Inscribing cubes and covering by rhombic dodecahedra via equivariant topology

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Abstract

First, we prove a special case of Knaster's problem, implying that each symmetric convex body in $\mathbb{R}^3$ admits an inscribed cube. We deduce it from a theorem in equivariant topology, which says that there is no S_4-equivariant map from $SO(3)$ to S^2, where S_4 acts on $SO(3)$ as the rotation group of the cube and on S^2 as the symmetry group of the regular tetrahedron. We also give some generalizations.

Second, we show how the above non-existence theorem yields Makeev's conjecture in $\mathbb{R}^3$ that each set in $\mathbb{R}^3$ of diameter 1 can be covered by a rhombic dodecahedron, which has distance 1 between its opposite faces. This reveals an unexpected connection between inscribing cubes into symmetric bodies and covering sets by rhombic dodecahedra.

Finally, we point out a possible application of our second theorem to the Borsuk problem in $\mathbb{R}^3$.

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1 Introduction

Problems of inscribing and circumscribing polyhedra to convex bodies have a vast literature; for a comprehensive recent survey cf. in [KlW], Ch. 11; cf. also [Mak0]. In [HMSz] we have surveyed some of the literature and announced some of the results of the present paper.

The proofs of such theorems are frequently based on theorems for continuous maps, such as some special cases of the so called Knaster problem.

The Knaster problem asks, for a given continuous function $F : S^{n-1} \to \mathbb{R}^k$ and a finite subset $X = \{x_1, ..., x_m\}$ of S^{n-1}, whether there is a rotation $A \in SO(n)$ such that $F(Ax_1) = \ldots = F(Ax_m)$. Observe that, choosing F linear, for a positive answer it is necessary that X should lie in an $(n-k)$-plane; correspondingly, this question is frequently asked for $m = n - k + 1$ only, as originally in [Knas]. The answer to the Knaster problem in general is negative: counterexamples were given by Makeev in [Mak2] and [Mak3], by Babenko and Bogatyi in [BaBo] and recently by Chen in [Chen].

However there are many positive results: a famous special case is the Borsuk-Ulam theorem which states that for every continuous map $F : S^{n-1} \to \mathbb{R}^{n-1}$ there is an $x \in S^{n-1}$ such that $F(x) = F(-x)$. This is Knaster's problem when $k = n - 1$, $m = 2$ and $X = \{e_1, -e_1\}$. It was generalized by Hopf in [Hopf] for any two points in S^{n-1}. Other positive results are proved when $k = 1$ and X is the set of vertices of any regular $(n-1)$-simplex inscribed to S^{n-1} in [BoKh], $X = \{e_1, \ldots, e_n\}$ [YaYu], $X = \{\pm e_1, ..., \pm e_{n-1}\}$ Corollary 7.5, [Yang], X any 3-element subset of S^2 [Floyd], $X = \{\pm x, \pm y\} \subset S^2$ [Dyson], [Liv], X being the vertex set of any planar rectangle inscribed to S^2 Theorem D, [Griff][1].

In this paper we settle the following

Theorem 1 *Let $F : S^2 \to \mathbb{R}$ be a continuous even function, i.e., $F(x) = F(-x)$, and let $\{\pm v_i \mid 1 \leq i \leq 4\}$ be the vertices of a cube inscribed to S^2. Then there exists an $A \in SO(3)$ such that $F(\pm Av_1) = F(\pm Av_2) = F(\pm Av_3) = F(\pm Av_4)$.*

Of course, this theorem also follows from Theorem D of [Griff] cited above. However the proof of that theorem is quite involved, and we hope our proof of this special case is considerably simpler. Our proof avoids calculations, and uses only elements of equivariant algebraic topology. Hopefully our theorems about non-existence of certain equivariant maps are of independent interest.

[1] We note, that a slight criticism of the Griffiths paper appeared in [Mak6]. This states that another theorem of [Griff], namely his Theorem C is correctly proved by calculating intersection numbers mod 2, not in $\mathbb{Z}$ as in [Griff]. Since [Griff] uses large computer calculations, we are unable to decide in this point. [Griff], Theorem C is not used in proving [Griff], Theorem D.

Note that when F is the Minkowski norm associated to a symmetric convex body in $\mathbb{R}^3$, Theorem 1 proves that each symmetric convex body in $\mathbb{R}^3$ contains an inscribed cube.

In Section 5 we will prove a more involved generalization of Theorem 1 which is no longer a consequence of Theorem D of [Griff]. It says:

Theorem 2 *Let $f : S^2 \to \mathbb{R}^3$ be a C^1 embedding of S^2 in $\mathbb{R}^3$, satisfying $f(-u) = -f(u)$ for $u \in S^2$. Further suppose that f is homotopic to the standard embedding, via a C^1 homotopy $H : S^2 \times [0,1] \to \mathbb{R}^3 \setminus \{0\}$ satisfying $H(-u,t) = -H(u,t)$ for $u \in S^2$ and $t \in [0,1]$. Then there is a cube in $\mathbb{R}^3$, with its centre in the origin, having all its vertices on the surface $f(S^2)$.*

The second theme of the present paper concerns *universal covers*. One says that a set $C \subset \mathbb{R}^n$ is a universal cover in $\mathbb{R}^n$ (cf. §10, 47, p. 87, [BoFe] and [Mak4]), if for any set $X \subset \mathbb{R}^n$ of diameter at most 1 there exist $A \in O(n)$ and $x \in \mathbb{R}^n$, such that $X \subset AC + x$. (If C has a plane of symmetry, we can write here $A \in SO(n)$. In any case, since $O(n)$ is disconnected, when proving the universal cover property of some $C \subset \mathbb{R}^n$, it is reasonable to use only $A \in SO(n)$.) Since the diameter of X and that of its closed convex hull are equal, we may assume that X is compact, convex, non-empty. For example, the unit ball is a universal cover, but also the circumball of a regular simplex with unit edges is a universal cover [Jung 1901], [Jung 1910], cf. also [BoFe], §10, 44, (7) (this ball has radius $\sqrt{n/(2n+2)}$).

Moreover Pál in [Pál] showed that a regular hexagon, with distance of opposite sides equal to 1, was also a universal cover in $\mathbb{R}^2$. It is an interesting open question, due to Makeev ([Mak4], p.129; cf. also our Section 6), whether an analogue of this holds in $\mathbb{R}^n$.

Here we prove Makeev's conjecture in $\mathbb{R}^3$:

Theorem 3 *A rhombic dodecahedron U_3, with distance of opposite faces equal to 1, is a universal cover in $\mathbb{R}^3$. I.e., any set of diameter 1 in $\mathbb{R}^3$ can be covered by U_3.*

We prove both Theorem 1 in Section 2 and Theorem 3 in Section 6, as a consequence of the following result in equivariant topology:

Proposition 4 *There is no equivariant map*

$$g : (SO(3), \rho_{S_4}) \to (S^2, \tau_{S_4}),$$

where ρ_{S_4} is the S_4 action on $SO(3)$ given by multiplication from right by the group of rotations of the cube: $S_4 \subset SO(3)$, and τ_{S_4} is the S_4 action on S^2 given by the symmetry group of a regular tetrahedron inscribed in S^2.

Following [Jer] and [Griff], the proof is based on the idea of *test functions*. Namely we show that one can inscribe exactly one cube in an ellipsoid E of different lengths of axes. From this in turn we get a test function $f_E : (SO(3), \rho_{S_4}) \to (\mathbb{R}^3, \tau_{S_4})$ which has exactly one transversal zero, which will prove the proposition.

In connection with this proposition, we investigate the existence of equivariant maps $SO(3) \to S^2$ for certain group actions of subgroups of S_4 on $SO(3)$ and S^2 in Section 3. As a consequence we deduce Theorem 10 in Section 3.1, which is Theorem 1 with the cube replaced by a square based box (a box is a rectangular parallelepiped). Thus one can inscribe a square based box, similar to any given one, into any symmetric convex body in $\mathbb{R}^3$.

In Section 4 we show that Theorem D in [Griff] easily yields a generalization of Theorem 1, namely the same statement with the vertices of the cube replaced by those of any box. Consequently each symmetric convex body in $\mathbb{R}^3$ admits an inscribed box similar to a given one.

In Section 5 we prove Theorems 2 and 16, that generalize Theorems 1 and 10, showing that certain centrally symmetric surfaces in $\mathbb{R}^3$ admit inscribed cubes, and inscribed similar copies of any square based box, respectively.

In the final Section 6 we prove Theorem 3 as a consequence of Proposition 4. We finish the paper by pointing out a possible application of Theorem 3 to Borsuk's problem in $\mathbb{R}^3$, which states that any set of diameter 1 in $\mathbb{R}^3$ can be divided into 4 sets of diameter smaller than 1.

We have announced the main results of this paper in [HMSz] in 1997. Since the present paper was written we learnt of the paper [Mak5] by V. V. Makeev from 1997 and of [Kup] by G. Kuperberg from 1998. Both papers consider similar questions. In [Kup] Proposition 4 is also proved and applied to problems similar to ours.

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2 Inscribed cubes: proof of Theorem 1

Eggleston in [Eggl1] has asserted that there exists a centrally symmetrical convex body in $\mathbb{R}^3$ that does not admit an inscribed cube. For this he has stated a lemma (p.79). Formula (2) in the proof of this lemma should stand correctly

$$x^2/A^2 + y^2/B^2 + z^2/C^2 + \lambda(xy + yz + zx) = d^2/A^2 + d^2/B^2 + d^2/C^2 - \lambda d^2.$$

In fact, the statement of the lemma itself is not correct. Namely, it asserts the following. Let

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \{-1, 1\},\ x + y + z \in \{-1, 1\}\}.$$

Suppose that the boundary of an ellipsoid $E \subset \mathbb{R}^3$ contains V. Then the lemma asserts that either E has two equal axes, or else its centre is 0, its axes are parallel to the coordinate axes, and its boundary also contains $(-1, -1, -1)$, $(1, 1, 1)$.

However, V consists of the vertices of an octahedron (i.e., an affine image of a regular octahedron, i.e., $V = \{\pm x, \pm y, \pm z\}$, with $x, y, z \in \mathbb{R}^3$ linearly independent). Further, any ellipsoid has lots of inscribed octahedra, with no connection to the directions of the axes of the ellipsoid.

To have a concrete counterexample, let us consider an ellipsoid E_0 of centre 0, of equation $0.5x^2 + y^2 + 1.5z^2 = 3$, whose boundary passes through the points $(\pm 1, \pm 1, \pm 1)$. Its section with the plane $z = 1$ has equation $z = 1$, $x^2/3 + 2y^2/3 = 1$. Then let $\epsilon > 0$ be sufficiently small, and let us consider a conic C in the plane $z = 1$ passing through the five points $(-1, -1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1 + \epsilon, 1, 1), (\sqrt{3}, 0, 1)$. This is uniquely determined, depends continuously on ϵ, is an ellipse, and does not pass through $(1, 1, 1)$, since it intersects the line $y = z = 1$ in two points only. Let the equation of C be $z = 1$, $\sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + 1 = 0$ $(a_{ij} = a_{ji})$, where $x_1 = x$, $x_2 = y$, and the coefficients depend continuously on ϵ. Let E be an ellipsoid of centre 0, with $\partial E \supset C$, of equation $\sum_{i,j} A_{ij} x_i x_j + C = 0$ $(A_{ij} = A_{ji})$, with $x_3 = z$. We may suppose $a_{ij} = A_{ij}$ $(i, j \leq 2)$, $b_i = 2A_{i3}$ $(i \leq 2)$, $1 = A_{33} + C$. Let now $A_{33} = 1.5$; this determines the equation of E uniquely, and the coefficients in this equation depend on ϵ continuously. For $\epsilon = 0$ we have the original ellipsoid E_0.

Therefore for ϵ sufficiently small, E is an ellipsoid, and the corresponding quadratic form has three different eigenvalues, hence E has three different axis lengths. By construction, $\partial E \supset V$, $(1, 1, 1) \notin \partial E$, contradicting the claim of the lemma of [Eggl1].

We will actually prove the opposite of Eggleston's statement as a corollary to Theorem 1:

Corollary 5 *Any centrally symmetric convex body K in $\mathbb{R}^3$ possesses an inscribed cube, centred at the centre of K.*

Before we proceed to the proof we recall the symmetry group of the regular tetrahedron and the rotation group (i.e., the group of orientation preserving symmetries) of the cube.

The symmetry group of the regular tetrahedron (which we think of as inscribed to S^2) is S_4, the symmetric group on four letters. It is because the symmetries of a regular tetrahedron are in bijective correspondence with the permutations of its vertices. Moreover the rotation group of the regular tetrahedron is clearly A_4, the alternating group on four letters.

The rotation group of the cube (which we also think of as inscribed to S^2) is S_4. It can be seen by considering the diagonals of the cube as follows. The cube has four diagonals and every rotation gives a permutation of these. Only the identity rotation maps each diagonal to itself. On the other hand one can choose a vertex of each diagonal yielding a regular tetrahedron of edge length $\sqrt{2}$. Every permutation of the diagonals gives rise to a permutation of the vertices of the tetrahedron, which in turn determines a symmetry of the tetrahedron and thus that of the cube. If it was not a rotation one could compose it with the antipodal map, yielding a rotation of the cube inducing the given permutation of the diagonals. We will think of this rotation group as $S_4 \subset SO(3)$ a subgroup of $SO(3)$.

Now, for the reader's convenience, we describe this embedding of S_4 to $SO(3)$. If $g \in S_4$ acts on the vertices of the regular tetrahedron as an even permutation (i.e., in disjoint cycle representation has one of the forms: identity, $(ij)(kl)$ or (ijk)), then its embedded image $\iota(g)$ in $SO(3)$ is that element of $SO(3)$, that is the extension of the permutation $g \in S_4$ of the vertices of the tetrahedron. Geometrically, if the cube has vertices $\{(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})\}$, these rotations are the identity, the rotations through π about the coordinate axes, and the rotations through $\pm 2\pi/3$ about the spatial diagonals of the cube, respectively. If $g \in S_4$ acts on the vertices of the regular tetrahedron as an odd permutation (i.e., has one of the forms (ij) or $(ijkl)$), then $\iota(g)$ is -1 times that element of $O(3)$, that is the extension of the permutation $g \in S_4$. Geometrically, these rotations are the rotations through π about all angle bisectors of all pairs of the coordinate axes, and the rotations through $\pm\pi/2$ about the coordinate axes, respectively. From now on, we will write g for $\iota(g)$.

Proof of Corollary 5. Let $C \subset \mathbb{R}^3$ be a fixed cube centred at the origin with diagonals $(v_i, -v_i)$, where $\|v_i\| = 1$. We suppose that the centre of K is the origin, and let $\| \, \|_K$ be the Minkowski norm in $\mathbb{R}^3$, associated to K. Define the map $f : SO(3) \to \mathbb{R}^4$ sending $A \in SO(3)$ to $(\|A(v_1)\|_K, \|A(v_2)\|_K, \|A(v_3)\|_K, \|A(v_4)\|_K)$. Our task now is to show that the image of f intersects the diagonal $\Delta := \{(\lambda, \lambda, \lambda, \lambda) : \lambda \in \mathbb{R}\}$ of $\mathbb{R}^4$ since if $f(A) = (\lambda, \lambda, \lambda, \lambda)$ $(\lambda > 0)$ is a point on the diagonal then the cube $\lambda^{-1}A(C)$ is inscribed in K, and is centred at the centre of K.

The function $F = \| \ \|_K \ |_{S^2}\colon S^2 \to \mathbb{R}$ is an even function because K is symmetric. Thus Corollary 5 is a corollary of Theorem 1 indeed. □

To prove Theorem 1 we begin with two lemmas. The first one is surely well-known; we include its short proof for the reader's convenience. Its statement and proof are slight variants of those of Theorem 1 of [DuKhSh]. A special case of its corollary is contained in [Jer], p. 240, proof of Lemma 6.

Lemma 6 *Let $P \subset \mathbb{R}^n$ be a box, and let $E \subset \mathbb{R}^n$ be an ellipsoid circumscribed about P. Then E has centre at the centre of P, and it has a system of n axes parallel to the edges of P.*

Proof. Let the vertices of P be $\{(\pm d_1, ..., \pm d_n)\}$. Let $\partial E = \{(x_1, ..., x_n) \mid F(x_1, ..., x_n) := \sum^n a_{ij}x_ix_j + \sum^n b_ix_i + c = 0\}$, where $a_{ij} = a_{ji}$. Then for any $\epsilon_1, ..., \epsilon_{n-1} \in \{-1, 1\}$ we have $0 = (F(\epsilon_1 d_1, ..., \epsilon_{n-1}d_{n-1}, d_n) - F(\epsilon_1 d_1, ..., \epsilon_{n-1}d_{n-1}, -d_n))/2 = d_n(\sum^{n-1} 2\epsilon_i a_{in} d_i + b_n)$, hence $0 = \sum^{n-1} 2\epsilon_i a_{in} d_i + b_n$, that readily implies $a_{in} = b_n = 0$ $(i \le n-1)$. Similarly one shows $a_{ij} = 0$ $(i < j \le n)$ and $b_i = 0$ $(i \le n)$, that implies the statement of the lemma. □

Corollary 7 *Let $E \subset \mathbb{R}^n$ be an ellipsoid with axes of different lengths, and $B \subset \mathbb{R}^n$ a box of size $a_1 \times ... \times a_n$, where $a_1 = ... = a_{n_1} < a_{n_1+1} = ... = a_{n_1+n_2} < ... < a_{n_1+...+n_{k-1}+1} = ... = a_n$ and $n = n_1 + ... + n_k$. Then there are $n!/(n_1!...n_k!)$ ways of inscribing a similar copy B' of B into E (and in each of these cases the centre of B' coincides with that of E).*

Proof. By Lemma 6 the centre of an inscribed similar copy B' of B is at the centre of E, and its edges of length $a_{n_1+...+n_{i-1}+1}$ are parallel to some n_i axes of E. This gives a partition of the n axes of E to classes of sizes $n_1, ..., n_k$; the number of such partitions is $n!/(n_1!...n_k!)$. Fixing one such partition, B' is determined, up to magnification from its centre. Its size is determined by the requirement that a given vertex of it should belong to ∂E (and then, by symmetry, all of its vertices belong to ∂E). □

Lemma 8 *Let $E \subset \mathbb{R}^n$ be an ellipsoid with axes of different lengths, and of equation $F(x_1, ..., x_n) := \sum^n a_{ii}x_i^2 - 1 = 0$. Let P be a box inscribed to E, of vertex set $\{v_1, ..., v_{2^n}\} = \{(\pm d_1, ..., \pm d_n)\}$. Let us consider the analytic map f, sending $A \in SO(n)$ to $(F(Av_1), ..., F(Av_{2^n})) \in \mathbb{R}^{2^n}$. Then at $A = I$ f has full rank $n(n-1)/2$, and the diagonal Δ of $\mathbb{R}^{2^n}$ intersects the tangent space of $f(SO(n))$ at $f(A) = f(I)$ transversally.*

Proof. We have $f(I) = 0$, and $T_I(SO(n))$ is the set of skew symmetric $n \times n$ matrices. Let $(d\alpha_{ij}) \in T_I(SO(n))$. Then $f(I + (d\alpha_{ij})) = (F(v_1 + (d\alpha_{ij})v_1, ..., F(v_{2^n} + (d\alpha_{ij})v_{2^n}))$. We have to show that for $0 \neq (d\alpha_{ij}) \in T_I(SO(n))$ we have that $f(I + (d\alpha_{ij}))$ does not lie on the diagonal of $T(\mathbb{R}^{2^n}) \cong \mathbb{R}^{2^n}$ (thus, in particular, is not 0). We will argue indirectly.

For $\{v_1, ..., v_{2^n}\} \ni v = (x_1, ..., x_n)$ we have $F(v + (d\alpha_{ij})v) = F(v) + F' \cdot ((d\alpha_{ij})v) = 2\sum_i a_{ii}x_i(\sum_j d\alpha_{ij}x_j) = 2\sum_{i<j}(a_{ii} - a_{jj})d\alpha_{ij}x_ix_j$, taking in account the skew symmetry of $(d\alpha_{ij})$. Here by hypothesis $a_{ii} - a_{jj} \neq 0$ for $i < j$. If $F(v_1 + (d\alpha_{ij})v_1) = ... = F(v_{2^n} + (d\alpha_{ij})v_{2^n})$, then the quadratic form $\sum_{i<j}(a_{ii} - a_{jj})d\alpha_{ij}x_ix_j$ assumes equal values for each $(x_1, ..., x_n) = (\pm d_1, ..., \pm d_n)$. Like in Lemma 6, this implies $(a_{ii} - a_{jj})d\alpha_{ij} = 0$ for each $i < j$, hence $(d\alpha_{ij}) = 0$ as asserted. □

Proof of Theorem 1. For any even map $F : S^2 \to \mathbb{R}$ we define the map $f : SO(3) \to \mathbb{R}^4$ sending $A \in SO(3)$ to $(F(\pm Av_1), ..., F(\pm Av_4))$. We are going to show that the image of f intersects the diagonal Δ of $\mathbb{R}^4$, which is equivalent to the statement of Theorem 1.

Of course maps from $SO(3)$ to $\mathbb{R}^4$ do exist whose images do not intersect the diagonal. Therefore we need some extra condition on the map f. Namely f as defined above should respect the rotation group of the cube in the sense that it should be equivariant with respect to the free S_4 action ρ_{S_4} on $SO(3)$ (given by right multiplication by $S_4 \subset SO(3)$) and $\tilde{\tau}_{S_4}$ on $\mathbb{R}^4$ (given by permuting the coordinates), i.e., $\tilde{\tau}_{S_4}(g)f = f\rho_{S_4}(g)$ for each $g \in S_4$. We show that the image of such an equivariant map intersects the diagonal Δ of $\mathbb{R}^4$.

Indirectly suppose that an equivariant map $g : (SO(3), \rho_{S_4}) \to (\mathbb{R}^4, \tilde{\tau}_{S_4})$ exists, whose image does not intersect the diagonal. Applying the projection along Δ to $\Delta^\perp \cong \mathbb{R}^3$ (the orthogonal complement of Δ) of $\mathbb{R}^4$ we get an equivariant map $p : (\mathbb{R}^4, \tilde{\tau}_{S_4}) \to (\mathbb{R}^3, \tau_{S_4})$, where the S_4 action $\tau_{S_4} : S_4 \to O(3)$ is the action of the symmetry group of the regular tetrahedron in $\mathbb{R}^3$, the four letters corresponding to the four vertices of the regular tetrahedron. Thus we have an equivariant map $p \circ g : (SO(3), \rho_{S_4}) \to (\mathbb{R}^3, \tau_{S_4})$ whose image does not contain the origin. Finally we have the equivariant map

$$h : (\mathbb{R}^3 \setminus \{0\}, \tau_{S_4}) \to (S^2, \tau_{S_4}),$$

which is the projection along the radial direction. Therefore we have an equivariant map $h \circ p \circ g : (SO(3), \rho_{S_4}) \to (S^2, \tau_{S_4})$. The following proof of Proposition 4 rules out the possibility of existence of such mappings, completing the proof of Theorem 1. □

Proof of Proposition 4. An equivariant map $(SO(3), \rho_{S_4}) \to (S^2, \tau_{S_4})$ would give by construction a section of the S^2 bundle $\eta_{\tau_{S_4}}$ defined as the sphere bundle

$$S^2 \to (SO(3) \times S^2)/(\rho_{S_4} \times \tau_{S_4}) \to SO(3)/\rho_{S_4}.$$

We show that this sphere bundle does not have a section by showing that the third Stiefel-Whitney class of the sphere bundle $w_3(\eta_{\tau_{S_4}}) \neq 0$ does not vanish. For this it is sufficient to show a section of the corresponding vector bundle

$$\bar{\eta}_{\tau_{S_4}} := (SO(3) \times \mathbb{R}^3)/(\rho_{S_4} \times \tau_{S_4}) \to SO(3)/\rho_{S_4}$$

which intersects transversally the zero section in an odd number of points. Namely it is well known that the cohomology class of the zero set of a section which is transversal to the zero section coincides with the Euler class of the vector bundle which in our case is the third Stiefel-Whitney class w_3.

For exhibiting such a section we consider an ellipsoid $E \subset \mathbb{R}^3$, centred at the origin, with axes of different lengths. Moreover we let $s_E : (SO(3), \rho_{S_4}) \to (\mathbb{R}^3, \tau_{S_4})$ be the section of $\bar{\eta}_{\tau_{S_4}}$ corresponding to the convex body E. By Corollary 7 E admits exactly one inscribed cube. Thus s_E vanishes at exactly one point of $SO(3)/\rho_{S_4}$, namely at the point corresponding to this inscribed cube. By Lemma 8, at this point the intersection is transversal.

Thus $w_3(\eta_{\tau_{S_4}}) \neq 0$ indeed, completing the proof of Proposition 4, hence of Theorem 1, and of Corollary 5. □

3 G-equivariant maps

The theorem above gives that there is no S_4-equivariant map from $SO(3)$ to S^2. One may wonder for which subgroup of S_4 we can find an equivariant map. To be more precise we need some more notation.

A subgroup $G \subset S_4$ gives the G-actions ρ_G and τ_G by restricting the S_4 actions ρ_{S_4} and τ_{S_4} to the subgroup G. Moreover we get the S^2 bundle η_{τ_G} as the sphere bundle $S^2 \to (SO(3) \times S^2)/(\rho_G \times \tau_G) \to SO(3)/\rho_G$. Our problem is thus to determine for which subgroup G of S_4 we get a section of the S^2 bundle η_{τ_G}.

The [GAP] algebraic program package tells that, up to conjugation, there are 11 different subgroups of S_4; these are given by it by generators, and are isomorphic to the cyclic groups C_1, C_2, C_2, C_3, C_4, the dihedral groups D_2, D_2, D_3, D_4, the alternative group A_4, and S_4, respectively. Since A_4 occurs in this list once, and is a normal subgroup of S_4, there is only one subgroup of S_4, isomorphic to A_4, and, for any subgroup G of S_4, the property $G \subset A_4$ is invariant under taking conjugates of G. Thus, by the above list, the subgroups of S_4, not contained in A_4, are, up to conjugation, the following: $[(ij)] \cong C_2$, $[(ijkl)] \cong C_4$ (corresponding to the group of rotations through multiples of $\pi/2$ about a coordinate axis), $[(ij),(kl)] \cong D_2 \cong C_2 \times C_2$, $D_3 \cong S_3$ the subgroup of permutations fixing some i, D_4 the rotation group of a non-cubical square-based box (when S_4 is represented in $SO(3)$ as above), and S_4. ($[B]$ is the subgroup generated by $B \subset S_4$, and i, j, k, l are different elements of $\{1, ..., 4\}$.)

1. Case $G \cong S_4$

The previous theorem has showed that in this case $\eta_{\tau_{S_4}}$ does not have a section.

3.1 Inscribed square based boxes

2. Case $G \cong D_4$

In this case we do not have a section as the following proposition shows. Still we note that the following three statements are generalizations of Proposition 4, Theorem 1 and Corollary 5, respectively. Like Theorem 1, also Theorem 10 follows from Theorem D of [Griff], too.

Proposition 9 *There is no equivariant map from* $(SO(3), \rho_{D_4})$ *to* (S^2, τ_{D_4}).

Proof. The proof of this proposition follows the proof of Proposition 4, with the exception that we need the following consequence of Corollary 7: there are three ways of inscribing a non-cubical square based box with given ratio of edges into an ellipsoid in $\mathbb{R}^3$ with axes of different lengths.

Now the proof of the proposition is complete noting that 3 is odd and therefore $w_3(\eta_{\tau_{D_4}})$ does not vanish. □

Theorem 10 *Let* $F : S^2 \to \mathbb{R}$ *be a continuous even map, and let* $\{\pm v_i \mid 1 \leq i \leq 4\}$ *be the vertex set of a square based box inscribed to* S^2. *Then there exists* $A \in SO(3)$ *such that* $F(\pm Av_1) = F(\pm Av_2) = F(\pm Av_3) = F(\pm Av_4)$.

Proof. As in the case of the cube we can define a map $f : SO(3) \to \mathbb{R}^4$ by setting $f(A) = (F(\pm Av_1), ..., F(\pm Av_4))$. We have to show that the image of f intersects the diagonal Δ in $\mathbb{R}^4$. Notice that the rotation group of a square based box contains D_4 ($\subset S_4$) (and in general equals D_4). Now an identical argument as in the case of the cube in Theorem 1 yields the statement of this theorem, using Proposition 9 instead of Proposition 4. □

Corollary 11 *Every centrally symmetric convex body* K *in* $\mathbb{R}^3$ *admits an inscribed square based box, with any given ratio of the height to the basic edge, and centred at the centre of* K.

Proof. Let us consider a square based box, with the given ratio of the height to the basic edge, that is inscribed to S^2, and has vertex set $\{\pm v_i \mid 1 \le i \le 4\}$. Like at the reduction of Corollary 5 to Theorem 1, we suppose that the centre of K is the origin, and we let $\| \ \|_K$ be the Minkowski norm in $\mathbb{R}^3$, associated to K. We apply Theorem 10 for $F = \| \ \|_K |_{S^2}$ and the v_i chosen above, obtaining $A \in SO(3)$ such that $(\| \pm Av_1\|_K, ..., \| \pm Av_4\|_K)$ is a point $(\lambda, ..., \lambda)$ $(\lambda > 0)$ on the diagonal Δ of $\mathbb{R}^4$. Then the square based box with vertex set $\{\lambda^{-1}(\pm Av_i) \mid 1 \le i \le 4\}$ is inscribed in K, has the given ratio of the height to the basic edge, and is centred at the centre of K. □

3.2 Existence of G-equivariant maps

3. Case $G \subset A_4$

Theorem 12 *For $G \subset A_4$ there exists a G-equivariant map from $(SO(3), \rho_G)$ to (S^2, τ_G).*

Proof. We will show that the S^2-bundle $\eta_{\tau_{A_4}}$ defined at the beginning of Section 3 is actually trivial. Then the S^2-bundle $\eta_{\tau_{A_4}}$ will have a section, that implies by construction the existence of A_4-equivariant maps, that are also G-equivariant for any $G \subset A_4$.

Think of $A \in SO(3)$ as an orthonormal basis. Let $p_i : SO(3) \to S^2$ be the forgetful map associating to $A \in SO(3)$ its i'th basis vector. These maps are A_4-equivariant as A_4 is the rotation group of the regular tetrahedron. In this way we get three linearly independent sections p_1, p_2 and p_3 of the vector bundle $\bar{\eta}_{\tau_{A_4}}$. Thus $\eta_{\tau_{A_4}}$ is trivial indeed. (Another way of seeing this is the following. If G is a discrete subgroup of $SO(3)$, then the bundle $\bar{\eta}_{\tau_G} = SO(3) \times_G \mathbb{R}^3 = T(SO(3)/G)$ is the tangent bundle of $SO(3)/G$ (left cosets), and this is trivial. Namely the tangent bundle of the quotient of a Lie group with respect to a discrete subgroup is trivial.) □

Remark. This argument does not work for $G \not\subset A_4$, because in these cases the action of G is given by $g(\hat{g}, x) = (g\hat{g}, \pm gx)$, where the sign is positive for $g \in A_4$ and negative for $g \in S_4 \setminus A_4$.

In what follows, we will investigate subgroups $G \not\subset A_4$.

4. Case $G \cong C_2, D_2, D_3$

In the following theorem we will use the notations introduced at the beginning of Section 3.

Theorem 13 *Let* $G = [(ij)] \cong C_2$, $G = [(ij),(kl)] \cong D_2$ *or let* G $(\cong D_3 \cong S_3)$ *be the subgroup of permutations fixing some* i. *Then there exists a* G*-equivariant map from* $(SO(3), \rho_G)$ *to* (S^2, τ_G).

Proof. Recall that $\tau_G : G \to O(3)$ is the action of the subgroup G of the symmetric group S_4 of the regular tetrahedron, the four letters corresponding to the four vertices of the regular tetrahedron. In other words, for $g \in G$ we have that $\tau_G(g)$ is that element of $O(3)$, whose restriction to the vertex set of the regular tetrahedron inscribed to S^2 equals the permutation $g \in G$ of these vertices.

For $G = [(ij)]$ any $x \in S^2$, equidistant to the i'th and j'th vertices, is a fixed point for τ_G. For $G = [(ij),(kl)]$ any vector $x \in S^2$, orthogonal to the lines connecting the i'th and j'th vertices, and the k'th and l'th vertices of the tetrahedron, respectively, is a fixed point of τ_G. For G being the subgroup of permutations fixing some i, the i'th vertex is a fixed point of τ_G. Then a G-equivariant map can be given in all three cases as a constant map $(SO(3), \rho_G) \to (S^2, \tau_G)$, having as value a fixed point x of τ_G. □

5. Case $G \cong C_4$

Theorem 14 *There does exist a* C_4*-equivariant map from* $(SO(3), \rho_{C_4})$ *to* (S^2, τ_{C_4}).

Proof. We show that there is a section of the S^2 bundle $\eta_\tau = (SO(3) \times S^2)/(\rho_{C_4} \times \tau_{C_4})$ over the orbit space $L_8 = SO(3)/\rho_{C_4}$. This is equivalent to the statement of the theorem.

Obstruction theory (cf. [MiSt], pp. 139-143) tells us that the existence of such a section is equivalent to the vanishing of the obstruction class $o_3(\eta_\tau) \in H^3(L_8, \pi_2(S^2))$ sitting in a cohomology space with twisted coefficients. We show the vanishing of $o_3(\eta_\tau)$. We use a theorem of Steenrod (cf. [Steen], 38.8) which claims that there is a homomorphism

$$\delta^* : H^2(L_8, \pi_1(SO(3)) \to H^3(L_8, \pi_2(S^2))$$

which sends the second obstruction class $o_2(\eta_\tau)$ to $o_3(\eta_\tau)$. Therefore it is sufficient to show that the second obstruction class vanishes. On the other hand the second obstruction class $o_2(\eta_\tau)$ coincides with the second Stiefel-Whitney class of η_τ, i.e., $o_2(\eta_\tau) = w_2(\eta_\tau)$ (cf. [MiSt], p.140). Thus the proof of $w_2(\eta_\tau) = 0$ gives the theorem.

The vector bundle $\eta_\tau = \eta_{\tau_1} \oplus \eta_{\tau_2}$ decomposes as the direct sum of a rank 1 and a rank 2 vector bundle, where τ_1 is the non-trivial representation of C_4 on $\mathbb{R}^1$ and τ_2 is an effective representation of C_4 on $\mathbb{R}^2$ (the generator is sent to the rotation through the angle $\frac{\pi}{2}$), and $\eta_{\tau_i} = (SO(3) \times \mathbb{R}^i)/(\rho \times \tau_i)$. Thus we get that

$$w_2(\eta_\tau) = w_1(\eta_{\tau_1}) \cup w_1(\eta_{\tau_2}) + w_2(\eta_{\tau_2}).$$

This formula shows that the vanishing of $w_1(\eta_{\tau_2})$ and $w_2(\eta_{\tau_2})$ yields the vanishing of $w_2(\eta_\tau)$. The first Stiefel-Whitney class $w_1(\eta_{\tau_2})$ vanishes since the representation τ_2 is orientation preserving and hence η_{τ_2} is oriented. The second Stiefel-Whitney class $w_2(\eta_{\tau_2})$ vanishes since, as we are going to show,

$$\eta_{\tau_2} = \eta_\gamma \otimes \eta_\gamma$$

with some S^1 principal bundle η_γ (here η_{τ_2} is considered as an S^1 principal bundle). Hence the Euler class $e(\eta_{\tau_2}) = 2e(\eta_\gamma)$ is even, yielding the desired vanishing of $w_2(\eta_{\tau_2})$.

Thus we are left with constructing an S^1 principal bundle η_γ with $\eta_{\tau_2} = \eta_\gamma \otimes \eta_\gamma$. The universal covering of L_8 is the composite covering $S^3 \to \mathbb{R}P^3 \cong SO(3) \to L_8$. Moreover, we claim that this covering is given by the C_8 action $\tilde{\rho}$ on S^3, sending the generator α to the transformation $q \mapsto e^{\frac{2\pi i}{8}} q$ of S^3, the unit sphere of the quaternionic algebra $\mathbb{H}$. To see this, divide out first by the $C_2 \subset C_8$ action defined by $\alpha^4(q) = e^{\frac{2\pi i}{2}} q = -q$. The quotient is $\mathbb{R}P^3$ which can be identified with $SO(3)$, by associating to any element $(q, -q) \in \mathbb{R}P^3$ a special orthogonal transformation of $\mathrm{Im}(\mathbb{H}) = \mathbb{R}^3$, namely the conjugation with q. Now we see that the $C_4 = C_8/C_2$ action on $\mathbb{R}P^3$ inherited from the C_8 action $\tilde{\rho}$ of S^3 corresponds to the C_4 action ρ of $SO(3)$ with respect to the axis $i \in \mathrm{Im}(\mathbb{H})$ (i.e., to the rotation of the space $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ through $\pi/2$ about the axis i), via the above identification. Hence indeed L_8 is a lens space.

Let $\eta_\gamma = (S^3 \times \mathbb{R}^2)/(\tilde{\rho} \times \gamma)$ be the principal S^1 bundle on L_8 given by the representation γ of C_8 on $\mathbb{R}^2$, sending the generator to the rotation through the angle $\frac{\pi}{4}$. Now the relation $\eta_\gamma \otimes \eta_\gamma = \eta_{\tau_2}$ is immediate. The proof is complete. □

4 Inscribed boxes

Now we will prove a theorem that contains both our previous Corollary 5 and its more general form Corollary 11 as special cases. However, as mentioned above, its proof uses the quite involved Theorem D in [Griff], rather than its special cases Theorem 1, and its more general form Theorem 10.

Theorem 15 *Every centrally symmetric convex body K in $\mathbb{R}^3$ admits an inscribed box, similar to any given box, and centred at the centre of K.*

Proof. Let us consider a box similar to the given box, that is inscribed to S^2, and one of whose faces has vertex set $\{v_1, ..., v_4\}$. Like at Corollaries 5 and 11, we suppose that the centre of K is the origin, and we let $\| \ \|_K$ be the Minkowski norm in $\mathbb{R}^3$, associated to K. We let $F = \| \ \|_K |_{S^2}$: $S^2 \to \mathbb{R}$. Then Theorem D of [Griff] (quoted in Section 1) implies for $F : S^2 \to \mathbb{R}$ and the set $\{v_1, ..., v_4\}$ of the vertices of a planar rectangle inscribed to S^2, that there exists $A \in SO(3)$ such that $((\|Av_1\|_K, ..., \|Av_4\|_K) =)$ $(F(Av_1), ..., F(Av_4))$ is a point $(\lambda, ..., \lambda)$ $(\lambda > 0)$ on the diagonal Δ of $\mathbb{R}^4$. Then the box with vertex set $\{\lambda^{-1}(\pm Av_i) \mid 1 \leq i \leq 4\}$ is inscribed in K, is similar to the given box, and is centred at the centre of K. $\square$

5 Inscribed cubes in C^1 surfaces

In this section we prove theorems about inscribability of cubes and square based boxes to certain centrally symmetric surfaces, that do not follow from Theorem D of [Griff]. We will roughly follow the proofs of Theorems 1 and 2 of [Jer], showing that a plane analytic Jordan curve admits an inscribed square, and in the "general" case (precised there) their number is odd. Cf. also [Gro], showing inscribability of homothets of simplices to hypersurfaces in $\mathbb{R}^n$, under topological and smoothness assumptions, and also solving a Knaster type problem. Another related theorem is [Griff], Theorem C, stating that each C^2–embedded S^2 in $\mathbb{R}^3$ admits for each $\rho > 0$ an inscribed "skew box of aspect ρ". This is obtained from a square based right prism with ratio of height to basic edge equal to ρ, by rotation of one base in its own plane about its centre through some angle.

First we prove Theorem 2:

Proof of Theorem 2. By Corollary 7 the statement is true if $f(S^2)$ is an ellipsoid with pairwise different lengths of axes. Moreover there is a single cube on such an ellipsoid and this is "transversally true", i.e., remains true for any sufficiently small perturbation in the C^1 topology (invariant under the map $x \mapsto -x$) of the ellipsoid (cf. Lemma 8).

The plan of the proof. We want to reduce the general statement of the theorem to the above special case. Namely we show that the cubes on the surface $f(S^2)$ are in bijective correspondence with the intersection points of two cycles (α and β_f) of complementary dimensions of a C^1 manifold with boundary. Here α is formed by (the equivalence classes of) those eight-tuples of points in $\mathbb{R}^3 \setminus \{0\}$ which are vertices of a cube (centred at 0). The cycle β_f is formed by (the equivalence classes of) those eight-tuples of points which are all on the surface $f(S^2)$. We may assume that their intersection is transversal, and consists of finitely many points, since else it is certainly not empty, and then there is nothing to prove. Now if we replace the surface $f(S^2)$ by any other one – $g(S^2)$, say, – satisfying the same assumptions and thus representing the same homotopy class, then the cycle β_f will be replaced by a cycle β_g, homological to it. Since α is a cycle, the intersection number (mod 2) $\alpha \cap \beta_g$ remains equal to $\alpha \cap \beta_f$. On the other hand, $\alpha \cap \beta_g$ is the number (mod

2) of the cubes on $g(S^2)$. If we choose $g(S^2)$ = an ellipsoid with different lengths of axes, with $g(-x) = -g(x)$ (let g be the radial projection of S^2 to the surface of the ellipsoid) then, by the special case proved earlier, $\alpha \cap \beta_g \neq 0 \mod 2$.

Preliminary remarks. The (oversimplified) plan described above will be modified by the following ways and following reasons.

1. Since $f(S^2)$ is invariant under the multiplication by -1 it will be sufficient to find four vertices v_1, v_2, v_3, v_4 of a face of a cube, in this cyclic order, because then the eight vectors $\pm v_i$ automatically will form a cube on $f(S^2)$. (In this proof by a cube we always mean one centred at the origin.) Therefore we will form the cycle α not from the 8-tuples of points but from quadruples, forming the vertices of a face of a cube. Similarly β_f will be formed not from 8-tuples but quadruples of points on $f(S^2)$. The dimensions of these cycles in $(\mathbb{R}^3)^4$ are 4 and 8, respectively, thus are complementary.

2. Naturally we do not consider two cubes different, if they differ only by a permutation of the vertices. (Otherwise we would get each cube 48 times and therefore the homological intersection index with $\mathbb{Z}_2$ coefficients would not give any information on their number.) Therefore we have to factorize out $(\mathbb{R}^3)^4$ by the "automorphism group of the cube", i.e., by the group $G := \{T \in O((\mathbb{R}^3)^4) \mid (Tv_1, Tv_2, Tv_3, Tv_4)$ is a permutation of (v_1, v_2, v_3, v_4), or of $(v_i, v_{i+1}, -v_{i+3}, -v_{i+2})$, $1 \leq i \leq 4$, indices meant cyclically, or of $(-v_1, -v_2, -v_3, -v_4)$, in each of these cases either *preserving* or *reversing* their above given cyclic order$\}$. (Observe that if $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ are the vertices of a face of a cube, in this cyclic order, then in a natural way these transformations bijectively correspond to the elements of the symmetry group ($\subset O(\mathbb{R}^3)$) of this cube.) This raises the problem that this action is not free at certain points of the set $\{(v_1, v_2, v_3, v_4) \in (\mathbb{R}^3)^4 \mid v_i = \pm v_j$ for some $1 \leq i < j \leq 4\}$. Since later we will have to exclude quadruples $(v_1, ..., v_4) \in (\mathbb{R}^3)^4$ having a zero coordinate v_i, we define the *singular set* S as the larger set

$$S := \{(v_1, v_2, v_3, v_4) \in (\mathbb{R}^3)^4 \mid v_i = 0 \text{ for some } 1 \leq i \leq 4$$
$$\text{or } v_i = \pm v_j \text{ for some } 1 \leq i < j \leq 4\} \subset (\mathbb{R}^3)^4.$$

The action of G is free on $(\mathbb{R}^3)^4 \setminus S$. By non-freeness of the action on $(\mathbb{R}^3)^4$ the quotient space $(\mathbb{R}^3)^4/G$ will not be a manifold, only an orbifold.

Let A be the set corresponding to the cycle α, i.e.,

$$A := \{(v_1, v_2, v_3, v_4) \mid v_i \in \mathbb{R}^3 \setminus \{0\} \text{ and } v_1, v_2, v_3, v_4 \text{ are the vertices of a face of a cube, in this cyclic order}\}.$$

Note that A is a cone in $(\mathbb{R}^3)^4$ with its vertex at the origin, with its vertex deleted, and with no point on the set S.

Of course $X := (\mathbb{R}^3)^4 \setminus S \subset (\mathbb{R}^3)^4$ is an open set, invariant under G, on which G acts freely. Thus we can consider the quotient manifold X/G. But the trouble is that this manifold is not compact.

3. So the final modification is that from the very beginning we shall work not with the whole set X, but only with some sufficiently large compact subset of it (to be specified later). The set X is an open set invariant under G, and G acts freely on it. Let $\varepsilon_1 > 0$ be a sufficiently small number, $K > 0$ be sufficiently large (to be specified later), and let $\varepsilon_2 = \varepsilon_1/100$. Let $\varphi : X \to \mathbb{R}$,

$$\varphi(v_1, ..., v_4) := \max \left\{ \frac{\varepsilon_1}{\|v_i\|} \ \ (1 \le i \le 4),\ \frac{\|v_i\|}{K} \ \ (1 \le i \le 4),\ \frac{\sqrt{2}\varepsilon_2}{\|v_i \pm v_j\|} \ \ (1 \le i < j \le 4) \right\}.$$

Observe that $\|v_i \pm v_j\|/\sqrt{2}$ is the distance of $(v_1, ..., v_4)$ to the subspace given by the equation $v_i \pm v_j = 0$ (since $(v_1, ..., v_4) \mapsto \big((v_1 + v_2)/\sqrt{2}, (v_1 - v_2)/\sqrt{2}, v_3, v_4\big)$ is an orthogonal transformation). Then φ is continuous, is invariant under G, and for any $r > 0$ the set $\varphi^{-1}([0, r])$ is a compact subset of X. However, we do not know whether $\varphi^{-1}([0, r]) \setminus \mathrm{int}\varphi^{-1}([0, r])$ is a topological manifold for a generic $r > 0$. Therefore we define $\psi : X \to \mathbb{R}$,

$$\psi(v_1, ..., v_4) := \left(\sum_{i=1}^{4} \frac{\varepsilon_1^2}{\|v_i\|^2} + \sum_{i=1}^{4} \frac{\|v_i\|^2}{K^2} + \sum_{1 \le i < j \le 4} \frac{2\varepsilon_2^2}{\|v_i \pm v_j\|^2} \right)^{1/2}.$$

Then ψ is an analytic, hence C^∞ function on X, is invariant under G, and

$$\varphi \le \psi \le \sqrt{20}\varphi.$$

By Sard's lemma, almost all $r \in \mathbb{R}$ is not a critical value of ψ (i.e., for each $(v_1, ..., v_4) \in \psi^{-1}(r)$ we have $\psi'(v_1, ..., v_4) \neq 0$ – that means at least one partial derivative is different from 0 –, and hence $\psi^{-1}(r)$ is a closed embedded C^∞ 11-submanifold of X, possibly empty), cf. [How], p. 66. Let $r \in (\sqrt{20}, 5)$ be a not critical value of ψ, and let

$$X_r := \psi^{-1}([0, r)) = \psi^{-1}((0, r)) \ \ (\supset \varphi^{-1}([0, 1])).$$

The set $Y_r := \psi^{-1}(r) \subset \varphi^{-1}((1, 5))$ is a non-empty closed embedded C^∞ 11-submanifold of X, and is invariant under G. Further, $Y_r = \partial X_r$, since r is not a critical value of ψ.

The factor space $\overline{X_r}/G = (X_r \cup Y_r)/G$ is a compact C^∞ manifold with boundary Y_r/G. Since both of the sets $A, f(S^2)^4 \subset (\mathbb{R}^3)^4$ are G-invariant C^1 manifolds, so their intersections with X_r and Y_r are G-invariant as well. The sets $A \cap X_r$, $f(S^2)^4 \cap X_r$ are C^1 manifolds as well (but we do not know if $A \cap \overline{X_r}$, $f(S^2)^4 \cap \overline{X_r}$ are C^1 manifolds with boundary, since we do not know anything about the sets $A \cap Y_r$, $f(S^2)^4 \cap Y_r$). Thus we can form the quotient C^1 manifolds $[A \cap X_r]/G$, $[f(S^2)^4 \cap X_r]/G$ (observe that $A \cap X_r$, $f(S^2)^4 \cap X_r$ are disjoint to the singular set S), and the quotient sets

$$A_r := [A \cap \overline{X_r}]/G, \; B_r := [f(S^2)^4 \cap \overline{X_r}]/G$$

(about which we do not know if they are C^1 manifolds with boundary). For the statement in the last brackets it would be necessary to know if $Y_r/G = \partial(X_r/G)$ intersects the manifold A/G and the orbifold $f(S^2)^4/G$, that is a manifold in a neighbourhood of Y_r/G, transversally.

However, by Thom's transversality theorem (cf. [Hir] Ch. 3, Theorem 2.5) this can be achieved, if we replace Y_r/G by a C^1 manifold $Q_r \subset X/G$, that is a small, generic C^1 perturbation of Y_r/G. Moreover, we may assume that

$$Q_r = \overline{P_r} \setminus P_r = \partial \overline{P_r},$$

for some open set $P_r \subset X/G$, whose symmetric difference with X_r/G lies in a small tubular neighbourhood of Y_r/G, included in $[\varphi^{-1}((1,5))]/G$. Further, we have

$$Q_r \subset \varphi^{-1}\left([1,5]\right)/G \text{ and } P_r \subset \varphi^{-1}\left([0,5)\right)/G.$$

Moreover, the modified sets

$$A'_r := [A/G] \cap \overline{P_r} \text{ and } B'_r := [f(S^2)^4/G] \cap \overline{P_r} = [\left(f(S^2)^4 \setminus S\right)/G] \cap \overline{P_r}$$

are C^1 manifolds with boundary, their boundaries $[A/G] \cap Q_r$ and $[(f(S^2)^4 \setminus S)/G] \cap Q_r$ lying in $\partial \overline{P_r}$; in other words, they are relative cycles in the chain complex $\mathcal{C}_*(\overline{P_r}, \partial \overline{P_r}) = \mathcal{C}_*(\overline{P_r}, Q_r)$.

Now we can finally start the proof. $\overline{P_r}$ is a compact C^1 manifold with boundary, and A'_r, B'_r are its C^1 submanifolds with boundaries, which are relative cycles in $\mathcal{C}_*(\overline{P_r}, \partial \overline{P_r})$.

Now we have to choose ε_1 and K properly. Observe that if a cube, having a face with vertices v_1, v_2, v_3, v_4, in this cyclic order, is inscribed in $f(S^2)$, then

$$\min_{u \in S^2} \|f(u)\| \leq \|v_1\| = \ldots = \|v_4\| \leq \max_{u \in S^2} \|f(u)\|,$$

moreover, for $1 \le i < j \le 4$, we have $\|v_i \pm v_j\| \in \{2\|v_i\|/\sqrt{3}, 2\sqrt{2}\|v_i\|/\sqrt{3}\}$, so $\|v_i \pm v_j\|/\sqrt{2} \ge \|v_i\|\sqrt{2/3}$. We have to choose ε_1 so small and K so large that, taking $\overline{P_r} \subset X/G$ rather than X/G does not exclude any inscribed cube. We have

$$\overline{P_r} \supset P_r \supset (X_r/G) \setminus \left(\varphi^{-1}\left((1,5)\right)/G\right) \supset$$

$$\left(\varphi^{-1}\left([0,1]\right)/G\right) \setminus \left(\varphi^{-1}\left((1,5)\right)/G\right) = \varphi^{-1}\left([0,1]\right)/G.$$

Hence for $x \in X$ and $x/G \notin \overline{P_r}$ we have $\varphi(x) > 1$, thus either $\|v_i\| < \varepsilon_1$ for some $1 \le i \le 4$, or $\|v_i\| > K$ for some $1 \le i \le 4$, or $\|v_i \pm v_j\|/\sqrt{2} < \varepsilon_2$ for some $1 \le i < j \le 4$. However, for $v_1, ..., v_4$ being vertices of a cube, like above, we have in the last case $\|v_i\| \le \left(\|v_i \pm v_j\|/\sqrt{2}\right)\sqrt{3/2} < \varepsilon_2\sqrt{3/2} < \varepsilon_1$, by the choice of ε_2. Hence the first or second inequality must hold. These however can be excluded by choosing

$$0 < \varepsilon_1 < \min_{u \in S^2} \|f(u)\| \le \max_{u \in S^2} \|f(u)\| < K.$$

Thus in fact taking $\overline{P_r} \subset X/G$ rather than X/G does not exclude any inscribed cube.

Now recall that A'_r and B'_r are C^1 manifolds with boundaries, with their boundaries lying in $\partial\overline{P_r} = Q_r$. Let

$$Q_{r1} := Q_r \cap \left[\left\{(v_1, ..., v_4) \in X \mid \max\{\frac{\varepsilon_1}{\|v_i\|}, \frac{\|v_i\|}{K} \mid 1 \le i \le 4\} < \max\{\frac{\sqrt{2}\varepsilon_2}{\|v_i \pm v_j\|} \mid 1 \le i < j \le 4\}\right\} \Big/ G\right]$$

and

$$Q_{r2} := Q_r \cap \left[\left\{(v_1, ..., v_4) \in X \mid \max\{\frac{\varepsilon_1}{\|v_i\|}, \frac{\|v_i\|}{K} \mid 1 \le i \le 4\} > \max\{\frac{\sqrt{2}\varepsilon_2}{\|v_i \pm v_j\|} \mid 1 \le i < j \le 4\}\right\} \Big/ G\right].$$

Then Q_{r1} and Q_{r2} are disjoint open subsets of Q_r. Actually we will attain that A'_r will be a relative cycle in $\mathcal{C}_*(\overline{P_r}, Q_{r2})$, and B'_r will be a relative cycle in $\mathcal{C}_*(\overline{P_r}, Q_{r1})$, i.e., $\partial A'_r \subset Q_{r2}$ and $\partial B'_r \subset Q_{r1}$. That is, we want to attain $(\partial A'_r) \cap (Q_r \setminus Q_{r2}) = \emptyset$ and $(\partial B'_r) \cap (Q_r \setminus Q_{r1}) = \emptyset$. However, by $\partial A'_r, \partial B'_r \subset Q_r \subset \varphi^{-1}\left([1,5]\right)/G$ we have

$$\partial A'_r \subset [A/G] \cap \left[\varphi^{-1}\left([1,5]\right)/G\right] \text{ and } \partial B'_r \subset \left[f(S^2)^4/G\right] \cap \left[\varphi^{-1}\left([1,5]\right)/G\right].$$

Therefore, it suffices to establish

$$[A/G]\cap\left[\varphi^{-1}\left([1,5]\right)/G\right]\cap(Q_r\setminus Q_{r2})=\emptyset \text{ and } \left[f(S^2)^4/G\right]\cap\left[\varphi^{-1}\left([1,5]\right)/G\right]\cap(Q_r\setminus Q_{r1})=\emptyset.$$

To obtain a contradiction, we assume that one of these sets is nonempty.

First assume that $(v_1,...,v_4)/G \in [A/G]\cap[\varphi^{-1}([1,5])/G]\cap(Q_r\setminus Q_{r2})$. Then $1\leq\varphi(v_1,...,v_4)\leq 5$, hence, by taking in consideration the definitions of φ and Q_{r2}, we have

$$\max\left\{\frac{\varepsilon_1}{\|v_i\|},\frac{\|v_i\|}{K}\mid 1\leq i\leq 4\right\}\in[0,5] \text{ and } \max\left\{\frac{\sqrt{2}\varepsilon_2}{\|v_i\pm v_j\|}\mid 1\leq i<j\leq 4\right\}\in[1,5].$$

These imply $\|v_i\|\geq\varepsilon_1/5$ for each $1\leq i\leq 4$, and $\|v_i\pm v_j\|/\sqrt{2}\leq\varepsilon_2$ for some $1\leq i<j\leq 4$ and some choice of the sign $\pm$. However, for vertices of a cube $v_1,...,v_4$ like above, we have $\|v_i\pm v_j\|/\sqrt{2}\geq\|v_i\|\sqrt{2/3}$, that implies for the above pair $\{i,j\}$ and the above choice of the sign that

$$\frac{\varepsilon_1}{100}=\varepsilon_2\geq\frac{\|v_i\pm v_j\|}{\sqrt{2}}\geq\|v_i\|\sqrt{\frac{2}{3}}\geq\frac{\varepsilon_1}{5}\sqrt{\frac{2}{3}},$$

a contradiction.

Second assume that $(v_1,...,v_4)/G\in[f(S^2)^4/G]\cap[\varphi^{-1}([1,5])/G]\cap(Q_r\setminus Q_{r1})$. Then $1\leq\varphi(v_1,...,v_4)\leq 5$, hence, by the definitions of φ and Q_{r1}, we have

$$\max\left\{\frac{\varepsilon_1}{\|v_i\|},\frac{\|v_i\|}{K}\mid 1\leq i\leq 4\right\}\in[1,5].$$

Then for some $1\leq i\leq 4$ we have $\|v_i\|\leq\varepsilon_1$ or $\|v_i\|\geq K$. However, by $(v_1,...,v_4)/G\in f(S^2)^4/G$ any of these possibilities contradicts the choice of ε_1 and K. This ends the proof of the statement that A_r' or B_r' are relative cycles in $\mathcal{C}_*(\overline{P_r},Q_{r2})$ or $\mathcal{C}_*(\overline{P_r},Q_{r1})$, respectively.

Actually, we shall choose the numbers ε_1 and K even more carefully as follows. We define g to be the embedding of the ellipsoid as in the plan of the proof. By genericity of Q_r we may assume that Q_r intersects the orbifold $g(S^2)^4/G$ (that is a manifold in a neighbourhood of Y_r/G) transversally — and hence in finitely many points — as well. By hypothesis we have an odd homotopy H (i.e., one satisfying $H(-u,t)=-H(u,t)$ for $u\in S^2,\ t\in[0,1]$) between f and the standard embedding of S^2 to $\mathbb{R}^3$. Similarly there is an odd homotopy, obtained by linear interpolation, between the standard embedding of S^2 to $\mathbb{R}^3$ and g. Putting these together we obtain an odd homotopy $H_1: S^2\times[0,1]\to\mathbb{R}^3\setminus\{0\}$ between f and g. Now we suppose

$$0<\varepsilon_1<\min\{\|H_1(u,t)\|\mid u\in S^2,\ t\in[0,1]\}\leq\max\{\|H_1(u,t)\|\mid u\in S^2,\ t\in[0,1]\}<K.$$

Now $H_1^4 : (S^2)^4 \times [0,1] \to (\mathbb{R}^3 \setminus \{0\})^4$, $H_1^4((u_1, ..., u_4), t) = (H_1(u_1, t), ..., H_1(u_4, t))$ is a homotopy (with four odd components) betwen $f \times f \times f \times f = f^4 : (S^2)^4 \to (\mathbb{R}^3 \setminus \{0\})^4$ and $g^4 : (S^2)^4 \to (\mathbb{R}^3 \setminus \{0\})^4$. Moreover, H_1^4 is G–equivariant, i.e., for $T \in G$ we have $H_1^4(T(u_1, ...u_4), t) = TH_1^4((u_1, ..., u_4), t)$ (this being true even for $T \in S_4 \times \mathbb{Z}_2^4$, the i-th $\mathbb{Z}_2$ acting on the i-th coordinate by $v \mapsto \pm v$, and on the other coordinates identically). Therefore H_1^4 induces a homotopy $H_1^4/G : (S^2)^4 \times [0,1])/G \to (\mathbb{R}^3 \setminus \{0\})^4/G$ (G acting on the factor $[0,1]$ trivially) between $f^4/G, g^4/G : (S^2)^4/G \to (\mathbb{R}^3 \setminus \{0\})^4/G$, by the formula $(H_1^4/G)(G(u_1, ..., u_4), t) = GH_1^4((u_1, ...u_4), t)$, where we put the quotient topology on each of these spaces. (There is one point to be clarified: the restriction of the quotient topology $\mathcal{T} = ((S^2)^4 \times [0,1])/G$ to $((S^2)^4 \times \{0\})/G$ is the quotient topology $\mathcal{T}_0 = (S^2)^4/G$, and a similar statement with $\{1\}$ rather than $\{0\}$. This follows from $\mathcal{T} = \mathcal{T}_0 \times [0,1]$. To see this last statement, observe that we have an evident mapping $(S^2)^4 \times [0,1] \to ((S^2)^4/G) \times [0,1]$. This factors through the quotient topology $\mathcal{T}$. However $\mathcal{T}$ is compact, and $((S^2)^4/G) \times [0,1]$ is T_2, so a bijective map between them is a homeomorphism. The T_2 property of $((S^2)^4/G) \times [0,1]$ follows since the quotient space of a T_2 space by a finite group of homeomorphisms is a T_2 space.)

Now recall that Q_r is transversal to $f(S^2)^4/G$ and $g(S^2)^4/G$, which are manifolds in a neighbourhood of Q_r. In other words, the maps f^4/G and g^4/G are transversal to Q_r. Therefore also the homotopy H_1^4/G connecting them is transversal to Q_r at $t = 0$ and $t = 1$.

Our aim will be to modify the map $H_1^4/G : ((S^2)^4/G) \times [0,1] \to (\mathbb{R}^3 \setminus \{0\})^4/G$ a bit, so as to obtain a map transversal to Q_r. For this aim first observe that $(S^2)^4/G$ is not a manifold. Therefore, analogously to $\varphi, \psi : X = (\mathbb{R}^3)^4 \setminus S \to \mathbb{R}$ we define $\Phi, \Psi : (S^2)^4 \setminus S \to \mathbb{R}$, by the formulas

$$\Phi(u_1, ...u_4) := \max \left\{ \frac{\sqrt{2}\delta}{\|u_i \pm u_j\|} \mid 1 \le i < j \le 4 \right\}$$

and

$$\Psi(u_1, ..., u_4) := \left(\sum_{1 \le i < j \le 4} \frac{2\delta^2}{\|u_i \pm u_j\|^2} \right)^{1/2},$$

where $\delta > 0$ is to be chosen later to be sufficiently small. We have

$$\Phi \le \Psi \le \sqrt{12}\Phi.$$

Let $s \in (\sqrt{12}, 4)$ be a not critical value of the C^∞ function Ψ, and let

$$M_s := \Psi^{-1}([0,s)) = \Psi^{-1}((0,s)) \supset \Phi^{-1}([0,1]) \text{ and } N_s := \Psi^{-1}(s) \subset \Phi^{-1}((1,4)).$$

Then $M_s \subset \Phi^{-1}([0,4))$ and $N_s = \partial \overline{M_s}$ is a non–empty closed embedded C^∞ 7–submanifold of $(S^2)^4 \setminus S$. Moreover M_s and N_s are G–invariant. The factor space $\overline{M_s}/G = (M_s \cup N_s)/G$ is a compact C^∞ manifold with boundary N_s/G.

Observe that the odd homotopy $H_1 : S^2 \times [0,1] \to \mathbb{R}^3 \setminus \{0\}$ has a compact domain, hence is uniformly continuous. Moreover we have $H_1^4\left(((S^2)^4 \setminus M_s) \times [0,1]\right) \subset H_1^4\left(((S^2)^4 \setminus \Phi^{-1}([0,1])) \times [0,1]\right)$. Now choose δ so that $u_i, u_j \in S^2$, $\|u_i - u_j\|/\sqrt{2} < \delta$ implies $\|H_1(u_i,t) - H_1(u_j,t)\|/\sqrt{2} < \varepsilon_2/7$. By oddness then we have that also $u_i, u_j \in S^2$, $\|u_i + u_j\|/\sqrt{2} < \delta$ implies $\|H_1(u_i,t) + H_1(u_j,t)\|/\sqrt{2} < \varepsilon_2/7$. Then we claim

$$H_1^4\left(((S^2)^4 \setminus M_s) \times [0,1]\right) \subset H_1^4\left(((S^2)^4 \setminus \Phi^{-1}([0,1])) \times [0,1]\right) \subset \varphi^{-1}((7,\infty)) \cup S.$$

In fact, for this it suffices to show that $u_1, \ldots, u_4 \in S^2$, $\Phi(u_1, \ldots, u_4) > 1$ and $(H_1(u_1,t), \ldots, H_1(u_4,t)) \notin S$ imply $\varphi(H_1(u_1,t), \ldots, H_1(u_4,t)) > 7$. We have $\Phi(u_1, \ldots, u_4) > 1 \iff \min_{i<j} \|u_i \pm u_j\|/\sqrt{2} < \delta$, hence at these conditions $\varphi(H_1(u_1,t), \ldots, H_1(u_4,t)) \geq \max_{i<j} (\sqrt{2}\varepsilon_2)/\|H_1(u_i,t) \pm H_1(u_j,t)\| > 7$, by the choice of δ and oddness of the homotopy H_1.

We have $Q_r \subset \varphi^{-1}([1,5])/G \subset V_1 := \varphi^{-1}((0,6))/G \subset \overline{V_1} \subset V_2 := \varphi^{-1}((0,7))/G$. By $H_1^4\left(((S^2)^4 \setminus M_s) \times [0,1]\right) \subset \varphi^{-1}((7,\infty)) \cup S$ we have $\left(((S^2)^4 \setminus M_s) \times [0,1]\right) \cap (H_1^4)^{-1} (\varphi^{-1}((0,7))) = \emptyset$, i.e., $(H_1^4)^{-1}(\varphi^{-1}((0,7))) \subset M_s \times [0,1]$. Here M_s is a C^1 manifold, thus $M_s \times [0,1]$ is a C^1 manifold with boundary. We have $(H_1^4/G)^{-1}(\varphi^{-1}((0,7))/G) = (H_1^4/G)^{-1}(V_2) \subset (M_s/G) \times [0,1]$, so $(H_1^4/G)^{-1}(V_2)$ is an open set in the C^1 manifold with boundary $(M_s/G) \times [0,1]$, so it is also a C^1 manifold with boundary. The set $(H_1^4)^{-1}(V_2 \setminus V_1)$ is a relatively closed set in $(H_1^4/G)^{-1}(V_2)$, and on it the map H_1^4/G is transversal to Q_r, since $(H_1^4/G)[(H_1^4/G)^{-1}(V_2 \setminus V_1)] \subset V_2 \setminus V_1$ is disjoint to Q_r. Moreover, as we have seen above, also on $[((S^2)^4/G) \times \{0,1\}] \cap (H_1^4/G)^{-1}(V_2)$ the map H_1^4/G is transversal to Q_r.

Hence by [BröJän], Theorem 14.7 there exist arbitrarily small C^1–perturbations H' of H_1^4/G on the set $(H_1^4/G)^{-1}(V_2)$ such that H' is transversal to Q_r, and H' coincides with H_1^4/G on a neighbourhood of the closed set $(H_1^4/G)^{-1}(V_2 \setminus V_1) \cup \{[((S^2)^4/G) \times \{0,1\}] \cap (H_1^4/G)^{-1}(V_2)\}$. Since the compact range of H_1 lies in the set $\{v \in \mathbb{R}^3 \setminus \{0\} \mid \varepsilon_1 < \|v\| < K\}$, therefore we may assume that $H'[(H_1^4/G)^{-1}(V_2)] \subset \{(v_1, \ldots, v_4) \in (\mathbb{R}^3 \setminus \{0\})^4 \mid \varepsilon_1 < \|v_i\| < K\}/G$. We can define H' also on $(H_1^4/G)^{-1}[((\mathbb{R}^3 \setminus \{0\})^4/G) \setminus (\overline{V_1})]$, as H_1^4/G. Then the last inclusion formula about the range of H' remains valid, in particular H' has a range in $(\mathbb{R}^3 \setminus \{0\})^4/G$. Thus we have defined H' on two open sets, coherently on their intersection, which will together give a perturbation H' of H_1^4/G, transversal to Q_r. In particular, $H'^{-1}(Q_r)$ is a manifold (here only $H'|((H_1^4/G)^{-1}(V_2))$ plays a role). Moreover, also $((S^2)^4/G \times \{0,1\}) \cap H'^{-1}(Q_r)$ is a manifold. Namely, Q_r is transversal to $f(S^2)^4/G$ and $g(S^2)^4/G$, i.e., $f^4/G, g^4/G : (S^2)^4/G \to (\mathbb{R}^3 \setminus \{0\})^4/G$ are transversal to Q_r, so $(f^4/G)^{-1}(Q_r), (g^4/G)^{-1}(Q_r) \subset (S^2)^4/G$ are manifolds. Further, $H'|((S^2)^4/G \times \{0\}) = (H_1^4/G)|((S^2)^4/G \times \{0\}) = f^4/G$ and $H'|((S^2)^4/G \times \{1\}) = g^4/G$.

Let us consider the restriction $H'|H'^{-1}(\overline{P_r})$. We assert that we may assume that $H'^{-1}(\overline{P_r})$ is a subset of the manifold with boundary $(M_s/G) \times [0,1]$, or in other words, $H'^{-1}\left(((\mathbb{R}^3)^4/G) \setminus \overline{P_r}\right) \supset (((S^2)^4 \setminus M_s)/G) \times [0,1]$, or in yet other words, $H'\{(((S^2)^4 \setminus M_s)/G) \times [0,1]\} \subset ((\mathbb{R}^3)^4/G) \setminus \overline{P_r}$. In fact, we have from above $(H_1^4/G)\{(((S^2)^4 \setminus M_s)/G) \times [0,1]\} \subset (\varphi^{-1}((7,\infty)) \cup S)/G)$ and also $\overline{P_r} = P_r \cup Q_r \subset \varphi^{-1}([0,5])/G$, hence $(\varphi^{-1}((5,\infty)) \cup S)/G \subset ((\mathbb{R}^3)^4/G) \setminus \overline{P_r}$. Now observe that $(\varphi^{-1}((5,\infty)) \cup S)/G$ is a neighbourhood of the closed set $(\varphi^{-1}([7,\infty)) \cup S)/G$ containing the open set $(\varphi^{-1}((7,\infty)) \cup S)/G$. The complement of this neighbourhood is a compact set, hence this neighbourhood contains even some metric η–neighbourhood of $(\varphi^{-1}([7,\infty)) \cup S)/G$. Thus a sufficiently small C^1–perturbation H' of H_1^4/G satisfies

$$H'\left\{\left(((S^2)^4 \setminus M_s)/G\right) \times [0,1]\right\} \subset \left(\varphi^{-1}((5,\infty)) \cup S\right)/G \subset ((\mathbb{R}^3)^4/G) \setminus \overline{P_r}.$$

Moreover, $H'^{-1}(\overline{P_r})$ $(\subset (M_s/G) \times [0,1])$ is itself a C^1–manifold with boundary. Namely, we consider $H'|((M_s/G) \times [0,1]) : (M_s/G) \times [0,1] \to (\mathbb{R}^3 \setminus \{0\})^4/G$. Here $(M_s/G) \times [0,1]$ is a manifold with boundary $(M_s/G) \times \{0,1\}$, $(\mathbb{R}^3 \setminus \{0\})^4/G$ is an orbifold, containing the manifold with boundary $\overline{P_r} \subset \varphi^{-1}([0,5])/G$, that is contained in the manifold $\varphi^{-1}([0,6))/G = \varphi^{-1}((0,6))/G$. Then $[H'|((M_s/G) \times [0,1])]^{-1}(\varphi^{-1}((0,6))/G)$ is an open subset of $(M_s/G) \times [0,1]$, hence is a manifold with boundary, and contains $[H'|((M_s/G) \times [0,1])]^{-1}(\overline{P_r}) = H'^{-1}(\overline{P_r})$. Now the restriction H'' of H' to $[H'|((M_s/G) \times [0,1])]^{-1}(\varphi^{-1}((0,6))/G)$ maps this manifold with boundary to the manifold $\varphi^{-1}((0,6))/G$. Then by [Hir], Ch. 1, Theorem 4.2, (ii) the inverse image by this map of the manifold with boundary $\overline{P_r}$, i.e., $H'^{-1}(\overline{P_r})$, is itself a manifold with boundary. In fact, for this we need that
(1) H'' is transversal to Q_r, and
(2) its restriction to the boundary of its domain, i.e., to $[H'|((M_s/G) \times \{0,1\})]^{-1}(\varphi^{-1}((0,6))/G)$ is also transversal to Q_r.
Here (1) follows, since H'' pointwise coincides with H', and H' is transversal to Q_r. Moreover (2) follows, since on $(M_s/G) \times \{0,1\}$ (even on $(S^2)^4/G \times \{0,1\}$) H'' and H' coincide with H_1^4/G, and from above H_1^4/G is transversal to Q_r at $t = 0$ and $t = 1$.

We make each calculation (boundary, homology, etc.) mod 2. Then we have

$$\partial(H'|H'^{-1}(\overline{P_r})) = H'|[((S^2)^4/G \times \{0,1\}) \cap H'^{-1}(\overline{P_r})] + H'|H'^{-1}(Q_r),$$

where $((S^2)^4/G \times \{0,1\}) \cap H'^{-1}(\overline{P_r}) \subset ((S^2)^4/G \times \{0,1\}) \cap ((M_s/G) \times [0,1]) = (M_s/G) \times \{0,1\}$, and $H'^{-1}(Q_r)$ is a manifold, from above. Further, $((S^2)^4/G \times \{0,1\}) \cap H'^{-1}(\overline{P_r})$ is a manifold with boundary. This can be seen analogously like above. We will show this for $((S^2)^4/G \times \{0\}) \cap H'^{-1}(\overline{P_r})$. (The other case is analogous.) Then $\overline{P_r}$ is a subset of the manifold $\varphi^{-1}((0,6))/G$, and we have to investigate the set $(f^4/G)^{-1}(\overline{P_r})$ $(\subset M_s/G)$. The restriction of f^4/G to M_s/G maps the manifold M_s/G to $(\mathbb{R}^3 \setminus \{0\})^4/G$. We consider the inverse image of $\varphi^{-1}((0,6))/G$ by this map, that is a manifold contained in M_s/G, and we further restrict the above restriction of f^4/G to it, obtaining a map to $\varphi^{-1}((0,6))/G$. By [Hir], Ch. 1, Theorem 4.2, (ii) we have that $(f^4/G)^{-1}(\overline{P_r})$ is a manifold with boundary, provided that our map is transversal to Q_r. However, our map pointwise coincides with f^4/G, so this follows from the choice of the manifold Q_r.

By the above paragraph the cycle

$$H'|[((S^2)^4/G \times \{0,1\}) \cap H'^{-1}(\overline{P_r})] + H'|H'^{-1}(Q_r) =$$
$$(f^4/G)|[(f^4/G)^{-1}(\overline{P_r})] + (g^4/G)|[(g^4/G)^{-1}(\overline{P_r})] +$$
$$H'|H'^{-1}(Q_r)$$

is homologous to 0.

Now we assert that the range of the map $H'|H'^{-1}(Q_r)$ is included in Q_{r1}. It will suffice to prove $H'((S^2)^4/G \times [0,1]) \cap (Q_r \setminus Q_{r1}) \subset (\{(v_1, ..., v_4) \mid \varepsilon_1 < \|v_i\| < K\}/G) \cap (Q_r \setminus Q_{r1}) = \emptyset$. Suppose that $(v_1, ..., v_4)/G$ belongs to the last set. Then $\max_i\{\varepsilon_1/\|v_i\|, \|v_i\|/K\} < 1$, and, by the definition of Q_{r1}, also $\max_{i<j}\{(\sqrt{2}\varepsilon_2)/\|v_i \pm v_j\|\} < 1$, hence $\varphi(v_1, ..., v_4) < 1$. However, $Q_r \setminus Q_{r1} \subset Q_r \subset \varphi^{-1}([1,5])/G$, so $\varphi(v_1, ..., v_4) \geq 1$, a contradiction.

Then by the above homological formula $(f^4/G)|[(f^4/G)^{-1}(\overline{P_r})]$ and $(g^4/G)|[(g^4/G)^{-1}(\overline{P_r})]$ are homologous, mod Q_{r1}. Now recall that $A'_r = [A/G] \cap \overline{P_r}$ is a relative cycle in $(\overline{P_r}, Q_{r2})$, and $B'_r = [f(S^2)^4/G] \cap \overline{P_r}$ is a relative cycle in $(\overline{P_r}, Q_{r1})$. Since the roles of f and g are symmetric, also $[g(S^2)^4/G] \cap \overline{P_r}$ is a relative cycle in $(\overline{P_r}, Q_{r1})$. By the above homologicity property the ranges of the injections $(f^4/G)|[(f^4/G)^{-1}(\overline{P_r})]$ and $(g^4/G)|[(g^4/G)^{-1}(\overline{P_r})]$, i.e., $[f(S^2)^4/G] \cap \overline{P_r}$ and $[g(S^2)^4/G] \cap \overline{P_r}$ will have the same intersection numbers (mod 2) with the relative cycle A'_r.

Therefore the intersection numbers of the relative cycle α realized by the set A'_r – which is a relative cycle in $(\overline{P_r}, Q_{r2})$ – with β_f, realized by the set B'_r, and β_g, realized analogously by the set $[g(S^2)^4/G] \cap \overline{P_r}$ – which in turn are relative cycles in $(\overline{P_r}, Q_{r1})$ – will be equal. □

(Alternatively one can give the same proof in the language of cohomologies by turning to the dual cohomology classes. Let $[\alpha]$ be the homology class of α in $H_*(\overline{P_r}, Q_{r2}; \mathbb{Z}_2)$, $[\beta_f] = [\beta_g]$ the homology class of β_f (and of β_g) in $H_*(\overline{P_r}, Q_{r1}; \mathbb{Z}_2)$. Their Poincaré duals will be denoted by

$$D[\alpha] \in H^*(\overline{P_r}, Q_{r1}; \mathbb{Z}_2) \text{ and } D[\beta_g] = D[\beta_f] \in H^*(\overline{P_r}, Q_{r2}; \mathbb{Z}_2).$$

Their product $D[\alpha] \cup D[\beta_g]$ can be considered as an element in $H^{\dim \overline{P_r}}(\overline{P_r}, \partial \overline{P_r}; \mathbb{Z}_2) = \mathbb{Z}_2$. Thus we have

$$\text{the number of cubes on } f(S^2) = D[\alpha \cap \beta_f] = D[\alpha] \cup D[\beta_f] =$$

$$D[\alpha] \cup D[\beta_g] = D[\alpha \cap \beta_g] = \text{ the number of cubes on } g(S^2) \neq 0.)$$

□

Actually the same arguments yield the following more general theorem (compare Theorem C of [Griff], cited at the beginning of this section).

Theorem 16 *Let f be as in Theorem 2. Then there is a square based box in $\mathbb{R}^3$, with its centre in the origin, and with any given ratio of the height to the basic edge, having all its vertices on the surface $f(S^2)$.*

Proof. We proceed identically as above, with the only difference that we apply Corollary 7 not for a cube, but use its consequence mentioned in the proof of Proposition 9, again observing that 3 is odd. □

Question. Can one generalize further Theorem 16, for any box, with any given ratios of edge-lengths (like in Theorem 15)? (Observe, that here Corollary 7 is of no use, since $3!/(1!)^3 = 6$ is even. Our proof fails for this case, because the cycles, whose intersection points give the cubes on the surface, were not oriented, i.e., integer cycles, because of the presence of orientation reversing elements in the group G.)

Remark. Our Theorem 2 generalizes Theorem 1 and Theorem 16 generalizes Theorem 10. In fact, for $\min F > 0$ we may choose the function $f(u)$ from Theorems 2 and 16 as $F(u) \cdot u$, where $F(u)$ is the function from Theorems 1 and 10. This shows Theorems 1 and 10 for even C^1 functions F (the condition $\min F > 0$ can evidently be suppressed). To obtain their statements for even continuous functions F, we use density of $C^1(S^2)$ in $C^0(S^2)$, and compactness of $SO(3)$.

6 Universal covers: proof of Theorem 3

As we have explained in Section 1, Pál in [Pál] proved that a regular hexagon with distance 1 between its opposite sides was a universal cover in $\mathbb{R}^2$. In this section we deal with Makeev's generalization of Pál's question.

Let $\Sigma^n \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, ..., v_{n+1}$. Let U_n be the intersection of $n(n+1)/2$ parallel strips S_{ij} $(1 \leq i < j \leq n+1)$ of width 1, where S_{ij} is bounded by the $(n-1)$-planes orthogonal to the segment $[v_i, v_j]$, and passing through v_i and v_j, respectively. For $n = 2$ we have that U_2 is the above mentioned regular hexagon. Makeev in [Mak4] proposes, as a generalization of Pál's result, the following

Conjecture 1 (Makeev) *U_n is a universal cover in $\mathbb{R}^n$.*

First we reformulate this conjecture, following [Mak1], to a problem concerning continuous functions.

Let $K \subset \mathbb{R}^n$ be a non-empty compact convex set, of diameter at most 1. One can define its support function $h : S^{n-1} \to \mathbb{R}$ as in [BoFe]. Let $A \in SO(n)$ and $1 \leq i < j \leq n+1$, and consider the parallel supporting $(n-1)$-planes

$$\{x \in \mathbb{R}^n \mid \langle x, A(v_j - v_i)\rangle = h(A(v_j - v_i))\}$$

and

$$\{x \in \mathbb{R}^n \mid \langle x, A(v_i - v_j)\rangle = h(A(v_i - v_j))\}$$

of K, whose distance is at most 1, because K has diameter at most 1. Consider also their mid-$(n-1)$-plane

$$\{x \in \mathbb{R}^n \mid \langle x, A(v_j - v_i)\rangle = [h(A(v_j - v_i)) - h(A(v_i - v_j))]/2\}.$$

Let these supporting $(n-1)$-planes bound the parallel strip S'_{ij}. Suppose that the above mid-$(n-1)$-planes are concurrent, with (unique) common point $x \in \mathbb{R}^n$. Then include each strip S'_{ij} to a strip of width 1, having the same mid-$(n-1)$-plane as S'_{ij}. These larger strips are of the form $AS_{ij} + x$. That is, $K \subset \cap S'_{ij} \subset A(\cap S_{ij}) + x = AU_n + x$, and thus U_n would be a universal cover. Therefore Makeev's conjecture would follow from

Conjecture 2 *Let $F : S^{n-1} \to \mathbb{R}$ be an odd function, and let $\Sigma_n \subset \mathbb{R}^n$ be a regular simplex of edge-length 1, with vertices $v_1, ..., v_{n+1}$. Then there exists an $A \in SO(n)$ such that the $n(n+1)/2$ $(n-1)$-planes*

$$\{x \in \mathbb{R}^n \mid \langle x, A(v_j - v_i)\rangle = F(A(v_j - v_i))\}\ (1 \leq i < j \leq n+1)$$

are concurrent.

We remark that [Mak1] has given this conjecture in a slightly different formulation.

Observe that F induces a map $\Phi : SO(n) \to \mathbb{R}^{n(n+1)/2}$, of coordinates $F(A(v_j - v_i))$, $1 \leq i < j \leq n+1$. We need $A \in SO(n)$, such that $\Phi(A)$ lies in the n-subspace

$$\begin{aligned}\{(\langle x, A(v_j - v_i)\rangle)_{1\leq i<j\leq n+1} \mid x \in \mathbb{R}^n\} &= \{(\langle A^* x, v_j - v_i\rangle)_{1\leq i<j\leq n+1} \mid x \in \mathbb{R}^n\} \\ &= \{(y, v_j - v_i\rangle)_{1\leq i<j\leq n+1} \mid y \in \mathbb{R}^n\}\end{aligned}$$

of $\mathbb{R}^{n(n+1)/2}$. Note that $\dim SO(n) = n(n+1)/2 - n$, hence the number of variables equals the number of constraints, and thus there is a chance to apply intersection theory.

Analogously as in §2, it would be sufficient to find a test odd function $F_0 : S^{n-1} \to \mathbb{R}$, for which we have a unique solution $A \in SO(n)$ (up to symmetries of the convex polyhedron U_n), and for which the intersection number at this unique solution is non-zero. A candidate for F_0 could be e.g. $(h_0(u) - h_0(-u))/2$, where h_0 is the support function of Σ^n, for which $A = I$ is a solution. (For $A = I$ the mid-$(n-1)$-planes are just the halving $(n-1)$-planes of the edges of Σ^n, all passing through the centre of Σ^n.) For $n = 2$ elementary considerations show unicity of the solution (up to symmetries of the regular hexagon), and in a natural sense the intersection is transversal there.

Though we could not complete the above program, for $n = 3$ we can deduce Makeev's conjecture from Proposition 4:

Theorem 17 *Let $F : S^2 \to \mathbb{R}$ be an odd function, and let $\Sigma_3 \subset \mathbb{R}^3$ be a regular simplex of edge-length 1 centred at the origin, with vertices v_1, v_2, v_3 and v_4. Then there exists an $A \in SO(3)$ such that the 6 planes*

$$\{x \in \mathbb{R}^3 \mid \langle x, A(v_j - v_i)\rangle = F(A(v_j - v_i))\} \ (1 \le i < j \le 4)$$

are concurrent.

Proof. Similarly as above F induces a map $\Phi : SO(3) \to \mathbb{R}^6$ given by coordinates $F(A(v_j - v_i))$, $1 \le i < j \le 4$. Let $\{e_{ij} \mid 1 \le i < j \le 4\}$ be the standard basis vectors of $\mathbb{R}^6$. By abuse of notation e_{ji} will stand for $-e_{ij}$ if $1 \le i < j \le 4$.

Let ρ_{S_4}, the action of S_4 on $SO(3)$ be given as right multiplication by the rotation group of U_3, the rhombic dodecahedron, which is easily seen to be isomorphic to the rotation group of the cube. Moreover define τ_6 to be the S_4 action on $\mathbb{R}^6$ given on the standard basis of $\mathbb{R}^6$ by the following rule: if $\sigma \in S_4$ is a permutation of the letters $1, 2, 3, 4$ then

$$\tau_6(\sigma)(e_{ij}) = sign(\sigma) e_{\sigma(i)\sigma(j)}.$$

The construction of the map Φ and the oddness of F imply that Φ has to be an S_4 equivariant map from $(SO(3), \rho)$ to $(\mathbb{R}^6, \tau_6)$.

As we have seen above, to prove the theorem we need $A \in SO(3)$ such that $\Phi(A)$ lies in the 3-subspace

$$\begin{aligned} V &:= \{(\langle x, A(v_j - v_i)\rangle)_{1 \le i < j \le 4} \mid x \in \mathbb{R}^3\} \\ &= \{(\langle A^* x, v_j - v_i\rangle)_{1 \le i < j \le 4} \mid x \in \mathbb{R}^3\} = \{(\langle y, v_j - v_i\rangle)_{1 \le i < j \le 4} \mid y \in \mathbb{R}^3\} \end{aligned}$$

of $\mathbb{R}^6$.

We prove more, namely that for any S_4 equivariant map $\Phi : (SO(3), \rho) \to (\mathbb{R}^6, \tau_6)$ there exists an $A \in SO(3)$ such that $\Phi(A) \in V$.

The 3-space V is spanned by any three of the four vectors $e_{12}+e_{13}+e_{14}, e_{21}+e_{23}+e_{24}, e_{31}+e_{32}+e_{34}, e_{41}+e_{42}+e_{43}$, as we get these when $y=-v_1,-v_2,-v_3,-v_4$ respectively. Note that V is an invariant subspace of $(\mathbb{R}^6,\tau_6)$. Let the 3-subspace W of $\mathbb{R}^6$ be spanned by any three of the four vectors $e_{23}+e_{34}+e_{42}, e_{31}+e_{14}+e_{43}, e_{12}+e_{24}+e_{41}, e_{21}+e_{13}+e_{32}$. Now one checks that W is the orthogonal complement of V with respect to the standard Euclidean scalar product of $\mathbb{R}^6$. As the action τ_6 preserves this standard scalar product, it follows that W, being the orthogonal complement of an invariant subspace V, is itself invariant under the action τ_6. Let τ_W denote the S_4 action $\tau_6|_W$ on W. Now finding $A\in SO(3)$ with $\Phi(A)\in V$ is equivalent to showing that the S_4 equivariant map $pr_W\Phi : (SO(3),\rho)\to(W,\tau_W)$ vanishes somewhere. By Proposition 4 this is the case as (W,τ_W) is isomorphic to $(\mathbb{R}^3,\tau)$, which is the action where S_4 acts as the symmetry group of the regular tetrahedron. This last statement can be seen by checking that τ_W faithfully permutes the four vectors $e_{23}+e_{34}+e_{42}, e_{31}+e_{14}+e_{43}, e_{12}+e_{24}+e_{41}, e_{21}+e_{13}+e_{32}$, which form a regular tetrahedron in W.

The result follows. □

Proof of Theorem 3. As we have explained above, Theorem 17 implies Makeev's conjecture for $n=3$, i.e., that U_3 is a universal cover in $\mathbb{R}^3$. Moreover U_3 is the intersection of the 6 strips corresponding to the 6 edges of a regular tetrahedron of edge length 1, which is a rhombic dodecahedron with distance of opposite faces equal to 1. □

A frequent application of universal covers is in the so-called Borsuk problem [Bor]: if $X\subset\mathbb{R}^n$ has diameter 1, can it be decomposed into $n+1$ sets $X_1,\dots,X_{n+1}$ of smaller diameters?

We note that for all sufficiently large n Borsuk's problem has a negative solution [KhKa] – even for finite sets X –, but the smallest n, for which a counterexample is known, is $n=561$, cf. [BoMaSo] pp.209-226, and [AiZi], pp. 83-88.

However for low dimensions the problem has an affirmative solution.

For $n=2$ the sharp answer is that one can even guarantee $\operatorname{diam}(X_i)\le\sqrt{3}/2$ ([Gale]). The proof goes by applying Pál's theorem to cover X with a regular hexagon U_2 with distance of opposite sides 1 and then cut U_2 into three congruent pentagons with diameter $\sqrt{3}/2$.

For $n=3$ the positive answer has been proved first by [Eggl2] and then [Hepp] and [Grün]. Heppes and Grünbaum used for their proof a universal cover in $\mathbb{R}^3$, the regular octahedron O_3, with distance of opposite faces equal to 1, then chopped off three vertices, still obtaining a universal cover. Then they decomposed this last set to four parts of diameters 0.998 and 0.989, respectively.

For a nice exposition of the story of Borsuk's problem cf. [BoMaSo], pp. 209-226.

The universal cover U_3, given by Theorem 3, is intuitively smaller than O_3 (e.g., $\mathrm{diam}(O_3) = \sqrt{3}$, $\mathrm{diam}(U_3) = \sqrt{2}$), but on the other hand is combinatorially more complex; so it is conceivable that with more amount of work one could substantially reduce the upper bound for $\mathrm{diam}(X_i)$. As we have been informed recently, [Mak5] already observed this point, moreover referred to work of A. Evdokimov, who in this way sharpened the results of Heppes and Grünbaum, to 0.98.

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