf{A}^1$ dont le fibre en zéro est M^d_{Dol} et dont les autres fibres sont M^d_{DR}; il existe de plus une action de $\mathrm{GL}(1)$ sur $\mathcal{M}^d_{\mathrm{Hod}}$ qui relève l'action linéaire sur $\mathbf{A}^1$;*
(3) *il existe un morphisme propre $\mu_d : M^d_{\mathrm{Dol}} \to V$, où V est un espace affine indépendant de d, qui fait de M^d_{Dol} un SHCIA.*

THÉORÈME 1. – *Ce résultat reste valable, sur une courbe C avec un nombre fini de points choisis, pour les espaces M^d_{Dol} des fibrés de Higgs semistables* paraboliques *avec des poids fixes, et pour les espaces M^d_{DR} des systèmes locaux* filtrés *avec la monodromie correspondante.*

Nous définissons les espaces de modules correspondants avec groupe de structure $\mathrm{PGL}(n)$ comme étant les quotients les quotients de leurs contreparties $\mathrm{SL}(n)$ par le groupe abélien fini $\Gamma = \mathrm{Pic}^0(C)[n] \cong \mathbf{Z}_n^{2g}$. Celui-ci opère par tensorisation, et préserve toutes les applications qui apparaissent dans le Théorème plus haut. Nous indiquerons par un accent circonflexe le passage au quotient par Γ: par exemple, le morphisme $\mu_d : M^d_{\mathrm{Dol}} \to V$ déscend à $\hat{\mu}_d : \hat{M}^d_{\mathrm{Dol}} \to V$.

PROPOSITION. – *Pour $v \in V$ générique, les fibres $\mu_d^{-1}(v)$ et $\hat{\mu}_d^{-1}(v)$ sont des espaces principaux homogènes des variétés abéliennes polarisées $\mu_0^{-1}(v)$ and $\hat{\mu}_0^{-1}(v)$, qui sont duales entre elles.*

Pour un d quelconque, M^0_{Dol} (resp. $\hat{M}^0_{\mathrm{Dol}}$) admet donc une fibration holomorphe par tores duaux à ceux de $\hat{M}^d_{\mathrm{Dol}}$ (resp. M^d_{Dol}). En appliquant le lemme, on trouve que le partenaire miroir de M^d_{DR} (resp. $\hat{M}^d_{\mathrm{DR}}$) doit être $\hat{M}^0_{\mathrm{DR}}$ (resp. M^0_{DR}). Puisque ces espaces sont des V-variétés hyperkähleriennes, à cause de la symétrie miroir [7], nous pouvons prédire que leurs polynômes de Hodge, ou plutot leurs E-polynômes de cordes [1], notés E_{st}, doivent etre égaux. On peut vérifier la prédiction dans deux cas, où tous les espaces qui interviennent sont des V-variétés.

Considérons d'abord le cas parabolique, en supposant qu'il existe un point avec structure parabolique où les poids sont suffisament génériques.

CONJECTURE. – *Dans ce cas, pour $c, d \in \mathbf{Z}$,*

$$E_{\mathrm{st}}(M^c_{\mathrm{DR}}) = E_{\mathrm{st}}(M^c_{\mathrm{Dol}}) = E_{\mathrm{st}}(\hat{M}^d_{\mathrm{Dol}}) = E_{\mathrm{st}}(\hat{M}^d_{\mathrm{D}}).$$

THÉORÈME 2. – *Ceci est valable pour $n = 2$ et $n = 3$.*

Considérons maintenant l'autre bon cas: celui où il n'y a pas de structure parabolique, mais où le rang n et le dégré d sont premiers entre eux. Dans ce cas, le partenaire miroir de M^d_{DR}, c'est-à-dire $\hat{M}^0_{\mathrm{DR}}$, a de mauvaises singularités, et la relation miroir n'est pas symétrique. Néanmoins, des calculs suggèrent la conjecture suivante.

Vafa [12] et Vafa-Witten [13] ont montré que, pour un groupe fini Γ, un élément $\rho \in H^2(\Gamma, \mathrm{U}(1))$ quelconque définit un E-polynôme de cordes, de façon tordue, d'une V-variété X/Γ. Ils appellent ce phénomène *torsion discrète*. Dans notre cas $\Gamma \cong \mathbf{Z}_n^{2g}$. Soit η le générateur standard de $H^2(\mathbf{Z}_n^2; \mathrm{U}(1)) \cong \mathbf{Z}_n$, et soit $\rho = \sum_{i=1}^{g} \pi_i^* \eta$ où $\pi_i : \Gamma \to \mathbf{Z}_n^2$ la projection sur les facteurs $(2i-1)$'ème and $(2i)$'ème.

CONJECTURE. – *Pour $c, d \in \mathbf{Z}$,*

$$E_{\mathrm{st}}(M_{\mathrm{DR}}^c) = E_{\mathrm{st}}(M_{\mathrm{Dol}}^c) = E_{\mathrm{st}}^{c\rho}(\hat{M}_{\mathrm{Dol}}^d) = E_{\mathrm{st}}^{c\rho}(\hat{M}_{\mathrm{DR}}^d).$$

THÉORÈME 3. – *Ceci est valable pour $n = 2$ et $n = 3$ si c et n, et d et n, sont premiers entre eux.*

The object of this note is to exhibit some striking mathematical evidence in support of the Strominger-Yau-Zaslow (SYZ) proposal for mirror symmetry in string theory [11], which however does not yet have a completely satisfactory precise mathematical formulation. In this note we describe some hyperkähler varieties which satisfy a mathematical version of some of the conditions laid down by these authors to be mirror partners, and whose stringy mixed Hodge numbers turn out to be equal. The latter could be considered as a hyperkähler analogue of the "topological mirror test", which was the first of the mathematically surprising findings of physicists pursuing the original mirror symmetry for Calabi-Yau 3-folds. The examples we present here also involve a complex reductive group G and its Langlands dual group $\hat{G}$. We hope G could be taken to be any reductive group, but at one point we will need to assume that G is $SL(2)$ or $SL(3)$.

An n-dimensional Kähler orbifold is said to be *Calabi-Yau* if it is Ricci-flat and it is equipped with a nowhere vanishing holomorphic n-form Ω. A submanifold L is *special Lagrangian* if it is Lagrangian and $\operatorname{Im}\Omega|_L = 0$.

Let M and $\hat{M}$ be Calabi-Yau orbifolds of complex dimension n. In this note we call them *weak SYZ mirror partners* if there exists an orbifold N of real dimension n and smooth maps $\pi : M \to N$, $\hat{\pi} : \hat{M} \to N$ so that, if x is a regular value of π and $\hat{\pi}$, $L_x = \pi^{-1}(x)$ and $\hat{L}_x = \hat{\pi}^{-1}(x)$ are special Lagrangian tori, and if the local systems of the $\pi_1(L_x)$ and $\pi_1(\hat{L}_x)$ are in duality, which is equivalent to say that there is a smoothly varying diffeomorphism $\hat{L}_x^* \simeq L_x^{**}$. Here for any torus L we define $L^* = \operatorname{Hom}(\pi_1(L), \mathrm{U}(1))$; the double dual is used to provide a distinguished basepoint. If in addition there are special Lagrangian sections s and $\hat{s}$ of π and $\hat{\pi}$, then the basepoints are automatically provided, so that $L_x = L_x^{**} \simeq \hat{L}_x^*$ and $\hat{L}_x = \hat{L}_x^{**} \simeq L_x^*$; in this case we call M and $\hat{M}$ *SYZ mirror partners*. We note that the original SYZ picture in [11] has extra conditions on the restricted metrics of these tori, but the metric they consider arises in a "large complex limit", which does not yet have a completely satisfactory mathematical formulation. Although we hope that our examples will also pass some appropriate metric conditions, for now we omit this, leaving the above fibrewise duality property rather weak. Note however that in our hyperkähler case we find something stronger: the tori are principal homogeneous spaces of abelian varieties in duality.

In general it is a hard analytical problem to find special Lagrangian tori on M. But in one case an easy lemma shows how to reduce it to a question of holomorphic geometry. Suppose that the Riemannian orbifold M^{4k} is *hyperkähler*, that is, it is Kähler with the three Kähler forms $\omega_1, \omega_2, \omega_3$ with respect to three complex structures $J_1, J_2, J_3 : TM \to TM$ satisfying the commutation relations of the imaginary quaternions. Then it is easy to check that the 2-form $\omega_1^c = \omega_2 + i\omega_3$ is a holomorphic symplectic form in the complex structure J_1. We denote $\Omega_1 = (\omega_1^c)^k$ the holomorphic volume form, which makes (M, J_1) into a Calabi-Yau manifold. We define ω_i^c and Ω_i cyclically.

LEMMA. – *In this situation, $L \subset M$ is special Lagrangian with respect to J_1 and Ω_1 provided that it is holomorphic Lagrangian with respect to J_2 and ω_2^c.*

If, in addition, M is an algebraically completely integrable Hamiltonian system (ACIHS), then it has a generic holomorphic Lagrangian torus fibration exactly as desired. A key example is furnished by the so-called *Hitchin system*, which can be described as follows.

Fix a smooth complex projective curve C of genus g, a basepoint $p \in C$, and a positive integer n, the rank. For any $d \in \mathbf{Z}$, let M_{Dol}^d be the moduli space of semistable Higgs bundles (E, ϕ) on C satisfying $\Lambda^n E \cong \mathcal{O}(dp)$ and $\operatorname{tr}\phi = 0$. (If $d = 0$, these conditions mean that (E, ϕ) has structure group $\mathrm{SL}(n)$.) Also let M_{DR}^d be the space of local systems with structure group $\mathrm{SL}(n)$ on $C \backslash \{p\}$, having monodromy $e^{2\pi i d/n}$ around p. We refer to Simpson [7,8] for the definitions of Higgs bundles, semistability, and so on.

THEOREM (Hitchin [4,5], Simpson [9,10]). – *With the above notation,*

(1) *there is a homeomorphism $M_{\mathrm{DR}}^d \simeq M_{\mathrm{Dol}}^d$ and a hyperkähler structure on the smooth locus so that the complex structures J_1 and J_2 correspond to those provided by M_{DR}^d and M_{Dol}^d;*
(2) *there is an equisingular algebraic family $\mathcal{M}_{\mathrm{Hod}}^d \to \mathbf{A}^1$ whose zero fiber is M_{Dol}^d and whose other fibers are M_{DR}^d, and a $\mathrm{GL}(1)$-action on $\mathcal{M}_{\mathrm{Hod}}^d$ lifting the standard action on $\mathbf{A}^1$;*
(3) *there is a proper morphism $\mu_d : M_{\mathrm{Dol}}^d \to V$, where V is an affine space independent of d, making M_{Dol}^d into an ACIHS. This morphism has a holomorphic Lagrangian section when $d = 0$.*

THEOREM 1. – *A similar result holds, over a curve C with a finite number of punctures, for the moduli spaces M_{Dol}^d of semistable* parabolic *Higgs bundles with fixed weights and M_{DR}^d of* filtered *local systems with the corresponding monodromy.*

In fact, similar spaces, and corresponding Theorems, exist for *principal* Higgs bundles with any reductive structure group [10]; the space M_{Dol}^0 is the case of $\mathrm{SL}(n)$. The spaces M_{Dol}^d for $d \neq 0$ are not only used for the construction of the $\mathrm{PGL}(n)$ moduli spaces below but also will play a decisive role in our main result. The only other group needed here is $\mathrm{PGL}(n)$; we define the corresponding moduli spaces as the quotients of their $\mathrm{SL}(n)$ counterparts by the finite abelian group $\Gamma = \mathrm{Pic}^0(C)[n] \cong \mathbf{Z}_n^{2g}$. This acts by tensorization, and preserves all of the maps appearing in the result of Hitchin and Simpson. The degree d corresponds to the class of the $\mathrm{PGL}(n)$-bundle in $H^2(C, \mathbf{Z}_n) = \mathbf{Z}_n$. We denote the quotient by Γ with a hat: for example, the morphism $\mu_d : M_{\mathrm{Dol}}^d \to V$ descends to $\hat{\mu}_d : \hat{M}_{\mathrm{Dol}}^d \to V$.

PROPOSITION. – *For generic $v \in V$, the fibers $\mu_d^{-1}(v)$ and $\hat{\mu}_d^{-1}(v)$ are principal homogeneous spaces of the polarized abelian varieties $\mu_0^{-1}(v)$ and $\hat{\mu}_0^{-1}(v)$. Indeed, for any c, d there are natural isomorphisms*

$$\mathrm{Pic}^d(\mu_c^{-1}(v)) = \hat{\mu}_d^{-1}(v)$$

and

$$\mathrm{Pic}^d(\hat{\mu}_c^{-1}(v)) = \mu_d^{-1}(v),$$

where Pic^d denotes the space of line bundles with first Chern class d times the polarization of the underlying polarized abelian variety.

The relevant abelian varieties can be constructed directly, using the method of spectral covers of Beauville-Narasimhan-Ramanan [2] and Hitchin [3], and can be seen to be the generalized Prym variety of the spectral cover and its dual.

Hence for any d, M_{Dol}^0 (resp. $\hat{M}_{\mathrm{Dol}}^0$) has a holomorphic torus fibration dual to that of $\hat{M}_{\mathrm{Dol}}^d$ (resp. M_{Dol}^d). Using the lemma, one finds that M_{DR}^0 and $\hat{M}_{\mathrm{DR}}^0$, with the corresponding holomorphic volume forms, are SYZ mirror partners, and that M_{DR}^c and $\hat{M}_{\mathrm{DR}}^d$ are weak SYZ mirror partners for any c and d.

If at any puncture the parabolic weights are distinct, then μ_d and $\hat{\mu}_d$ have holomorphic Lagrangian sections, so the dependence on c and d becomes unimportant.

For a compact hyperkähler manifold M, the Hodge numbers are easily seen to be self-mirror in that $h^{p,q}(M) = h^{n-p,q}(M)$. One would therefore hope that the mirror $\hat{M}$ satisfies $h^{p,q}(M) = h^{p,q}(\hat{M})$. We will see that (with a suitable notion of Hodge numbers) the latter equality holds for many of our mirror partners, even though they are not compact or self-mirror in the sense above. We have no explanation as to why this equality holds, rather than the usual mirror relationship $h^{p,q}(M) = h^{n-p,q}(\hat{M})$.

For spaces which are non-compact and have quotient singularities, the right notion of Hodge polynomial seems to be the *stringy E-polynomial* of Vafa, Zaslow, and Batyrev-Dais [1]. This is a polynomial in two variables x and y, and will be denoted E_{st}.

For the present, we assume either (a) that the weights are sufficiently generic in the parabolic case, or (b) that n is coprime to both c and d in the non-parabolic case. In these cases the moduli spaces without hats are smooth, so their E_{st}-polynomials reduce to the ordinary E-polynomials defined in terms of Deligne's mixed Hodge structures on the compactly supported cohomology:

$$E(M) = \sum_{k,p,q} (-1)^{k-p-q} h^{p,q}(H^k_{\mathrm{cpt}}(M)) x^p y^q.$$

The moduli spaces with hats are global quotients of the smooth spaces by Γ, so their E_{st}-polynomials can be defined by the orbifold method, according to which

$$E_{\mathrm{st}}(X/\Gamma) = \sum_{\{\gamma\}} E(X^\gamma / C(\gamma))(xy)^{F(\gamma)}. \tag{1}$$

Here the sum runs over conjugacy classes of $\gamma \in \Gamma$, X^γ is the fixed-point set of γ on the smooth variety X, $C(\gamma)$ is the centralizer of γ, and $F(\gamma)$ is an integer called the *fermionic shift*. See Batyrev-Dais [1] for details.

Consider first case (a), when C has at least one puncture where the weights are sufficiently generic.

CONJECTURE. – *In this case, for any $c, d \in \mathbf{Z}$,*

$$E_{\mathrm{st}}(M^c_{\mathrm{DR}}) = E_{\mathrm{st}}(M^c_{\mathrm{Dol}}) = E_{\mathrm{st}}(\hat{M}^d_{\mathrm{Dol}}) = E_{\mathrm{st}}(\hat{M}^d_{\mathrm{DR}}).$$

THEOREM 2. – *This is true for $n = 2$ and $n = 3$.*

Sketch of proof. – The proof of the first and last equalities uses the family $\mathcal{M}_{\mathrm{Hod}} \to \mathbf{A}^1$ described above, together with a fiberwise compactification constructed as a quotient $(\mathcal{M}_{\mathrm{Hod}} \times \mathbf{P}^1)/\mathrm{GL}(1)$. This part of the argument is in fact valid for any n and d.

The proof of the middle equality is by explicit calculation. It is convenient to subtract $E(\hat{M}^c_{\mathrm{Dol}})$ from both sides. On the left, what remains is the contribution of cohomology on M^c_{Dol} *not* invariant under the Γ-action. This can be computed from the Białynicki-Birula stratification, using the methods of Hitchin [4] to describe the fixed points of the GL(1)-action. On the right, the remaining terms are simply those in equation (1) above with $\gamma \neq 1$. These can be calculated using the methods of Narasimhan-Ramanan [8] to describe the fixed points of the Γ-action. The construction again uses spectral covers, but since they are unramified it has quite a different flavor from before.

In the rank 2 case, one finds that

$$E_{\mathrm{st}}(M^d_{\mathrm{Dol}}) - E(\hat{M}^d_{\mathrm{Dol}}) = E_{\mathrm{st}}(\hat{M}^d_{\mathrm{Dol}}) - E(\hat{M}^d_{\mathrm{Dol}}) = 2^{m-1}(2^{2g}-1)(xy)^{3g-3+m}(1+x)^{g-1}(1+y)^{g-1},$$

where m is the number of punctures whose two weights are distinct.

In the rank 3 case, the formula is similar but more complicated.

Now turn to case (b), where there are no punctures, but the rank n is coprime to the degrees c and d. In this case $M^c_{\rm DR}$ and $\hat{M}^d_{\rm DR}$ are merely weak mirror partners, and the stringy E-polynomials are different. Hitchin has suggested [6] that weak mirror partners should be understood as being equipped with "B-fields," that is, flat gerbes on M and $\hat{M}$. Motivated by this, we expect that the right stringy Hodge numbers should somehow be twisted with these gerbes. We suggest that these twistings are occuring due to "discrete torsion" as sketched below.

Vafa [12] and Vafa-Witten [13] have shown that, for a finite group Γ, any element $\rho \in H^2(\Gamma, {\rm U}(1))$ can be used to define stringy E-polynomials, in a twisted sense, of an orbifold X/Γ. They refer to this as turning on the *discrete torsion*. In fact, ρ induces homomorphisms $C(\gamma) \to {\rm U}(1)$ for each $\gamma \in \Gamma$, which give rise to local systems on $X^\gamma/C(\gamma)$; the $E_{\rm st}$-polynomial of X/Γ then generalizes to a polynomial $E^\rho_{\rm st}(X/\Gamma)$ where cohomology with coefficients in these local systems is used in equation (1).

In the present case $\Gamma \cong \mathbf{Z}_n^{2g}$. Let η be the standard generator of $H^2(\mathbf{Z}_n^2; {\rm U}(1)) \cong \mathbf{Z}_n$, and let $\rho = \sum_{i=1}^g \pi_i^* \eta$ where $\pi_i : \Gamma \to \mathbf{Z}_n^2$ is projection on the $(2i-1)$th and $(2i)$th factors. This is in some sense the natural class corresponding to the symplectic form on C.

Conjecture. – *For any $c, d \in \mathbf{Z}$,*

$$E_{\rm st}(M^c_{\rm DR}) = E_{\rm st}(M^c_{\rm Dol}) = E^{c\rho}_{\rm st}(\hat{M}^d_{\rm Dol}) = E^{c\rho}_{\rm st}(\hat{M}^d_{\rm DR}).$$

Theorem 3. – *This is true for $n = 2$ and $n = 3$ when c and d are coprime to n.*

The proof is similar to that of Theorem 2.

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