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Abelianization for hyperkähler quotients

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Abstract

We study an integration theory in circle equivariant cohomology in order to prove a theorem relating the cohomology ring of a hyperkähler quotient to the cohomology ring of the quotient by a maximal abelian subgroup, analogous to a theorem of Martin for symplectic quotients. We discuss applications of this theorem to quiver varieties, and compute as an example the ordinary and equivariant cohomology rings of a hyperpolygon space.

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Let X be a symplectic manifold equipped with a hamiltonian action of a compact Lie group G. Let $T \subseteq G$ be a maximal torus, let $\Delta \subset \mathfrak{t}^*$ be the set of roots of G, and let $W = N(T)/T$ be the Weyl group. Suppose that $0 \in \mathfrak{g}^*$ and $0 \in \mathfrak{t}^*$ are regular values for the two moment maps. If the symplectic quotients $X/\!/G$ and $X/\!/T$ are both compact, Martin's theorem [21, Theorem A] relates the cohomology[1] of $X/\!/G$ to the cohomology of $X/\!/T$. Specifically, it says that

$$H^*(X/\!/G) \cong \frac{H^*(X/\!/T)^W}{ann(e_0)},$$

where

$$e_0 = \prod_{\alpha \in \Delta} \alpha \in (\mathrm{Sym}\, \mathfrak{t}^*)^W \cong H_T^*(pt)^W,$$

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[1] In this paper cohomology means cohomology with rational coefficients.

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which acts naturally on $H^*(X/\!/T)^W \cong H_T^*(\mu_T^{-1}(0))^W$. In the case where X is a complex vector space and G acts linearly on X, a similar result was obtained by Ellingsrud and Strømme [4] using different techniques.

Our goal is to state and prove an analog of this theorem for hyperkähler quotients [12]. There are two main obstacles to this goal. First, hyperkähler quotients are rarely compact. The assumption of compactness in Martin's theorem is crucial because his proof involves integration. Generalizing an idea of Moore et al. [22] and Paradan [25], our answer to this problem is to work with equivariant cohomology of *circle compact* manifolds, by which we mean oriented manifolds with an action of S^1 such that the fixed point set is oriented and compact. By the localization theorem of Atiyah–Bott [1] and Berline–Vergne [2], integration in rationalized S^1-equivariant cohomology of circle compact manifolds can be defined in terms of integration on their fixed point sets. Section 1 is devoted to making this statement precise by defining a well-behaved push forward in the rationalized S^1-equivariant cohomology of circle compact manifolds.

The second obstacle is that Martin's result uses surjectivity [13] of the Kirwan map from $H_G^*(X)$ to $H^*(X/\!/G)$. The analogous map for circle compact hyperkähler quotients is surjective only conjecturally. Our approach is to assume that the rationalized Kirwan map is surjective, which is equivalent to saying that the cokernel of the nonrationalized Kirwan map

$$\kappa_G : H^*_{S^1\times G}(X) \to H^*_{S^1}(X/\!/\!/\!/G)$$

is torsion as a module over $H^*_{S^1}(pt)$. This is a weaker assumption than surjectivity of K_G; in particular, we show in Section 3 that this assumption holds for quiver varieties, as a consequence of the work of Nakajima.

Under this assumption, Theorem 2.3 computes the rationalized equivariant cohomology of $X/\!/\!/\!/G$ in terms of that of $X/\!/\!/\!/T$. We show that, at regular values of the hyperkähler moment maps,

$$\widehat{H}^*_{S^1}(X/\!/\!/\!/G) \cong \frac{\widehat{H}^*_{S^1}(X/\!/\!/\!/T)^W}{ann(e)},$$

where

$$e = \prod_{\alpha\in\Delta} \alpha(x-\alpha) \in (\mathrm{Sym}\,\mathfrak{t}^*)^W \otimes \mathbb{Q}[x] \cong H^*_{S^1\times T}(pt)^W.$$

Theorem 2.4 describes the image of the nonrationalized Kirwan map in a similar way

$$H^*_{S^1}(X/\!/\!/\!/G) \supseteq \mathrm{Im}\,(\kappa_G) \cong \frac{(\mathrm{Im}\,\kappa_T)^W}{ann(e)},$$

where $\kappa_T : H^*_{S^1\times T}(X) \to H^*_{S^1}(X/\!/\!/\!/T)$ is the Kirwan map for the abelian quotient. In many situations, such as when $X = T^*\mathbb{C}^n$, κ_T is known to be surjective.

In Section 3, we show that all of the hypotheses of Theorems 2.3 and 2.4 are satisfied for Nakajima's quiver varieties. This way we can reduce questions about the (rationalized) equivariant cohomology of quiver varieties to questions about the (rationalized) equivariant cohomology of toric hyperkähler varieties (also called hypertoric varieties in [6]). The cohomology rings of toric hyperkähler varieties are well understood, as in [3,6,8,15]. When the hyperkähler Kirwan map is known to be surjective, for example in the case of the Hilbert scheme of points on an ALE space, Theorem 2.4 gives an explicit description of the cohomology ring of the quiver variety. Such cases are discussed in Remarks 3.3 and 4.3.

We conclude in Section 4 by demonstrating how the ideas of the present paper work in the case of a particular quiver variety, the so-called hyperpolygon space. We show that the hyperkähler Kirwan map is surjective, and therefore our machinery reproduces, by different means, the results of Konno [16, Section 7] and Harada and Proudfoot [7, Section 3].

1. Integration

The localization theorem of Atiyah–Bott [1] and Berline–Vergne [2] says that given a manifold M with a circle action, the restriction map from the circle equivariant cohomology of M to the circle equivariant cohomology of the fixed point set F is an isomorphism modulo torsion. In particular, integrals on a compact M can be computed in terms of integrals on F. If F is compact, it is possible to use the Atiyah–Bott–Berline–Vergne formula to *define* integrals on M.

We will work in the category of *circle compact* manifolds, by which we mean oriented S^1-manifolds with compact and oriented fixed point sets. Maps between circle compact manifolds are required to be equivariant.

Definition 1.1. Let $\mathbb{K} = \mathbb{Q}(x)$, the rational function field of $H^*_{S^1}(pt) \cong \mathbb{Q}[x]$. For a circle compact manifold M, let $\widehat{H}^*_{S^1}(M) = H^*_{S^1}(M) \otimes \mathbb{K}$, where the tensor product is taken over the ring $H^*_{S^1}(pt)$. We call $\widehat{H}^*_{S^1}(M)$ the *rationalized* S^1-equivariant cohomology of M. Note that because $\deg(x) = 2$, $\widehat{H}^*_{S^1}(M)$ is supergraded, and supercommutative with respect to this supergrading.

An immediate consequence of Atiyah and Bott [1] is that restriction gives an isomorphism

$$\widehat{H}^*_{S^1}(M) \cong \widehat{H}^*_{S^1}(F) \cong H^*(F) \otimes_{\mathbb{Q}} \mathbb{K}, \tag{1}$$

where $F = M^{S^1}$ denotes the compact fixed point set of M. In particular $\widehat{H}^*_{S^1}(M)$ is a finite-dimensional vector space over $\mathbb{K}$, and trivial if and only if F is empty.

Let $i: N \hookrightarrow M$ be a closed embedding. There is a standard notion of proper pushforward

$$i_*: H^*_{S^1}(N) \to H^*_{S^1}(M)$$

given by the formula $i_* = r \circ \Phi$, where $r: H^*_{S^1}(M, M \setminus N) \to H^*_{S^1}(M)$ is the restriction map, and $\Phi: H^*_{S^1}(N) \to H^*_{S^1}(M, M \setminus N)$ is the Thom isomorphism. We will also denote the induced map $\widehat{H}^*_{S^1}(N) \to \widehat{H}^*_{S^1}(M)$ by i_*. Geometrically, i_* can be understood as the inclusion of cycles in Borel–Moore homology.

This map satisfies two important formal properties [1]:

$$\textit{Functoriality}: \quad (i \circ j)_* = i_* \circ j_*, \tag{2}$$

$$\textit{Module homomorphism}: \quad i_*(\gamma \cdot i^*\alpha) = i_*\gamma \cdot \alpha \ \text{ for all } \alpha \in \widehat{H}^*_{S^1}(M), \gamma \in \widehat{H}^*_{S^1}(N). \tag{3}$$

We will denote the Euler class $i^* i_*(1) \in \widehat{H}^*_{S^1}(N)$ by $e(N)$. If a class $\gamma \in \widehat{H}^*_{S^1}(N)$ is in the image of i^*, then property (3) tells us that $i^* i_* \gamma = e(N)\gamma$. Since the pushforward construction is local in a neighborhood of N in M, we may assume that i^* is surjective, hence this identity holds for all $\gamma \in \widehat{H}^*_{S^1}(N)$.

Let $F = M^{S^1}$ be the fixed point set of M. Since M and F are each oriented, so is the normal bundle to F inside of M. The following result is standard, see e.g. [13].

Lemma 1.2. *The Euler class $e(F) \in \widehat{H}^*_{S^1}(F)$ of the normal bundle to F in M is invertible.*

Proof. Let $\{F_1, \ldots, F_d\}$ be the connected components of F. Since $\widehat{H}^*_{S^1}(F) \cong \oplus \widehat{H}^*_{S^1}(F_i)$ and $e(F) = \oplus e(F_i)$, our statement is equivalent to showing that $e(F_i)$ is invertible for all i. Since S^1 acts trivially on F_i, $\widehat{H}^*_{S^1}(F_i) \cong H^*(F_i) \otimes_{\mathbb{Q}} \mathbb{K}$. We have $e(F_i) = 1 \otimes ax^k + nil$, where $k = \mathrm{codim}(F_i)$, a is the product of the weights of the S^1 action on any fiber of the normal bundle, and nil consists of terms of positive degree in $H^*(F_i)$. Since F_i is a component of the fixed point set, S^1 acts freely on the complement of the zero section of the normal bundle, therefore $a \neq 0$. Since ax^k is invertible and nil is nilpotent, we are done. □

Definition 1.3. For $\alpha \in \widehat{H}^*_{S^1}(M)$, let

$$\int_M \alpha = \int_F \frac{\alpha|_F}{e(F)} \in \mathbb{K}.$$

Note that this definition does not depend on our choice of orientation of F. Indeed, reversing the orientation of F has the effect of negating $e(F)$, *and* introducing a second factor of -1 coming from the change in fundamental class. These two effects cancel.

For this definition to be satisfactory, we must be able to prove the following lemma, which is standard in the setting of ordinary cohomology of compact manifolds.

Lemma 1.4. *Let $i : N \hookrightarrow M$ be a closed immersion. Then for any $\alpha \in \widehat{H}^*_{S^1}(M), \gamma \in \widehat{H}^*_{S^1}(N)$, we have $\int_M \alpha \cdot i_*\gamma = \int_N i^*\alpha \cdot \gamma$.*

Proof. Let $G = N^{S^1}$, let $j : G \to F$ denote the restriction of i to G, and let $\phi : F \to M$ and $\psi : G \to N$ denote the inclusions of F and G into M and N, respectively.

$$\begin{array}{ccc} N & \xrightarrow{\ i\ } & M \\ {\scriptstyle\psi}\big\uparrow & & \big\uparrow{\scriptstyle\phi} \\ G & \xrightarrow{\ j\ } & F \end{array}$$

Then

$$\int_M \alpha \cdot i_*\gamma = \int_F \frac{\phi^*\alpha \cdot \phi^* i_*\gamma}{e(F)}$$

and

$$\int_N i^*\alpha \cdot \gamma = \int_G \frac{\psi^* i^*\alpha \cdot \psi^*\gamma}{e(G)} = \int_G \frac{j^*\phi^*\alpha \cdot \psi^*\gamma}{e(G)} = \int_F \phi^*\alpha \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right),$$

where the last equality is simply the integration formula applied to the map $j : G \to F$ of compact manifolds [1]. Hence it will be sufficient to prove that

$$\phi^* i_*\gamma = e(F) \cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right) \in \widehat{H}^*_{S^1}(F).$$

To do this, we will show that the difference of the two classes lies in the kernel of ϕ_*, which we know is trivial because the composition $\phi^*\phi_*$ is given by multiplication by the invertible class $e(F) \in \widehat{H}^*_{S^1}(F)$. On the left-hand side we get

$$\phi_*\phi^* i_*\gamma = \phi_*(1) \cdot i_*\gamma \quad \text{by (3)}$$

and on the right-hand side we get

$$\begin{aligned}\phi_*\left(e(F)\cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right)\right) &= \phi_*\left(\phi^*\phi_*(1)\cdot j_*\left(\frac{\psi^*\gamma}{e(G)}\right)\right)\\ &= \phi_*(1)\cdot\phi_* j_*\left(\frac{\psi^*\gamma}{e(G)}\right) \quad \text{by (3)}\\ &= \phi_*(1)\cdot i_*\psi_*\left(\frac{\psi^*\gamma}{e(G)}\right) \quad \text{by (2).}\end{aligned}$$

It thus remains only to show that $\gamma = \psi_*(\psi^*\gamma/e(G))$. This is seen by applying ψ^* to both sides, which is an isomorphism (working over the field $\mathbb{K}$) by Atiyah and Bott [1]. □

For $\alpha_1, \alpha_2 \in \widehat{H}^*_{S^1}(M)$, consider the symmetric, bilinear, $\mathbb{K}$-valued pairing

$$\langle \alpha_1, \alpha_2 \rangle_M = \int_M \alpha_1\alpha_2.$$

Lemma 1.5 (Poincaré Duality). *This pairing is nondegenerate.*

Proof. Suppose that $\alpha \in \widehat{H}^*_{S^1}(M)$ is nonzero, and therefore $\phi^*\alpha \neq 0$. Since F is compact, there must exist a class $\gamma \in \widehat{H}^*_{S^1}(F)$ such that $0 \neq \int_F \phi^*\alpha\cdot\gamma = \int_M \alpha\cdot\phi_*\gamma = \langle \alpha, \phi_*\gamma\rangle_M$. □

Definition 1.6. For an arbitrary equivariant map $f : N \to M$, we may now define the pushforward

$$f_* : \widehat{H}^*_{S^1}(N) \to \widehat{H}^*_{S^1}(M)$$

to be the adjoint of f^* with respect to the pairings $\langle\cdot,\cdot\rangle_N$ and $\langle\cdot,\cdot\rangle_M$. This is well defined because, according to (1), $\widehat{H}^*_{S^1}(M)$ and $\widehat{H}^*_{S^1}(N)$ are finite-dimensional vector spaces over the field $\mathbb{K}$. Lemma 1.4 tells us that this definition generalizes the definition for closed immersions. Furthermore, properties (2) and (3) for pushforwards along arbitrary maps are immediate corollaries of the definition. If f is a projection, then f_* will be given by integration along the fibers. Using the fact that every map factors through its graph as a closed immersion and a projection, we always have a geometric interpretation of the pushforward.

As an application, let us consider the manifold $M \times M$, along with the two projections π_1 and π_2, and the diagonal map $\Delta : M \to M \times M$. Suppose that we can write

$$\Delta_*(1) = \sum \pi_1^* a_i \cdot \pi_2^* b_i$$

for a finite collection of classes $a_i, b_i \in \widehat{H}^*_{S^1}(M)$. The following proposition will be used in Section 3.

Proposition 1.7. *The set* $\{b_i\}$ *is an additive spanning set for* $\widehat{H}^*_{S^1}(M)$.

Proof. For any $\alpha \in \widehat{H}^*_{S^1}(M)$, we have

$$\begin{aligned}\alpha &= \mathrm{id}_*\mathrm{id}^*\alpha \\ &= (\pi_2 \circ \Delta)_*(\pi_1 \circ \Delta)^*\alpha \\ &= \pi_{2*}(\Delta_*(1 \cdot \Delta^*\pi_1^*\alpha)) \\ &= \pi_{2*}(\pi_1^*\alpha \cdot \Delta_*(1)) \\ &= \pi_{2*}\left(\sum \pi_1^*(a_i\alpha) \cdot \pi_2^* b_i\right) \\ &= \sum \pi_{2*}\pi_1^*(a_i\alpha) \cdot b_i \\ &= \sum \langle a_i, \alpha\rangle \cdot b_i,\end{aligned}$$

hence α is in the span of $\{b_i\}$. □

2. An analog of Martin's theorem

Let X be a hyperkähler manifold with a circle action, and suppose that a compact Lie group G acts hyperhamiltonianly on X. We will assume that the circle action preserves a given complex structure I. Having chosen a particular complex structure on X, we may write the hyperkähler moment map in the form

$$\mu_G = \mu_{\mathbb{R}} \oplus \mu_{\mathbb{C}} : X \to \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}},$$

where $\mu_{\mathbb{C}}$ is holomorphic with respect to I [6]. We require that the action of G commute with the action of S^1, that $\mu_{\mathbb{R}}$ is S^1-invariant, and that $\mu_{\mathbb{C}}$ is S^1-equivariant with respect to the action of S^1 on $\mathfrak{g}^*_{\mathbb{C}}$ by complex multiplication. We do *not* ask the action of S^1 on X to preserve the hyperkähler structure.

Let $T \subseteq G$ be a maximal torus, and let $pr : \mathfrak{g}^* \to \mathfrak{t}^*$ be the natural projection. Then T acts on X with hyperkähler moment map

$$\mu_T = pr \circ \mu_{\mathbb{R}} \oplus pr_{\mathbb{C}} \circ \mu_{\mathbb{C}} : X \to \mathfrak{t}^* \oplus \mathfrak{t}^*_{\mathbb{C}}.$$

Let $\xi \in \mathfrak{g}^*$ be a central element such that $(\xi, 0)$ is a regular value of μ_G and $(pr(\xi), 0)$ is a regular value of μ_T. Assume further that G acts freely on $\mu_G^{-1}(\xi, 0)$, and T acts freely on $\mu_T^{-1}(pr(\xi), 0)$.[2] Let

$$X////G = \mu_G^{-1}(\xi, 0)/G \quad \text{and} \quad X////T = \mu_T^{-1}(pr(\xi), 0)/T$$

be the hyperkähler quotients of X by G and T, respectively. Because μ_G and μ_T are circle equivariant, the action of S^1 on X descends to actions on the hyperkähler quotients. Note that $X////T$ also inherits an action of the Weyl group W of G.

[2] We make this simplifying assumption in order to talk about manifolds, rather than orbifolds. In fact, Theorems 2.3 and 2.4 generalize easily to the orbifold case, as in [21, Section 6].

Example 2.1. Suppose that G acts linearly on $\mathbb{C}^n$ with moment map $\mu:\mathbb{C}^n \to \mathfrak{g}^*$, and let X be the hyperkähler manifold $T^*\mathbb{C}^n \cong \mathbb{H}^n$. The action of G on $\mathbb{C}^n$ induces an action of G on X with hyperkähler moment map

$$\mu_{\mathbb{R}}(z,w) = \mu(z) - \mu(w) \quad \text{and} \quad \mu_{\mathbb{C}}(z,w)(v) = w(\hat{v}_z),$$

where $w \in T_z^*\mathbb{C}^n \cong \mathbb{C}^n$, $v \in \mathfrak{g}_{\mathbb{C}}^*$, and $\hat{v}_z$ the element of $T_z\mathbb{C}^n$ induced by v [6]. The action of G commutes with the action of S^1 on X given by scalar multiplication on each fiber, and the hyperkähler moment map is equivariant. The quotient $X////G$ is a partial compactification of the cotangent bundle $T^*(X//G)$, and is circle compact if μ is proper [6, 1.3].

Consider the Kirwan maps

$$\kappa_G : H^*_{S^1\times G}(X) \to H^*_{S^1}(X////G) \quad \text{and} \quad \kappa_T : H^*_{S^1\times T}(X) \to H^*_{S^1}(X////T)$$

induced by the inclusions of $\mu_G^{-1}(\xi,0)$ and $\mu_T^{-1}(pr(\xi),0)$ into X, along with their rationalizations

$$\hat{\kappa}_G : \widehat{H}^*_{S^1\times G}(X) \to \widehat{H}^*_{S^1}(X////G) \quad \text{and} \quad \hat{\kappa}_T : \widehat{H}^*_{S^1\times T}(X) \to \widehat{H}^*_{S^1}(X////T).$$

Let

$$r_T^G : \widehat{H}^*_{S^1\times G}(X) \to \widehat{H}^*_{S^1\times T}(X)^W$$

be the standard isomorphism.

Let $\Delta = \Delta^+ \sqcup \Delta^- \subset \mathfrak{t}^*$ be the set of roots of G. Let

$$e = \prod_{\alpha\in\Delta} \alpha(x-\alpha) \in (\operatorname{Sym}\mathfrak{t}^*)^W \otimes \mathbb{Q}[x] \cong H_{S^1\times G}(pt) \subseteq \widehat{H}_{S^1\times G}(pt)$$

and

$$e' = \prod_{\alpha\in\Delta^-} \alpha \cdot \prod_{\alpha\in\Delta}(x-\alpha) \in \operatorname{Sym}\mathfrak{t}^* \otimes \mathbb{Q}[x] \cong H_{S^1\times T}(pt) \subseteq \widehat{H}_{S^1\times T}(pt).$$

The following two theorems are analog of Theorems B and A of Martin [21], adapted to circle compact hyperkähler quotients. Our proofs follow closely those of Martin.

Theorem 2.2. *Suppose that $X////G$ and $X////T$ are both circle compact. If $\gamma \in \widehat{H}^*_{S^1\times G}(X)$, then*

$$\int_{X////G} \hat{\kappa}_G(\gamma) = \frac{1}{|W|}\int_{X////T} \hat{\kappa}_T \circ r_T^G(\gamma)\cdot e.$$

Theorem 2.3. *Suppose that $X////G$ and $X////T$ are both circle compact, and that the rationalized Kirwan map $\hat{\kappa}_G$ is surjective. Then*

$$\widehat{H}^*_{S^1}(X////G) \cong \frac{\widehat{H}^*_{S^1}(X////T)^W}{ann(e)} \cong \left(\frac{\widehat{H}^*_{S^1}(X////T)}{ann(e')}\right)^W.$$

Proof of Theorem 2.2. Consider the following pair of maps:

$$\begin{array}{ccc} \mu_G^{-1}(\xi,0)/T & \stackrel{i}{\to} & \mu_T^{-1}(pr(\xi),0)/T \cong X////T \\ \pi\downarrow & & \\ \mu_G^{-1}(\xi,0)/G \cong X////G. & & \end{array}$$

Each of these spaces is a complex S^1-manifold with a compact, complex fixed point set, and therefore satisfies the hypotheses of Section 1. Let

$$b = \prod_{\alpha\in\Delta^+} \alpha \in H_{S^1\times T}(pt)$$

be the product of the positive roots of G, which we will think of as an element of $\widehat{H}^*_{S^1}(X////T)$. Martin shows that $\pi_* i^* b = |W|$, and that $i^*\circ\hat{\kappa}_T\circ r_T^G = \pi^*\hat{\kappa}_G$ [21], hence we have

$$\begin{aligned} \int_{X////G} \hat{\kappa}_G(\gamma) &= \frac{1}{|W|}\int_{X////G} \hat{\kappa}_G(\gamma)\cdot \pi_* i^* b \\ &= \frac{1}{|W|}\int_{\mu_G^{-1}(\xi,0)/T} \pi^*\hat{\kappa}_G(\gamma)\cdot i^* b \quad \text{by Definition 1.6} \\ &= \frac{1}{|W|}\int_{\mu_G^{-1}(\xi,0)/T} i^*\circ\hat{\kappa}_T\circ r_T^G(\gamma)\cdot i^* b \\ &= \frac{1}{|W|}\int_{X////T} \hat{\kappa}_T\circ r_T^G(\gamma)\cdot b\cdot i_*(1) \quad \text{by Lemma 1.4.} \end{aligned}$$

It remains to compute $i_*(1)\in\widehat{H}^*_{S^1}(X////T)$. For $\alpha\in\Delta$, let

$$L_\alpha = \mu_T^{-1}((pr(\xi),0)\times_T \mathbb{C}_\alpha$$

be the line bundle on $X////T$ with S^1-equivariant Euler class α. Similarly, let L_x be the (topologically trivial) line bundle with S^1-equivariant Euler class x. Following the idea of Martin [21, 1.2.1], we observe that the restriction of $\mu_G-(\xi,0)$ to $\mu_T^{-1}(pr(\xi),0)$ defines an $S^1\times T$-equivariant map

$$s:\mu_T^{-1}(pr(\xi),0)\to V\oplus V_{\mathbb{C}},$$

where $V=pr^{-1}(0)$ and $V_{\mathbb{C}}=pr_{\mathbb{C}}^{-1}(0)$. This descends to an S^1-equivariant section of the bundle $E=\mu_T^{-1}(pr(\xi),0)\times_T(V\oplus V_{\mathbb{C}})$ with zero locus $\mu_G^{-1}(\xi,0)/T$. The fact that $(\xi,0)$ is a regular value implies that this section is generic, hence the equivariant Euler class $e(E)\in\widehat{H}^*_{S^1}(X////T)$ is equal to $i_*(1)$.

The vector space V is isomorphic as a T-representation to $\bigoplus_{\alpha\in\Delta^-}\mathbb{C}_\alpha$, with S^1 acting trivially. Similarly, $V_{\mathbb{C}}$ is isomorphic to $V\otimes\mathbb{C}\cong V\oplus V^*$, with S^1 acting diagonally by scalars. Hence

$$\begin{aligned} E &\cong \bigoplus_{\alpha\in\Delta^-} L_\alpha \oplus \bigoplus_{\alpha\in\Delta^-}(L_x\otimes L_\alpha)\oplus(L_x\otimes L_{-\alpha}) \\ &\cong \bigoplus_{\alpha\in\Delta^-} L_\alpha \oplus \bigoplus_{\alpha\in\Delta} L_x\otimes L_{-\alpha} \end{aligned}$$

and therefore

$$i_*(1) = e(E) = \prod_{\alpha \in \Delta^-} \alpha \cdot \prod_{\alpha \in \Delta} (x - \alpha) = e'.$$

Multiplying by b we obtain e, and the theorem is proved. □

Proof of Theorem 2.3. Observe that the restriction of π^* to the Weyl-invariant part $\widehat{H}^*_{S^1}(\mu_G^{-1}(\xi,0)/T)^W$ is given by the composition of isomorphisms

$$\widehat{H}^*_{S^1}(\mu_G^{-1}(\xi,0)/T)^W \cong \widehat{H}^*_{S^1 \times T}(\mu_G^{-1}(\xi,0))^W \cong \widehat{H}^*_{S^1 \times G}(\mu_G^{-1}(\xi,0)) \cong \widehat{H}^*_{S^1}(X////G),$$

hence we may define

$$i_W^* := (\pi^*)^{-1} \circ i^* : \widehat{H}^*_{S^1}(X////T)^W \to \widehat{H}^*_{S^1}(\mu_G^{-1}(\xi,0)/T)^W.$$

Furthermore, we have $\hat{\kappa}_G = i_W^* \circ \hat{\kappa}_T \circ r_T^G$, hence i_W^* is surjective. As in [21, Section 3],

$$\begin{aligned}
i_W^*(a) = 0 &\Leftrightarrow \forall c \in \widehat{H}^*_{S^1}(X////T)^W, \int_{X////G} i_W^*(c) \cdot i_W^*(a) = 0 \quad \text{by 1.5 and surjectivity of } i_W^* \\
&\Leftrightarrow \forall c \in \widehat{H}^*_{S^1}(X////T)^W, \int_{X////T} c \cdot a \cdot e = 0 \quad \text{by Theorem 2.2} \\
&\Leftrightarrow \forall d \in \widehat{H}^*_{S^1}(X////T), \int_{X////T} d \cdot a \cdot e = 0 \quad \text{by using } W \text{ to average } d \\
&\Leftrightarrow a \cdot e = 0 \quad \text{by Lemma 1.5},
\end{aligned}$$

hence $\ker i_W^* = ann(e)$. By surjectivity of i_W^*,

$$\widehat{H}^*_{S^1}(X////G) \cong \frac{\widehat{H}^*_{S^1}(X////T)^W}{\ker i_W^*} \cong \frac{\widehat{H}^*_{S^1}(X////T)^W}{ann(e)}.$$

By a second application of Lemma 1.5, for any $a \in \widehat{H}^*_{S^1}(X////T)$, we have

$$\begin{aligned}
i^*(a) = 0 &\Rightarrow \forall f \in \widehat{H}^*_{S^1}(\mu_G^{-1}(\xi,0)/T), \int_{\mu_G^{-1}(\xi,0)/T} f \cdot i^*(a) = 0 \\
&\Rightarrow \forall c \in \widehat{H}^*_{S^1}(X////T), \int_{\mu_G^{-1}(\xi,0)/T} i^*(c) \cdot i^*(a) = 0 \\
&\Rightarrow \forall c \in \widehat{H}^*_{S^1}(X////T), \int_{X////T} c \cdot a \cdot i_*(1) = 0 \quad \text{by Lemma 1.4} \\
&\Rightarrow a \cdot e' = a \cdot i_*(1) = 0 \quad \text{by Lemma 1.5},
\end{aligned}$$

hence $\ker i^* \subseteq ann(e')$. This gives us a natural surjection

$$\frac{\widehat{H}^*_{S^1}(X////T)^W}{ann(e)} = \frac{\widehat{H}^*_{S^1}(X////T)^W}{\ker i_W^*} \cong \left(\frac{\widehat{H}^*_{S^1}(X////T)}{\ker i^*}\right)^W \to \left(\frac{\widehat{H}^*_{S^1}(X////T)}{ann(e')}\right)^W,$$

which is also injective because e' divides e. This completes the proof of Theorem 2.3. □

For the nonrationalized version of Theorem 2.3, we make the additional assumption that $X////G$ and $X////T$ are *equivariantly formal* S^1-manifolds, i.e. that $H^*_{S^1}(X////G)$ and $H^*_{S^1}(X////T)$ are free modules over $H^*_{S^1}(pt)$. This is the case whenever the circle action is hamiltonian and its moment map is proper and bounded below (see [13,6, 4.7]).

Theorem 2.4. *Suppose that $X////G$ and $X////T$ are equivariantly formal, circle compact, and that the rationalized Kirwan map $\hat\kappa_G$ is surjective. Then*

$$H^*_{S^1}(X////G) \supseteq \operatorname{Im}(\kappa_G) \cong \frac{(\operatorname{Im}\kappa_T)^W}{ann(e)} \cong \left(\frac{\operatorname{Im}\kappa_T}{ann(e')}\right)^W.$$

Remark 2.5. In the context of Example 2.1 with $pr \circ \mu$ proper, $X////G$ and $X////T$ are both circle compact and equivariantly formal, and κ_T is always surjective [6]. Note that this applies throughout Sections 3 and 4.

Proof of Theorem 2.4. Consider the following exact commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & H^*_{S^1}(X////T)^W & \xrightarrow{i^*_W} & H^*_{S^1}(X////G) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{H}^*_{S^1}(X////T)^W & \xrightarrow{i^*_W} & \widehat{H}^*_{S^1}(X////G). \end{array}$$

Equivariant formality implies that the downward maps in the above diagram are inclusions, hence the map on top labeled i^*_W is simply the restriction of the map on the bottom to the subring $H^*_{S^1}(X////T) \subseteq \widehat{H}^*_{S^1}(X////T)$. We therefore have

$$A = \widehat{A} \cap H^*_{S^1}(X////T)^W = ann(e).$$

Just as in the rationalized case, we have $\kappa_G = i^*_W \circ \kappa_T \circ r^G_T$, hence

$$\operatorname{Im}(\kappa_G) \cong i^*_W(\operatorname{Im}\kappa_T \circ r^G_T) \cong \frac{(\operatorname{Im}\kappa_T)^W}{ann(e)}.$$

Now consider the analogous diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & H^*_{S^1}(X////T) & \xrightarrow{i^*} & H^*_{S^1}(\mu_G^{-1}(\xi,0)/T) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widehat{B} & \longrightarrow & \widehat{H}^*_{S^1}(X////T) & \xrightarrow{i^*} & H^*_{S^1}(\mu_G^{-1}(\xi,0)/T). \end{array}$$

Since we have not assumed that $\mu_G^{-1}(\xi,0)/T$ is equivariantly formal, we only know that the first two downward arrows are inclusions, and hence can only conclude that B is contained in the annihilator of e'. Since e' divides e, we have a series of natural surjections

$$\frac{(\operatorname{Im}\kappa_T)^W}{ann(e)} \cong \frac{(\operatorname{Im}\kappa_T)^W}{A} \cong \left(\frac{\operatorname{Im}\kappa_T}{B}\right)^W \to \left(\frac{\operatorname{Im}\kappa_T}{ann(e')}\right)^W \to \left(\frac{\operatorname{Im}\kappa_T}{ann(e)}\right)^W.$$

The composition of these maps is an isomorphism, hence so is each one. □

3. Quiver varieties

Let Q be a quiver with vertex set I and edge set $E \subseteq I \times I$, where $(i,j) \in E$ means that Q has an arrow pointing from i to j. We assume that Q is connected and has no oriented cycles. Suppose given two collections of vector spaces $\{V_i\}$ and $\{W_i\}$, each indexed by I, and consider the affine space

$$A = \bigoplus_{(i,j)\in E} \mathrm{Hom}(V_i, V_j) \oplus \bigoplus_{i\in I} \mathrm{Hom}(V_i, W_i).$$

The group $U(V) = \prod_{i\in I} U(V_i)$ acts on A by conjugation, and this action is hamiltonian. Given an element

$$(B,J) = \bigoplus_{(i,j)\in E} B_{ij} \oplus \bigoplus_{i\in I} J_i$$

of A, the $\mathfrak{u}(V_i)^*$ component of the moment map is

$$\mu_i(B,J) = J_i^\dagger J_i + \sum_{(i,j)\in E} B_{ij}^\dagger B_{ij},$$

where $\dagger$ denotes adjoint, and $\mathfrak{u}(V_i)^*$ is identified with the set of hermitian matrices via the trace form. Given a generic central element $\xi \in \mathfrak{u}(V)^*$, the Kähler quotient $A/\!/_\xi U(V)$ parameterizes stable, framed representations of Q of fixed dimension [23]. If $W_i = 0$ for all i, then the diagonal circle $U(1)$ in the center of $U(V)$ acts trivially, and we instead quotient by $PU(V) = U(V)/U(1)$.

Consider the hyperkähler quotient

$$\mathfrak{M} = T^*A/\!/\!/\!/_{(\xi,0)} U(V).$$

As in Example 2.1, $\mathfrak{M}$ has a natural circle action induced from scalar multiplication on the fibers of T^*A. We now show that $X = T^*A$ satisfies the hypotheses of Theorems 2.3 and 2.4.

Proposition 3.1. *Let $T(V) \subseteq U(V)$ be a maximal torus, and let $pr : \mathfrak{u}(V)^* \to \mathfrak{t}(V)^*$ be the natural projection. The moment maps $\mu = \oplus_{i\in I} \mu_i : A \to \mathfrak{u}(V)^*$ and $pr \circ \mu : A \to \mathfrak{t}(V)^*$ are each proper.*

Proof. To show that μ and $pr \circ \mu$ is proper, it suffices to find an element $t \in T(V) \subseteq U(V)$ such that the weights of the action of t on A are all strictly positive. Let $\lambda = \{\lambda_i \mid i \in I\}$ be a collection of integers, and let $t \in T(V)$ be the central element of $U(V)$ that acts on V_i with weight λ_i for all i. Then t acts on $\mathrm{Hom}(V_i, V_j)$ with weight $\lambda_j - \lambda_i$, and on $\mathrm{Hom}(V_i, W_i)$ with weight $-\lambda_i$. Hence we have reduced the problem to showing that it is possible to choose λ such that $\lambda_i < 0$ for all $i \in I$ and $\lambda_i < \lambda_j$ for all $(i,j) \in E$.

We proceed by induction on the order of I. Since Q has no oriented cycles, there must exist a source $i \in I$; a vertex such that for all $j \in I$, $(j,i) \notin E$. Deleting i gives a smaller (possibly disconnected) quiver with no oriented cycles, and therefore we may choose $\{\lambda_j \mid j \in I \setminus \{i\}\}$ such that $\lambda_j < 0$ for all $j \in I \setminus \{i\}$ and $\lambda_j < \lambda_k$ for all $(j,k) \in E$. We then choose $\lambda_i < \min\{\lambda_j \mid j \in I \setminus \{i\}\}$, and we are done. □

Proposition 3.2. *The rationalized Kirwan map $\hat{\kappa}_{U(V)} : \widehat{H}^*_{S^1 \times U(V)}(T^*A) \to \widehat{H}^*_{S^1}(\mathfrak{M})$ is surjective.*

Proof. Nakajima [24, Section 7.3] shows that there exist cohomology classes a_i, b_i in the image of $\hat{\kappa}_{U(V)}$ such that $\Delta_*(1) = \sum \pi_1^* a_i \cdot \pi_2^* b_i$. (Nakajima uses a slightly modified circle action, but his proof is easily adapted to the circle action that we have defined.) It follows from Proposition 1.7 that the classes $\{b_i\}$ generate $\widehat{H}^*_{S^1}(\mathfrak{M})$. □

Remark 3.3. This proposition shows that the assumptions of Theorems 2.2–2.4 are satisfied for Nakajima's quiver varieties. Thus integration in equivariant cohomology yields a description of the rationalized S^1-equivariant cohomology, and also of the image of the nonrationalized Kirwan map κ_G. Therefore if we know that κ_G is surjective for a particular quiver variety, then we have a concrete description of the (S^1-equivariant) cohomology ring of that quiver variety. It is known that κ_G is surjective for Hilbert schemes of n points on an ALE space, so our theory applies and gives a description of the cohomology ring of these quiver varieties. It would be interesting to compare our result in this case with that of Lehn and Sorger [17] and Li et al. [19]. More examples of quiver varieties with surjective Kirwan map are given in Remark 4.3.

Remark 3.4. Another interesting application of Proposition 1.7 is for the moduli space of Higgs bundles. It is an easy exercise to write down the cohomology class of the diagonal in $\mathcal{M} \times \mathcal{M}$ as an expression in the tautological classes for the equivariantly formal and circle compact moduli space $\mathcal{M}$ of stable rank n and degree 1 Higgs bundles on a genus $g > 1$ smooth projective algebraic curve C. Therefore Proposition 1.7 implies that the rationalized S^1-equivariant cohomology ring $\widehat{H}^*_{S^1}(\mathcal{M})$ is generated by tautological classes. In fact the same result follows from the argument of Hausel and Thaddeus [10]. There $\mathcal{M}$ was embedded into a circle compact manifold $\mathcal{M}_\infty$, whose cohomology is the free algebra on the tautological classes. The argument in [10] then goes by showing that the embedding of the S^1-fixed point set of $\mathcal{M}$ in that of $\mathcal{M}_\infty$ induces a surjection on cohomology. This already implies that $\widehat{H}^*_{S^1}(\mathcal{M}_\infty)$ surjects onto $\widehat{H}^*_{S^1}(\mathcal{M})$. In [10] it is shown that in the Rank 2 case this embedding also implies the surjection on nonrationalized cohomology, and then a companion paper [9] describes the cohomology ring of $\mathcal{M}$ explicitly. However for higher rank this part of the argument of [10] breaks down. Later Markman [20] used similar diagonal arguments on certain compactifications of $\mathcal{M}$ and Hironaka's celebrated theorem on desingularization of algebraic varieties to deduce that the cohomology ring of $\mathcal{M}$ is generated by tautological classes for all n.

Example 3.5. Here, we present an example of an embedding of circle compact manifolds, due to Thaddeus [26], where surjection on rationalized S^1-equivariant cohomology does not imply surjection on S^1-equivariant cohomology. Let S^1 act on $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$((x:y),(u:v)) \mapsto ((\lambda x:y),(u:v))$$

and on $\mathbb{P}^3$ by

$$(z_1:z_2:z_3:z_4) \mapsto (\lambda z_1:\lambda z_2:z_3:z_4).$$

Then the Segré embedding $i:\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by

$$i((x:y),(u:v)) = (xu:xv:yu:yv)$$

is S^1-equivariant, and clearly induces an isomorphism on the fixed point sets of the S^1 action. Therefore $i^*:\widehat{H}^*_{S^1}(\mathbb{P}^3) \to \widehat{H}^*_{S^1}(\mathbb{P}^1 \times \mathbb{P}^1)$ is surjective, and in fact an isomorphism, however $i^*:H^*_{S^1}(\mathbb{P}^3) \to H^*_{S^1}(\mathbb{P}^1 \times \mathbb{P}^1)$ is only an injection and therefore cannot be surjective.

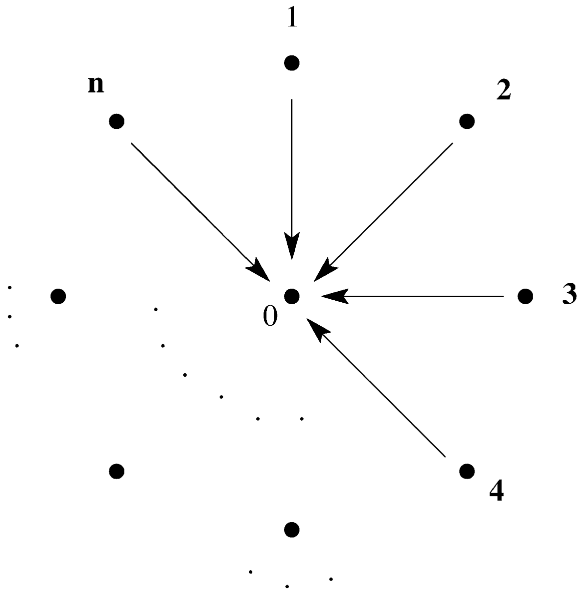

Fig. 1. The quiver for a hyperpolygon space.

4. Hyperpolygon spaces

We conclude by illustrating Theorem 2.4 with a computation of the equivariant cohomology ring of a hyperpolygon space. Proposition 4.4 first appeared in [7], and Corollary 4.5 in [16], both obtained by geometric arguments completely different from those used here.

A hyperpolygon space, introduced in [16], is a quiver variety associated to the following quiver (Fig. 1), with $V_0 = \mathbb{C}^2$, $V_i = \mathbb{C}^1$ for $i \in \{1, \ldots, n\}$, and $W_i = 0$ for all i. It is so named because, for

$$\xi = \left(-\frac{1}{2} \sum_{i=1}^{n} \xi_i; \xi_1, \ldots, \xi_n \right) \in \mathfrak{pu}(V)^* \subseteq \mathfrak{u}(2)^* \oplus \mathfrak{u}(1)^n,$$

the Kähler quotient $A/\!/_{\xi} PU(V) \cong (\mathbb{C}^2)^n /\!/_{\xi} PU(V)$ parameterizes n-sided polygons in $\mathbb{R}^3$ with edge lengths $\xi_1, \ldots, \xi_n$, up to rotation [11].

We will simplify our computations by dividing first by the torus $\prod_{i=1}^{n} U(V_i)$. We have

$$\begin{aligned} \mathfrak{M} &= (T^*\mathbb{C}^2)^n /\!/\!/\!/ PU(V) \\ &\cong \left((T^*\mathbb{C}^2)^n /\!/\!/\!/ \prod_{i=1}^{n} U(V_i) \right) /\!/\!/\!/ SU(2) \\ &\cong \prod_{i=1}^{n} T^*\mathbb{C}P^1 /\!/\!/\!/ SU(2), \end{aligned}$$

where the action of $SU(2)$ on each copy of $T^*\mathbb{C}P^1$ is induced by the rotation action on $\mathbb{C}P^1 \cong S^2$.

Proposition 4.1. *The nonrationalized Kirwan map* $\kappa_{U(V)}: H^*_{S^1\times U(V)}(T^*\mathbb{C}^{2n}) \to H^*_{S^1}(\mathfrak{M})$ *is surjective.*

Proof. The map $\kappa_{U(V)}$ factors as a composition

$$H^*_{S^1\times U(V)}(T^*\mathbb{C}^{2n}) \to H^*_{S^1\times SU(2)}\left(\prod_{i=1}^n T^*\mathbb{C}P^1\right) \stackrel{\kappa_{SU(2)}}{\to} H^*_{S^1}(\mathfrak{M}),$$

where the first map is the Kirwan map for a toric hyperkähler variety, and therefore surjective by Harada and Proudfoot [6]. Hence it suffices to show that $\kappa_{SU(2)}$ is surjective.

The level set $\mu_{\mathbb{C}}^{-1}(0)$ for the action of $SU(2)$ on $\prod_{i=1}^n T^*\mathbb{C}P^1$ is a subbundle of the cotangent bundle, given by requiring the n cotangent vectors to add to zero after being restricted to the diagonal $\mathbb{C}P^1$. In particular this set is smooth, and its $S^1\times SU(2)$-equivariant cohomology ring is canonically isomorphic to that of $\prod_{i=1}^n T^*\mathbb{C}P^1$. Hence $\kappa_{SU(2)}$ factors as

$$H^*_{S^1\times SU(2)}\left(\prod_{i=1}^n T^*\mathbb{C}P^1\right) \cong H^*_{S^1\times SU(2)}(\mu_{\mathbb{C}}^{-1}(0)) \to H^*_{S^1}(\mu_{\mathbb{C}}^{-1}(0)/\!/SU(2)) \cong H^*_{S^1}(\mathfrak{M}),$$

where the map in the middle is the Kähler Kirwan map. Surjectivity of this map follows from the following more general lemma, applied to the manifold $\mu_{\mathbb{C}}^{-1}(0)$.

Lemma 4.2. *Suppose given a hamiltonian action of* $S^1\times G$ *on a symplectic manifold* X, *such that the* S^1 *component of the moment map is proper and bounded below with finitely many critical values. Then the Kähler Kirwan map* $\kappa: H^*_{S^1\times G}(X) \to H^*_{S^1}(X/\!/G)$ *is surjective.*

Proof. Extend the action of S^1 to an action on $X\times\mathbb{C}$ by letting S^1 act only on the left-hand factor. On the other hand, consider a second copy of the circle, which we will call $\mathbb{T}$ to avoid confusion, acting diagonally on $X\times\mathbb{C}$. Choose $r\in \mathrm{Lie}(\mathbb{T})^*\cong\mathbb{R}$ greater than the largest critical value of the $\mathbb{T}$-moment map, and consider the space

$$Cut(X/\!/G) := (X\times\mathbb{C})/\!/_r\mathbb{T}\times G \cong ((X/\!/G)\times\mathbb{C})/\!/_r\mathbb{T}.$$

This space, which is called the *symplectic cut* of $X/\!/G$ [18], is an S^1-equivariant (orbifold) compactification of $X/\!/G$. We then have a commutative diagram

$$\begin{array}{ccc} H^*_{S^1\times G\times\mathbb{T}}(X\times\mathbb{C}) & \longrightarrow & H^*_{S^1\times G}(X) \\ \downarrow & & \downarrow \kappa \\ H^*_{S^1}(Cut(X/\!/G)) & \longrightarrow & H^*_{S^1}(X/\!/G). \end{array}$$

The vertical map on the left is surjective because the $G\times\mathbb{T}$ moment map is proper, and the map on the bottom is surjective because the long exact sequence in cohomology for $X/\!/G\subseteq Cut(X/\!/G)$ splits naturally, hence κ is surjective as well. □

By applying Lemma 4.2 to $X=\mu_{\mathbb{C}}^{-1}(0)$, this completes the proof of Proposition 4.1. □

Remark 4.3. The argument in Proposition 4.1 generalizes immediately to show that the hyperkähler Kirwan map for the quotient

$$\left(\prod_{i=1}^{n} T^*Flag(\mathbb{C}^k)\right) //// \; SU(k)$$

is surjective. This is itself a quiver variety, and like the hyperpolygon space, it has a moduli theoretic interpretation. The Kähler quotient

$$\left(\prod_{i=1}^{n} Flag(\mathbb{C}^k)\right) // \; SU(k)$$

is isomorphic to the space of n-tuples of $k \times k$ hermitian matrices with fixed eigenvalues adding to zero, modulo conjugation. This space has been studied by many authors. The classical problem, due to Horn, of determining the values of the moment map for which it is nonempty, has only recently been solved [14]. For a survey, see [5].

To compute the kernel of the hyperkähler Kirwan map for the hyperpolygon space, we first need to study the abelian quotient

$$\mathfrak{N} := \prod_{i=1}^{n} T^*\mathbb{C}P^1 //// \; T,$$

where $T \cong U(1) \subseteq SU(2)$ is a maximal torus. The space $\prod_{i=1}^{n} T^*\mathbb{C}P^1$ is a toric hyperkähler manifold [3], given by an arrangement of $2n$ hyperplanes in $\mathbb{R}^n$, where the $(2i-1)$st and $(2i)$th hyperplanes are given by the equations $x_i = \pm\xi_i$. Taking the hyperkähler quotient by T corresponds on the level of arrangements to restricting this arrangement to the hyperplane $\{x \in \mathbb{R}^n \mid \sum x_i = 0\}$.

Call a subset $S \subseteq \{1, \ldots, n\}$ *short* if $\sum_{i \in S} \xi_i < \sum_{j \in S^c} \xi_j$. Requiring that ξ is a regular value of the hyperkähler moment map is equivalent to requiring that for every $S \subseteq \{1, \ldots, n\}$, either S or S^c is short [16]. Applying [6, 4.5], we have

$$H^*_{S^1}(\mathfrak{N}) \cong \mathbb{Q}[a_1, b_1, \ldots, a_n, b_n, \alpha, x]/\langle a_i - b_i - \alpha,\; a_i b_i \mid i \leqslant n\rangle + \langle A_S, B_S \mid S \text{ short}\rangle,$$

where

$$A_S = \prod_{i \in S}(x - a_i) \prod_{j \in S^c} b_j \quad \text{and} \quad B_S = \prod_{i \in S}(x - b_i) \prod_{j \in S^c} a_j.$$

Here α is the image in $H^*_{S^1}(\mathfrak{N})$ of the unique positive root of $SU(2)$. The Weyl group W of $SU(2)$, isomorphic to $\mathbb{Z}/2\mathbb{Z}$, acts on this ring by fixing x and switching a_i and b_i for all i. Let $c_i = a_i + b_i$, and let $C_S = A_S + B_S$. Let

$$P = \mathbb{Q}[c_1, \ldots, c_n, \alpha, x]/\langle c_i^2 - \alpha^2 \mid i \leqslant n\rangle$$

and

$$Q = P^W = \mathbb{Q}[c_1, \ldots, c_n, \alpha^2, x]/\langle c_i^2 - \alpha^2 \mid i \leqslant n\rangle.$$

Let

$$\mathscr{I} = \langle A_S, B_S \mid S \text{ short}\rangle \subseteq P \quad \text{and} \quad \mathscr{J} = \langle C_S \mid S \text{ short}\rangle \subseteq Q,$$

so that

$$H^*_{S^1}(\mathfrak{N}) \cong P/\mathscr{I} \quad \text{and} \quad H^*_{S^1}(\mathfrak{N})^W \cong Q/\mathscr{J}.$$

Note that all odd powers of α in the expression for $C_S = A_S + B_S$ cancel out.

Then by Theorem 2.4 and Remark 2.5,

$$H^*_{S^1}(\mathfrak{M}) \cong \frac{H^*_{S^1}(\mathfrak{N})^W}{ann(e)} \cong \frac{Q}{(e:\mathscr{J})},$$

where $e = \alpha^2(x^2 - \alpha^2)$, and $(e:\mathscr{J})$ is the ideal of elements of Q whose product with e lies in $\mathscr{J}$.

If S is a nonempty short subset, let m_S be the smallest element of S, n_S the smallest element of S^c, and

$$D_S = \prod_{m_S \neq i \in S} (c_i - x) \cdot \prod_{n_S \neq j \in S^c} (c_{n_S} + c_j) \in Q.$$

Proposition 4.4. *The equivariant cohomology ring $H^*_{S^1}(\mathfrak{M})$ is isomorphic to*[3]

$$Q/\langle D_S \mid \emptyset \neq S \text{ short}\rangle.$$

Proof. We begin by proving that $e \cdot D_S \in \mathscr{J}$ for all nonempty short subsets $S \subseteq \{1,\ldots,n\}$. We will in fact prove the slightly stronger statement

$$e \cdot D_S \in \langle C_T \mid T \subseteq S \text{ short}\rangle \subseteq \mathscr{J},$$

proceeding by induction on $|S|$. We will assume, without loss of generality, that $n \in S$. The base case occurs when $S = \{n\}$, and in this case we observe that

$$e \cdot D_S = 2^{n-3} \cdot (x + c_n) \cdot ((2x - c_n) \cdot C_\emptyset - c_n \cdot C_S).$$

We now proceed to the inductive step, assuming that the proposition is proved for all short subsets of size less than $|S|$, and all values of n. For all $T \subseteq S \setminus \{n\}$, we have

$$\frac{1}{2}(C_T - C_{T\cup\{n\}}) = (c_n - x) \cdot C'_T,$$

where C'_T is the polynomial in the variables $\{c_1,\ldots,c_{n-1},\alpha^2\}$ corresponding to the short subset $T \subseteq \{1,\ldots,n-1\}$. Since $S \setminus \{n\}$ is a short subset of $\{1,\ldots,n-1\}$ of size strictly smaller than S, our inductive hypothesis tells us that $e \cdot D_S/(c_n - x)$ can be written as a linear combination of polynomials C'_T, where the coefficients are quadratic polynomials in $\{c_1,\ldots,c_{n-1},\alpha^2\}$. Replacing C'_T with $\frac{1}{2}(C_T - C_{T\cup\{n\}}) = (c_n - x)\cdot C'_T$, we have expressed $e \cdot D_S$ in terms of the appropriate polynomials. This completes the induction.

Suppose that $F \in Q$ is an element of degree less than $n-2$ such that $e \cdot F \in \mathscr{J}$. By the second isomorphism of Theorem 2.4, this implies that $e' \cdot F \in \mathscr{I} \subseteq P$, where $e' = \alpha(x^2 - \alpha^2)$. Consider the quotient ring R of P obtained by setting $a_i^2 = b_i^2 = x = 0$ for all i. (Recall that $a_i = \frac{1}{2}(c_i + \alpha)$ and $b_i = \frac{1}{2}(c_i - \alpha)$.) Then element e' maps to zero in R, while the generators $\{A_S, B_S\}$ of $\mathscr{I}$ descend to a basis for the nth degree part of R. This means that we must have $e' \cdot F = 0 \in P$. Using the additive

[3] The class denoted by c_i in Harada and Proudfoot [7] differs from our c_i by a sign, hence to recover the presentation of [7] we must replace $c_i - x$ with $c_i + x$ in the expression for D_S.

basis for P consisting of monomials that are squarefree in the variables $c_1,\ldots,c_n$, it is easy to check that e' is not a zero divisor in P, and therefore that $F=0$.

Finally, we must show that $\{D_S \mid \emptyset \neq S \text{ short}\}$ generates all elements of $(e:\mathscr{J})$ of degree at least $n-2$. Let F be an element of minimal degree $k \geq n-2$ that is in $(e:\mathscr{J})$ but not $\langle D_S \mid \emptyset \neq S \text{ short}\rangle$. In the proof of Harada and Proudfoot [7, 3.2] it is shown that $\{D_S \mid \emptyset \neq S \text{ short}\}$ descends to a basis for the degree $n-2$ part of the quotient ring $Q/\langle x\rangle$, hence F differs from an element of $\langle D_S \mid \emptyset \neq S \text{ short}\rangle$ by $x\cdot F'$ for some F' of degree $k-1$. By equivariant formality of $H^*_{S^1}(\mathfrak{M})$,

$$x\cdot F' = F \in (e:\mathscr{J}) \;\Rightarrow\; F' \in (e:\mathscr{J}),$$

which contradicts the minimality of $k=\deg F$. Hence $\langle D_S \mid \emptyset \neq S \text{ short}\rangle = (e:\mathscr{J})$, and the proposition is proved. □

Corollary 4.5. *The ordinary cohomology ring $H^*(\mathfrak{M})$ is isomorphic to*

$$\mathbb{Q}[c_1,\ldots,c_n]/\langle c_i^2 - c_j^2 \mid i,j \leqslant n\rangle + \langle \text{ all monomials of degree } n-2\rangle.$$

Proof. This follows from the fact that $H^*(M) \cong H^*_{S^1}(M)/\langle x\rangle$ for any equivariantly formal space M, and the observation in [7] that $\{D_S \mid \emptyset \neq S \text{ short}\}$ descends to a basis for the degree $n-2$ part of $Q/\langle x\rangle$. □

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