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GENERATORS FOR THE COHOMOLOGY RING OF THE MODULI SPACE OF RANK 2 HIGGS BUNDLES

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A central object of study in gauge theory is the moduli space of unitary flat connections on a compact surface. Thanks to the efforts of many people, a great deal is understood about the ring structure of its cohomology. In particular, the ring is known to be generated by the so-called *universal classes* [**1**, **32**], and, in rank 2, all the relations between these classes are also known [**2**, **23**, **36**, **48**].

If instead of just unitary connections one allows *all* flat connections, one obtains larger moduli spaces of equal importance and interest. However, these spaces are not compact and so very little was known about the ring structure of their cohomology.

This paper will show that, in the rank 2 case, the cohomology ring of this non-compact space is again generated by universal classes. A companion paper [**19**] gives a complete set of explicit relations between these generators.

The non-compact spaces studied here have significance extending well beyond gauge theory. They play an important role in 3-manifold topology: see for example the work of Culler and Shalen [**6**]. And they are the setting for much of the geometric Langlands program: see for example the work of Beilinson and Drinfeld [**3**]. But they have received perhaps the most attention from algebraic geometers, in the guise of moduli spaces of *Higgs bundles*. A Higgs bundle is a holomorphic object, related to a flat connection by a correspondence theorem similar to that of Narasimhan and Seshadri in the unitary case. This point of view has been exploited to great effect, notably by Hitchin [**20–22**] and Simpson [**37–40**].

The Higgs point of view predominates in this paper also. Indeed, it is strongly influenced by, and occasionally parallel to, the works of Hitchin [**20**] and Atiyah and Bott [**1**]. It is true that the moduli spaces of Higgs bundles and of flat connections carry different complex structures, but thanks to the correspondence theorem, they are diffeomorphic, and hence interchangeable topologically. What is important for our purposes is that the Higgs moduli space carries a holomorphic action of the complex torus $\mathbb{C}^\times$. This allows the topology of the moduli space, in some sense, to be determined from that of the fixed-point set.

Most of the results in this paper are valid for bundles of arbitrary rank. We state them this way in the hope that they may hold independent interest. We especially have in mind the construction in § 9 of the classifying space of the gauge group as a direct limit of spaces of Higgs bundles.

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We originally believed (and claimed in an early version of this paper) that the entire argument extended to arbitrary rank. However, a key lemma, (10.5), is definitely false for rank greater than 2. Nevertheless, the main theorem, (1.1), turns out to hold true for rank greater than 2: this was established in a paper of Markman [**28**] not long after this paper was originally written.

Outline of the paper. Section 1 gives statements of the main results, cast in terms of flat connections. Section 2 reviews the correspondence theorem relating flat connections to Higgs bundles. The next two sections give some necessary background on Higgs bundles: § 3 is about their deformation theory, while § 4 is about the existence and uniqueness of universal families. Section 5 then shows how the statements of § 1 will follow from the corresponding (and more general) statements for Higgs bundles. From then on Higgs bundles are used exclusively.

Section 6 re-states the main theorem, on the generation of the rational cohomology ring by universal classes, in terms of Higgs bundles. The proof occupies the remainder of the paper. It consists of four sections. Sections 7 and 8 describe how families of Higgs bundles can be decomposed into strata where the Harder–Narasimhan filtrations have fixed type. Section 7 is about finite-dimensional algebraic families and is largely parallel to the work of Shatz [**35**]. Section 8 transfers this stratification to the setting of an infinite-dimensional space acted on by the gauge group: a story familiar to readers of Atiyah and Bott [**1**].

In § 9, however, our approach takes a different turn. The moduli spaces under examination are nested in one another, and their direct limit is shown to have the homotopy type of the classifying space of the gauge group. This implies that the cohomology of the direct limit is indeed generated by universal classes. It now suffices to show that this surjects onto the cohomology of the original moduli spaces. This is proved in § 10 by algebro-geometric methods. It is here that the rank 2 hypothesis becomes necessary.

Notation and conventions. Throughout the paper, C denotes the compact surface, or smooth projective curve, of genus g over which we work. Its cohomology has the usual generators $e_1, \ldots, e_{2g} \in H^1$, and $\sigma = e_j e_{j+g} \in H^2$. The Jacobian of degree d line bundles on C is denoted $\mathrm{Jac}^d C$; if $d = 0$, we write simply $\mathrm{Jac}\, C$. Likewise, the dth symmetric product of C is denoted $\mathrm{Sym}^d C$. The letters $\mathcal{H}$ and $\mathcal{M}$ denote moduli spaces over C, respectively, of $\mathrm{GL}(r, \mathbb{C})$-connections of central constant curvature on a bundle of rank r and degree d, and of connections in $\mathcal{H}$ with fixed determinant. The expressions $\mathcal{H}_n$ and $\mathcal{M}_n$ denote moduli spaces over C, respectively, of Higgs bundles of rank r and degree d with values in $K(n) = K \otimes \mathcal{O}(np)$, and of Higgs bundles $(E, \phi) \in \mathcal{M}_n$ having $\Lambda^r E$ isomorphic to a fixed line bundle Ξ and $\mathrm{tr}\, \phi = 0$. Groups are denoted $T = \mathbb{C}^\times$, $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$, and $\Sigma = \mathbb{Z}_2^{2g}$.

All cohomology is with rational coefficients unless otherwise stated. Our methods do not obviously produce any information on the integral cohomology of the moduli spaces.

We do *not* assume $g \geqslant 2$: the moduli spaces $\mathcal{M}_0$ and $\mathcal{H}_0$ are trivial or empty if $g = 0$ or 1, but $\mathcal{M}_n$ and $\mathcal{H}_n$ for $n > 0$ are not so trivial, and they play an important role even for understanding $g \geqslant 2$.

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1. *Statement of results in terms of flat connections*

In one respect the summary given in the introduction is slightly inaccurate. The space we study is not exactly the moduli space of *flat* connections on a compact surface. For that space is generally singular due to the presence of reducible connections. This problem is circumvented by shifting attention to connections of constant central curvature whose degrees are coprime to their ranks. The bait-and-switch is perhaps regrettable, but it is standard practice in the subject.

So let C be a surface of genus g. The fundamental group of C has presentation

$$\pi_1(C) = \frac{\langle a_1, b_1, \ldots, a_g, b_g \rangle}{\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}}.$$

Let G be the non-compact group $\mathrm{GL}(r, \mathbb{C})$. A flat G-connection on C is determined by a representation $\pi_1(C) \to G$. So if $\mu : G^{2g} \to G$ is defined by

$$\mu(A_1, B_1, \ldots, A_g, B_g) = \prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1},$$

then any element of $\mu^{-1}(I)$ defines a flat G-connection. The quotient $\mu^{-1}(I)/G$, where G acts by conjugation on all factors, therefore parametrizes all flat G-connections modulo gauge equivalence. However, as mentioned above, this is generally a singular space. An exception is when $r = 1$, but then it is nothing but the complex torus $\mathcal{T} = (\mathbb{C}^\times)^{2g}$.

Instead of struggling with the singularities, choose any integer d coprime to r, and consider the space $\mathcal{H} = \mu^{-1}(e^{2\pi i d/r} I)/G$. This is a non-compact complex manifold, indeed, a smooth affine variety [**11**, **13**], and will be our main object of study. It parametrizes gauge equivalence classes of G-connections on C of constant central curvature $di\omega I$, where ω is a 2-form on C chosen so that $\int_C \omega = 2\pi/r$, and I is the $r \times r$ identity matrix. Indeed, such a connection is again determined by its holonomies (A_j, B_j) around a_j and b_j, subject to the constraint that

$$\prod_{j=1}^{g} A_j B_j A_j^{-1} B_j^{-1} = \exp \int_C di\omega I = e^{2\pi d i/r} I.$$

Alternatively, $\mathcal{H}$ may be regarded as the space of flat G-connections on $C \setminus p$ having holonomy $e^{2\pi d i/r} I$ around p, modulo gauge. This has the advantage that no choice of ω is necessary, but it is less compatible with the Higgs bundle interpretation coming up.

The subject of this paper is the ring structure of the rational cohomology of $\mathcal{H}$. To begin this study, some sources of cohomology classes on $\mathcal{H}$ are needed.

The simplest thing to do is pull back the generators of $H^*(\mathcal{T})$ by the obvious determinant map $\det : \mathcal{H} \to \mathcal{T}$. This produces classes $\varepsilon_1, \dots, \varepsilon_{2g} \in H^1(\mathcal{H})$, but they are not very interesting. In fact, the subring they generate can be split off in the following way.

Let $\mathcal{M}$ be the space constructed in the same way as $\mathcal{H}$, but with $\mathrm{SL}(r, \mathbb{C})$ substituted for $\mathrm{GL}(r, \mathbb{C})$. Certainly $\mathcal{M}$ is a subspace of $\mathcal{H}$; but also, scalar multiplication induces a map $\mathcal{T} \times \mathcal{M} \to \mathcal{H}$ which is easily seen to be a free quotient by the abelian group $\Sigma = \mathbb{Z}_r^{2g}$. According to a theorem of Grothendieck [**12**, **26**], the rational cohomology of such a quotient satisfies

$$H^*(\mathcal{H}) = H^*(\mathcal{T} \times \mathcal{M})^{\Sigma},$$

where the right-hand side denotes the Σ-invariant part. Now Σ acts on $\mathcal{T}$ by scalar multiplications, so it acts trivially on cohomology and hence as rings

$$H^*(\mathcal{H}) = H^*(\mathcal{T}) \otimes H^*(\mathcal{M})^{\Sigma}.$$

Furthermore the composition of the Σ-quotient with the determinant is the map $\mathcal{T} \times \mathcal{M} \to \mathcal{T}$ given simply by projecting to $\mathcal{T}/\Sigma = \mathcal{T}$. The subring of $H^*(\mathcal{H})$ generated by $\varepsilon_1, \dots, \varepsilon_{2g}$ is therefore nothing but the first factor of the tensor product.

To define more interesting cohomology classes on $\mathcal{H}$, construct a principal bundle over $\mathcal{H} \times C$ as follows. Let $\overline{G} = \mathrm{PGL}(r, \mathbb{C})$. Any $\rho \in \mu^{-1}(e^{2\pi d i/r} I)$ induces a well-defined homomorphism $\pi_1(C) \to \overline{G}$. Let $\widetilde{C}$ be the universal cover of C, which is acted on by $\pi_1(C)$ via deck transformations. There is then a free action of $\pi_1(C) \times G$ on $\overline{G} \times \mu^{-1}(e^{2\pi d i/r} I) \times \widetilde{C}$ given by

$$(p, g) \cdot (h, \rho, x) = (\overline{g}\rho(p)h, \overline{g}\rho\overline{g}^{-1}, p \cdot x),$$

where $\overline{g}$ denotes the image of g in $\overline{G}$. The quotient is the desired principal $\overline{G}$-bundle. Like any principal $\overline{G}$-bundle, it has characteristic classes $\overline{c}_2, \dots, \overline{c}_r$, where $\overline{c}_i \in H^{2i}(\mathcal{H} \times C)$. In terms of formal Chern roots ξ_k, $\overline{c}_i$ can be described as the ith elementary symmetric polynomial in the $\xi_k - \zeta$, where ζ is the average of the ξ_k.

Now let $\sigma \in H^2(C)$ be the fundamental cohomology class, and let $e_1, \dots, e_{2g}$ be the basis of $H^1(C)$ Poincaré dual to $a_1, \dots, a_g, b_1, \dots, b_g$. In terms of these, each of the characteristic classes has a Künneth decomposition

$$c_i = \alpha_i \sigma + \beta_i + \sum_{j=1}^{2g} \psi_{i,j} e_j,$$

defining classes $\alpha_i \in H^{2i-2}(\mathcal{H})$, $\beta_i \in H^{2i}(\mathcal{H})$, and $\psi_{i,j} \in H^{2i-1}(\mathcal{H})$. The pull-back of these classes to $\mathcal{T} \times \mathcal{M}$, by the way, is easily seen to come entirely from $H^*(\mathcal{M})^{\Sigma}$.

It is convenient to refer to the entire collection of classes α_i, β_i, $\psi_{i,j}$, and ε_j as the *universal classes*.

Now specialize, for the remainder of this section, to the case $r = 2$. Then d must be odd, so that $\mathcal{H} = \mu^{-1}(-I)/G$. The main result of this paper is then the following.

(1.1) *When $r = 2$, the rational cohomology ring of $\mathcal{H}$ is generated by the universal classes.*

Equivalently, $\mathcal{H}^*(\mathcal{M})^\Sigma$ is generated by the classes $\alpha = \frac{1}{2}\alpha_2 \in H^2(\mathcal{M})$, $\beta = -\frac{1}{4}\beta_2 \in H^4(\mathcal{M})$, and $\psi_j = \psi_{2,j} \in H^3(\mathcal{M})$ for $j = 1, \ldots, 2g$. (These normalizations are by now standard in the literature.) It is worth mentioning that $H^*(\mathcal{M})$ is generally *not* entirely Σ-invariant [**20**, 7.6], and hence is not generated by the universal classes. However, by the aforementioned theorem of Grothendieck, $H^*(\mathcal{M})^\Sigma$ is the rational cohomology of $\mathcal{M}/\Sigma$, which is a component of the moduli space of flat $\overline{G}$-connections on C.

For completeness, we recount here the main result of our companion paper [**19**] giving all the relations between these generators. In the light of the discussion earlier it suffices to work with $H^*(\mathcal{M})^\Sigma$. Let $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$. The action of diffeomorphisms on C will be shown to induce an action of Γ on $H^*(\mathcal{M})^\Sigma$, fixing α and β and acting as the standard representation on the span $V = H^3(\mathcal{M})$ of the ψ_j. Thus $\gamma = -2\sum_{j=1}^g \psi_j\psi_{j+g}$ is a Γ-invariant element of $H^6(\mathcal{M})$. Let $\Lambda_0^n(\psi)$ be the kernel of the natural map $\Lambda^n V \to \Lambda^{2g+2-n} V$ given by the wedge product with γ^{g+1-n}, or equivalently, the Γ-invariant complement of $\gamma\Lambda^{n-2}V$ in $\Lambda^n V$.

For any $g, n \geqslant 0$, let I_n^g be the ideal within the polynomial ring $\mathbb{Q}[\alpha, \beta, \gamma]$ generated by γ^{g+1} and the polynomials

$$\rho^c_{r,s,t} = \sum_{i=0}^{\min(c,r,s)} (c-i)!\, \frac{\alpha^{r-i}}{(r-i)!}\, \frac{\beta^{s-i}}{(s-i)!}\, \frac{(2\gamma)^{t+i}}{i!},$$

where

$$c = r + 3s + 2t - 2g + 2 - n,$$

for all $r, s, t \geqslant 0$ such that $r + 3s + 3t > 3g - 3 + n$ and $r + 2s + 2t \geqslant 2g - 2 + n$.

The main result of our companion paper is then the following.

(1.2) *If the rank $r = 2$, then as an algebra acted on by Γ,*

$$H^*(\mathcal{M})^\Sigma = \bigoplus_{n=0}^{g} \Lambda_0^n(\psi) \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_n^{g-n}.$$

Together, the two main theorems completely describe the ring structure of $H^*(\mathcal{H})$ when $r = 2$. They do not completely describe $H^*(\mathcal{M})$ because of the classes that are not invariant under Σ. However, these form a relatively minor and simple part of the cohomology, and can be dealt with by hand; this will be carried out in a forthcoming paper [**45**].

The main theorems will be proved in the language not of flat connections but rather of Higgs bundles. Indeed, it proved most convenient to deduce them from more general results applying to an infinite sequence of spaces of Higgs bundles, of which $\mathcal{H}$ is only the first. We shall next review the definition of Higgs bundles, and the correspondence theorem relating them to flat connections.

2. *Higgs bundles*

A major advance in the study of these representation spaces was made by Hitchin [**20**] and Simpson [**39**], who discovered that they can alternatively be viewed as moduli spaces of holomorphic objects. So now, and for the remainder of the paper, let C be a smooth complex projective curve of genus g.

A *Higgs bundle* or *Higgs pair* on C with values in a holomorphic line bundle L is a pair (E, ϕ), where E is a holomorphic vector bundle over C, and ϕ, called a *Higgs field*, is any element of $H^0(\operatorname{End} E \otimes L)$. Its *slope* is the rational number

$\deg E/\mathrm{rk}\, E$. A holomorphic subbundle $F \subset E$ is *ϕ-invariant* if $\phi(F) \subset F \otimes L$. A Higgs bundle is *semistable* if $\mathrm{slope}\, F \leqslant \mathrm{slope}\, E$ for all proper ϕ-invariant holomorphic subbundles $F \subset E$, and *stable* if this inequality is always strict.

For example, a pair of the form $(E, 0)$ is stable if and only if the bundle E is stable.

We will be concerned entirely with the case when the line bundle L is $K(n) = K \otimes \mathcal{O}(np)$, where K is the canonical bundle of C, p is a distinguished point in C, and $n \geqslant 0$.

The moduli space of Higgs bundles with values in K was constructed by Hitchin [**20**] and Simpson [**39**], and generalized to an arbitrary line bundle by Nitsure [**34**]. Their work implies the following.

(2.1) *For fixed rank r, degree d coprime to r, and $n \geqslant 0$, there exists a moduli space $\mathcal{H}_n$ of Higgs bundles with values in $K(n)$, which is a smooth quasi-projective variety of dimension $r^2(2g-2+n)+2$. For a fixed holomorphic line bundle Ξ of degree d, the locus $\mathcal{M}_n$ where $\Lambda^r E \cong \Xi$ and $\mathrm{tr}\, \phi = 0$ is a smooth subvariety of dimension $(r^2-1)(2g-2+n)$.*

In the case $n = 0$, Higgs bundles are related to connections of constant central curvature in the following way. Suppose that C is equipped with a Kähler metric, and let ω be the Kähler form, again normalized so that $\int_C \omega = 2\pi/r$. Then Hitchin showed the following.

(2.2) *Suppose that r and d are coprime and that $n = 0$. Then a Higgs bundle (E, ϕ) is stable if and only if it admits a Hermitian metric so that the metric connection A satisfies the equation $F_A + [\phi, \phi^*] = di\omega I$. This metric is unique up to rescaling and depends smoothly on (E, ϕ).*

Here $F_A \in \Omega^2(\mathrm{End}\, E)$ is the curvature, the Higgs field ϕ is regarded as a section of $\Omega^{0,1}(\mathrm{End}\, E)$, and I is the identity in $\mathrm{End}\, E$. For such a connection A, an easy calculation shows that the $\mathrm{GL}(r, \mathbb{C})$ connection $A + \phi + \phi^*$ has constant central curvature $di\omega I$. Hence there is a natural smooth map from the space $\mathcal{H}_0$ of Higgs bundles to the space $\mathcal{H}$ discussed in the previous section.

In fact, this map is a diffeomorphism, as is the restriction $\mathcal{M}_0 \to \mathcal{M}$. The inverse map is provided by a result of Corlette [**5**] and Donaldson [**8**].

(2.3) *Any $\mathrm{GL}(r, \mathbb{C})$ connection on C with constant central curvature $di\omega I$ is gauge equivalent to one of the form $A + \phi + \phi^*$, where $\overline{\partial}_A \phi = 0$ and $F_A + [\phi, \phi^*] = di\omega I$.*

Both $\mathcal{H}_0$ and $\mathcal{H}$ carry natural complex structures, but these are *not* identified by the diffeomorphism. Rather, they are different members of the family of complex structures which comprises a *hyperkähler structure* on the moduli space.

Our approach does not use this hyperkähler structure. Indeed, it will be shown in (5.1) that the cohomology classes defined above in terms of flat connections can also be obtained from the universal family of Higgs bundles. From then on the flat connection point of view will vanish, and the moduli space will be regarded exclusively as a Higgs space.

As pointed out by Simpson [**37**], the moduli space of Higgs bundles actually retracts onto a highly singular Lagrangian subvariety, the *nilpotent cone*.

Therefore our results could be viewed as describing the cohomology ring of the nilpotent cone. However, this seems to be only a curiosity and is not relevant to our approach.

The advantage of the Higgs moduli space is that it admits a holomorphic action of the group $T = \mathbb{C}^\times$, given simply by $\lambda \cdot (E, \phi) = (E, \lambda\phi)$. This of course fixes all stable pairs of the form $(E, 0)$, which are parametrized by the moduli space of stable bundles of rank r and degree d. But the fixed-point set has other components as well, and they will play a crucial role in what follows.

3. *Deformation theory of Higgs pairs*

This section and the next summarize, mostly without proof, some basic facts about Higgs pairs that will be needed later on. The omitted proofs are entirely straightforward, along the lines of Markman [**27**, 7.3], Welters [**46**] or the second author [**44**, 2.1]. In the rank 2 case, some details are worked out in the first author's PhD thesis [**16**].

The deformation space of a holomorphic bundle E is well known to be $H^1 \operatorname{End} E$; that of a Higgs pair (E, ϕ) is similar, but involves hypercohomology.

Let (E, ϕ) be a Higgs pair, and let $\mathbf{End}(E, \phi)$ denote the two-term complex on C,

$$\operatorname{End} E \xrightarrow{[\ ,\phi]} \operatorname{End} E \otimes K(n).$$

(3.1) *The space of infinitesimal deformations of* (E, ϕ) *is the first hypercohomology group* $\mathbf{H}^1 \mathbf{End}(E, \phi)$. *The space of endomorphisms of* E preserving ϕ is $\mathbf{H}^0 \mathbf{End}(E, \phi)$. □

Similarly, let $\mathbf{Hom}((E', \phi'), (E, \phi))$ denote the complex

$$\operatorname{Hom}(E', E) \longrightarrow \operatorname{Hom}(E', E) \otimes K(n)$$

given by $\psi \mapsto \psi\phi' - \phi\psi$.

(3.2) *The space of homomorphisms* $E' \to E$ *intertwining* ϕ *with* ϕ' *is the zeroth hypercohomology group* $\mathbf{H}^0 \mathbf{Hom}((E', \phi'), (E, \phi))$. *The space of extensions of* (E', ϕ') *by* (E, ϕ) *is* $\mathbf{H}^1 \mathbf{Hom}((E', \phi'), (E, \phi))$. □

Here an *extension* of one Higgs pair by another is a Higgs pair (E'', ϕ'') and a short exact sequence

$$0 \longrightarrow E \longrightarrow E'' \longrightarrow E' \longrightarrow 0$$

such that ϕ'' restricts to ϕ on E and projects to ϕ' on E'.

One more variation on the theme will be needed in § 7. Let (E, ϕ) be a Higgs pair containing a flag of ϕ-invariant subbundles. (In practice this flag will always be the *Harder–Narasimhan filtration* defined in § 7.) Let $\operatorname{End}' E$ be the subbundle of $\operatorname{End} E$ consisting of endomorphisms fixing the flag, and let $\mathbf{End}'(E, \phi)$ be the two-term complex

$$\operatorname{End}' E \xrightarrow{[\ ,\phi]} \operatorname{End}' E \otimes K(n).$$

(3.3) *The space of infinitesimal deformations of the Higgs pair* (E, ϕ) *together with the* ϕ*-invariant flag is* $\mathbf{H}^1\,\mathbf{End}'(E, \phi)$. □

4. *Universal families of stable Higgs pairs*

(4.1) *Let* (E, ϕ) *and* (E', ϕ') *be stable Higgs pairs with* $\operatorname{slope} E' \geqslant \operatorname{slope} E$. *Then the dimension of* $\mathbf{H}^0\,\mathbf{Hom}((E', \phi'), (E, \phi))$ *is* 1 *if* (E', ϕ') *and* (E, ϕ) *are isomorphic, and* 0 *otherwise.* □

In particular, the space of endomorphisms $\mathbf{H}^0\,\mathbf{End}(E, \phi)$ consists only of scalar multiplications.

(4.2) *Let* $(\mathbf{E}, \mathbf{\Phi})$ *and* $(\mathbf{E}', \mathbf{\Phi}') \to X \times C$ *be families of stable Higgs pairs parametrized by* X *such that for all* $x \in X$, $(\mathbf{E}, \mathbf{\Phi})_x \cong (\mathbf{E}', \mathbf{\Phi}')_x$. *Then there exists a line bundle* $L \to X$ *such that* $(\mathbf{E}, \mathbf{\Phi}) \cong (\mathbf{E}' \otimes \pi_1^* L, \mathbf{\Phi})$. *In particular,* $\mathbb{P}\mathbf{E}$ *and* $\operatorname{End} \mathbf{E}$ *are canonical.*

Proof. By (4.1), $\mathbf{H}^0\,\mathbf{Hom}((\mathbf{E}', \mathbf{\Phi}')_x, (\mathbf{E}, \mathbf{\Phi})_x)$ is 1-dimensional for all x. Hence the hyperdirect image $(\mathbf{R^0}\pi_\mathbf{1})_*\,\mathbf{Hom}((\mathbf{E}', \mathbf{\Phi}'), (\mathbf{E}, \mathbf{\Phi}))$ is a line bundle $L \to X$. It is then easy to construct the desired isomorphism. □

It is clear from the proof that the above proposition holds true not only for algebraic families of Higgs pairs, but even for smooth families, that is, C^∞ bundles $(\mathbf{E}, \mathbf{\Phi}) \to X \times C$ for any smooth parameter space X, endowed with a partial holomorphic structure in the C-directions.

(4.3) *Let* $(\mathbf{E}, \mathbf{\Phi})$ *be a family of stable Higgs pairs parametrized by* X, *and let* $\mathbb{C}^\times$ *act on* X. *If there are two liftings of the action to* $\mathbf{E}$ *so that the induced action on* $\mathbf{\Phi}$ *is* $\operatorname{Ad}(\lambda)\mathbf{\Phi} = \lambda^{-1}\mathbf{\Phi}$, *then one is the tensor product of the other with an action of* $\mathbb{C}^\times$ *on a trivial line bundle. In particular, there are canonical* $\mathbb{C}^\times$*-actions on* $\mathbb{P}\mathbf{E}$ *and* $\operatorname{End} \mathbf{E}$.

Proof. Compose one lifting with the inverse of the other. This gives a lifting of the trivial action on $\mathcal{H}_n$ to $\mathbf{E}$ which preserves $\mathbf{\Phi}$. By (4.1), this acts on each fiber via scalar multiplications. □

(4.4) *There exists a universal family* $(\mathbf{E}, \mathbf{\Phi})$ *over* $\mathcal{H}_n \times C$, *and a lifting of the* $\mathbb{C}^\times$*-action on* $\mathcal{H}_n$ *to* $\mathbf{E}$ *whose induced action on* $\mathbf{\Phi}$ *is* $\operatorname{Ad}(\lambda)\mathbf{\Phi} = \lambda^{-1}\mathbf{\Phi}$.

That is, $\mathcal{H}_n$ is a fine moduli space for the Higgs bundles of degree d and rank r with values in $K(n)$.

Proof. This follows in a standard way (cf. Newstead [**33**]), from the geometric invariant theory construction of $\mathcal{H}_n$ due to Nitsure [**34**]. Alternatively, the universal pair can be constructed gauge-theoretically just as in the paper by Atiyah and Bott [**1**, § 9]. In the rank 2 case, both methods are explained in detail by the first author [**17**, 5.3; **16**, 5.2.3]. □

5. *Equivalence of the two sets of universal classes*

The main results were stated in § 1 for the moduli space of flat connections $\mathcal{H}$. But what is actually proved are similar results for the Higgs moduli spaces $\mathcal{H}_n$. To show that these imply the statements of § 1 in the case $n = 0$, it suffices to check that the relevant cohomology classes correspond under the diffeomorphism $\mathcal{H}_0 \to \mathcal{H}$.

Let $(\mathbf{E}, \mathbf{\Phi})$ be a universal family on $\mathcal{H}_n \times C$. There is a morphism $\mathcal{H}_n \to \mathrm{Jac}^d C$ given by $(E, \phi) \mapsto \Lambda^r E$, so the generators of $H^*(\mathrm{Jac}^d C)$ pull back to classes $\varepsilon_1, \ldots, \varepsilon_{2g} \in H^1(\mathcal{H}_n)$. Also, let $c_2, \ldots, c_r$ be the characteristic classes of $\mathbb{P}\mathbf{E}$. These are elements of rational cohomology; they can be regarded as the Chern classes of the tensor product of $\mathbf{E}$ with a formal rth root of $\Lambda^r \mathbf{E}^*$.

Each of these classes has a Künneth decomposition

$$c_i = \alpha_i \sigma + \beta_i + \sum_{j=1}^{2g} \psi_{i,j} e_j,$$

defining classes $\alpha_i \in H^{2i-2}(\mathcal{H})$, $\beta_i \in H^{2i}(\mathcal{H})$, and $\psi_{i,j} \in H^{2i-1}(\mathcal{H})$. The entire collection of classes α_i, β_i, $\psi_{i,j}$, and ε_j will be referred to as the *universal classes.* What requires proof is then the following.

(5.1) *When $n = 0$, these classes correspond under the diffeomorphism $\mathcal{H}_0 \to \mathcal{H}$ to their counterparts defined in* § 1.

Proof. First, there is a diagram

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\det"] \arrow[d, "\cong"'] & \mathcal{T} \arrow[d] \\ \mathcal{H}_0 \arrow[r, "\Lambda^r"] & \mathrm{Jac}^d C \end{tikzcd}$$

relating the map det of § 1 to the map $(E, \phi) \mapsto \Lambda^r E$ mentioned above. It is easy to see from (2.3) that this commutes, and hence that the classes $\varepsilon_j \in \mathcal{H}$ correspond under the diffeomorphism.

To show that the higher-degree classes α_i, β_i, and $\psi_{i,j}$ correspond, it suffices to show that the principal $\mathrm{PGL}(r, \mathbb{C})$-bundle associated to $\mathbb{P}\mathbf{E}$ corresponds under the diffeomorphism $\mathcal{H}_0 \to \mathcal{H}$ to the principal bundle

$$\frac{\overline{G} \times S \times \widetilde{C}}{\pi_1(C) \times G}$$

of § 1, where $S = \mu^{-1}(e^{2\pi i d/r} I)$. (Recall that $G = \mathrm{GL}(r, \mathbb{C})$, $\overline{G} = \mathrm{PGL}(r, \mathbb{C})$, and $\widetilde{C}$ is the universal cover of C.) In fact, we will construct a principal G-bundle R over $\mathcal{H}_0$ and show that the principal $\overline{G}$-bundle U associated to the pull-back of $\mathbb{P}\mathbf{E}$ to $R \times C$ is G-equivariantly isomorphic to

$$V = \frac{\overline{G} \times S \times \widetilde{C}}{\pi_1(C)}.$$

Let R be the total space of the principal $\overline{G}$-bundle over $\mathcal{H}_0$ associated to $\mathbb{P}\mathbf{E}|_{\mathcal{H}_0 \times \{p\}}$. Then R parametrizes stable pairs (E, ϕ) equipped with a frame for the fiber E_p, up to rescaling. On the other hand, S parametrizes connections of constant central curvature, together with a frame for the fiber at a base point, up to rescaling. The diffeomorphism $\mathcal{H}_0 \to \mathcal{H}$ therefore lifts to a G-equivariant diffeomorphism $R \to S$.

Let $\mathbf{F}$ be the pull-back to $R \times C$ of $\mathbf{E}$. Then $\mathbb{P}\mathbf{F}$ admits a natural G-action lifting that on R, and it is canonically trivialized on $R \times \{p\}$. Moreover, by (2.2), $\mathbf{F}$ admits a Hermitian metric so that the restriction of the metric connection $\mathbf{A}$ to each slice $\{(E, \phi)\} \times C$ satisfies the self-duality equation. Hence $\mathbf{A} + \mathbf{\Phi} + \mathbf{\Phi}^*$ determines a G-connection on $\mathbf{F}$ whose restriction to each slice has constant central curvature. In particular, the associated $\overline{G}$-connection on the associated $\overline{G}$-bundle U is flat on each slice.

On the other hand, the bundle V over $S \times C$ defined above is trivial on $S \times \{p\}$, and carries a flat connection on each slice $\{r\} \times C$, which has the same holonomy as the one just mentioned and is preserved by the action of G. These flat connections can be used to extend the isomorphism $U|_{R\times\{p\}} \cong V|_{S\times\{p\}}$ of trivial bundles with G-action to a G-equivariant isomorphism $U \cong V$ lifting the G-equivariant diffeomorphism $R \to S$. □

This marks the last appearance of flat connections in our story. From now on it is all about Higgs bundles.

6. *Statement of the generation theorem*

Let ε_j, α_i, β_i, $\psi_{i,j}$ be the universal classes, defined in § 5, on the Higgs moduli space $\mathcal{H}_n$. The goal of the paper will be to prove this, its main result.

(6.1) *The rational cohomology ring of $\mathcal{H}_n$ is generated by the universal classes.*

The proof of this generation theorem has several parts, with quite different flavors.

First, we study the stratification of families of Higgs bundles according to their Harder–Narasimhan type. Section 7 is devoted to finite-dimensional families, and § 8 to an infinite-dimensional family analogous to that of Atiyah and Bott [**1**]. The aim is to show that the strata are smooth of the expected dimension, but this turns out to be true only in a stable sense. We need to consider not only a single $\mathcal{H}_n$, but the chain of inclusions $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$.

We are therefore led to consider in § 9 the direct limit $\mathcal{H}_\infty$ of the $\mathcal{H}_n$, and to show that its cohomology is generated by universal classes. Indeed, topological arguments show that it has the homotopy type of the classifying space of the gauge group, and the generation then follows from a theorem of Atiyah and Bott [**1**].

Having done this, we then show in § 10 that, when $r = 2$, the cohomology of $\mathcal{H}_\infty$ surjects on that of $\mathcal{H}_n$ for every n, and hence in particular on that of $\mathcal{H}$ itself. This part of the proof is algebro-geometric in nature.

7. *The finite-dimensional stratification*

We wish to adopt the point of view taken by Atiyah and Bott [**1**], in which the objects of interest – for us, Higgs pairs – are parametrized by an infinite-dimensional, contractible space. The whole space will be divided into strata on which the level of instability is in some sense constant. So we will first study the analogue of this stratification in finite-dimensional algebraic families, then transfer it to our infinite-dimensional setting. Except for some subtleties surrounding smoothness, the results of §§ 7 and 8 are mostly analogous to those of Shatz [**35**] and Atiyah and Bott [**1**]; readers familiar with those papers may be willing to skip directly to § 9.

Let (E, ϕ) be a Higgs pair with values in a line bundle L. A filtration by ϕ-invariant subbundles

$$0 = E^0 \subset E^1 \subset \ldots \subset E^l = E$$

is said to be a *Harder–Narasimhan filtration* (hereinafter HN filtration) if the pairs (F^i, ϕ^i) are semistable with slope strictly decreasing in i, where $F^i = E^i/E^{i-1}$ and ϕ^i is induced by ϕ.

(7.1) *Any Higgs pair defined over any field of characteristic* 0 *possesses a unique HN filtration.*

Proof. The analogous statement for bundles without a Higgs field is proved by Harder and Narasimhan [**14**] and Shatz [**35**]. The proof for Higgs pairs is entirely parallel: just substitute ϕ-invariant subbundles for ordinary subbundles everywhere in either proof. Shatz assumes that the ground field is algebraically closed, but his proof of this theorem does not require it. □

For a given (E, ϕ), the *type* μ is the l-tuple $(r_1, d_1), \ldots, (r_l, d_l)$ of ranks and degrees of the F^i appearing in its HN filtration. For example, the type of a semistable pair is the 1-tuple (r, d).

Since the slope d_i/r_i is strictly decreasing, the pairs $(0, 0), (r_1, d_1), (r_1 + r_2, d_1 + d_2), \ldots, (r, d)$ consisting of partial sums form the vertices of a convex polygon $\mathrm{Pol}(\mu)$ in $\mathbf{R}^2$, as shown in Figure 1.

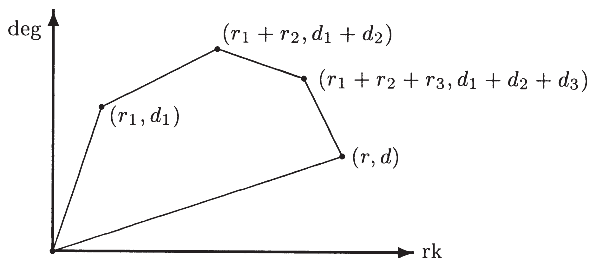

FIGURE 1.

Let S be a scheme of finite type over $\mathbb{C}$, and $(\mathbf{E}, \mathbf{\Phi}) \to S \times C$ a family, parametrized by S, of Higgs pairs on C with values in L.

(7.2) *The set* $\{s \in S \mid (\mathbf{E}, \mathbf{\Phi})_s$ *is semistable*$\}$ *is open in* S.

Proof. See [**34**] by Nitsure. □

For any μ, let S^μ be the set of those $s \in S$ such that $(\mathbf{E}, \mathbf{\Phi})_s$ has type μ. Recall that a *constructible* subset is a finite union of locally closed sets in the Zariski topology.

(7.3) *For any family of Higgs pairs over S and any type μ, S^μ is a constructible set. Moreover, $S^\mu \neq \emptyset$ for only finitely many μ, and each S^μ is covered by constructible sets where the HN filtration varies algebraically, that is, it determines a filtration of $\mathbf{E}$ by $\mathbf{\Phi}$-invariant subbundles.*

Proof. Without loss of generality assume S is irreducible. Given a family $(\mathbf{E}, \mathbf{\Phi}) \to S \times C$, let $(\mathbf{E}, \mathbf{\Phi})_\xi$ be the fiber over the generic point $\xi \in S$. This is a Higgs pair defined over the function field $\mathbb{C}(S)$. It therefore has a HN filtration, of some type μ, by $\mathbf{\Phi}_\xi$-invariant subbundles. By Lemma 5 of Shatz [**35**] there exists an open $U \subset S$ such that this filtration extends to a filtration of $\mathbf{E}|_{U\times C}$ by subbundles $\mathbf{E}^i$. They are $\mathbf{\Phi}$-invariant, since this is a closed condition and the closure of ξ is all of S.

On the other hand, (7.2) implies that, since the quotient pairs $(\mathbf{F}^i, \mathbf{\Phi}^i)_\xi$ are semistable, after restricting to a smaller U if necessary, $(\mathbf{F}^i, \mathbf{\Phi}^i)_s$ are also semistable for all $s \in U$. Hence this is the HN filtration at every $s \in U$, so $U \subset S^\mu$.

Now pass to $S \setminus U$ and proceed by induction on the dimension of the parameter space. $\square$

Following Shatz [**35**], define a partial ordering on the set of types by declaring $\mu \leqslant \nu$ if $\mathrm{Pol}(\mu) \subset \mathrm{Pol}(\nu)$. Then let $S^{\geqslant\mu} = \bigcup_{\nu\geqslant\mu} S^\nu$.

(7.4) *For S and μ as above, $S^{\geqslant\mu} \subset S$ is closed.*

The proof requires the following lemma.

(7.5) *Let (E, ϕ) be a Higgs pair of type μ, and let $F \subset E$ be a ϕ-invariant subbundle. Then* $(\mathrm{rk}\, F, \deg F) \in \mathrm{Pol}(\mu)$.

Proof. The analogous statement without a Higgs field is Theorem 2 of Shatz [**35**], and the proof of this is entirely parallel. One simply has to note that since the filtration and the subbundle F are ϕ-invariant, so are the subsheaves $E^i \cap F$ and $E^i \vee F$ considered by Shatz. $\square$

Proof of (7.4). If $F \subset E$ is any inclusion of torsion-free sheaves, define $\widehat{F} \subset E$ to be the inverse image under the projection $E \to E/F$ of the torsion subsheaf. Then $\widehat{F}$ and $E/\widehat{F}$ are torsion-free, $F = \widehat{F}$ on the locus where E/F is torsion-free, and $F \mapsto \widehat{F}$ preserves inclusions of subsheaves of E.

By (7.3) $S^{\geqslant\mu}$ can be regarded as a reduced subscheme of S. To show that it is closed, by the valuative criterion [**15**, Chapter II, 4.7] it suffices to show that, if X is any smooth curve, and $f : X \to S$ any morphism taking a non-empty open set into S^μ, then $f(X) \subset S^\mu$.

The generic point of X maps to one of the constructible sets named in (7.3), where the HN filtrations are parametrized by subbundles. Hence there is a non-empty open set $V \subset X$ such that the restriction of $(f\times 1)^*\mathbf{E}$ to $V \times C$ is filtered by subbundles restricting over every point in V to the HN filtration.

Like any coherent subsheaf defined on an open set [**15**, Chapter II, Exercise 5.15(d)], these bundles extend to coherent subsheaves $\mathbf{E}^i$ of $(f\times 1)^*\mathbf{E}$ over all of $X \times C$. These can be chosen to remain nested, and as subsheaves of $\mathbf{E}$ they are of course torsion-free. Furthermore, they can be chosen so that $\mathbf{E}/\mathbf{E}^i$ are torsion-free also, by replacing $\mathbf{E}^i$ with $\widehat{\mathbf{E}}^i$.

Since torsion-free sheaves on a smooth surface such as $X \times C$ are locally free except on a set of codimension 2 [**10**, Corollary 2.38], it follows that the $\mathbf{E}^i$ are subbundles except at finitely many points in the fibers over $X \setminus V$.

Now on a smooth curve such as one of these fibers, torsion-free sheaves are locally free, and so the procedure of the first paragraph implies the following: every subsheaf of a locally free sheaf is contained in a subbundle having the same rank and no less degree, with equality if and only if it was a subbundle to begin with.

When restricted to $\{x\} \times C$ for any $x \in X \setminus V$, then, the nested subsheaves $\mathbf{E}^i_x$ determine a sequence of subbundles, which only differ from $\mathbf{E}^i_x$ at finitely many points and hence remain nested and $\mathbf{\Phi}_x$-invariant, and have the same rank. Since the degrees may have risen, they determine a polygon which contains $\mathrm{Pol}(\mu)$. By (7.5), if the type of $(\mathbf{E}, \mathbf{\Phi})_x$ is ν_x, then $\mathrm{Pol}(\nu_x)$ contains this polygon. Hence $\nu_x \geqslant \mu$, so $f(x) \in S^{\geqslant \mu}$. □

As in the paper by Atiyah and Bott [**1**], the last statement of (7.3) can be refined.

(7.6) *The HN filtration varies algebraically on all of S^μ; that is, there exists a filtration of $\mathbf{E}|_{S^\mu \times C}$ by subbundles restricting to the HN filtration on each fiber.*

The proof again requires a lemma.

(7.7) *If E^1 is the first term in the HN filtration of (E, ϕ) and $F \subset E$ is another ϕ-invariant subbundle of the same rank and degree, then $F = E^1$.*

Proof. The corresponding statement for ordinary bundles is a special case of Lemma 3 of Shatz [**35**]. The proof of this is again entirely parallel. □

Proof of (7.6). By (7.3), the HN filtrations determine a constructible subset of the product of Grassmannian bundles $\times_i \mathrm{Grass}_{r_i} \mathbf{E}|_{S^\mu \times C}$. It must be shown that it is closed. By the valuative criterion, it suffices to show that for any morphism $f : X \to S^\mu$, where X is a smooth curve, the HN filtrations determine a filtration of $(f \times 1)^* \mathbf{E}$ by subbundles.

As in the proof of (7.4), a filtration by subbundles does exist over an open $V \subset X$, and the subbundles extend to nested, $\mathbf{\Phi}$-invariant torsion-free sheaves $\mathbf{E}^i$ over $X \times C$.

The restrictions of these to the fibers over $x \in X \setminus V$ generate nested, $\mathbf{\Phi}$-invariant subbundles whose ranks and degrees span a polygon containing $\mathrm{Pol}(\mu)$. But now since $(\mathbf{E}, \mathbf{\Phi})_x$ also has type μ, by (7.5) this polygon is contained in $\mathrm{Pol}(\mu)$ as well. Hence it equals $\mathrm{Pol}(\mu)$, so the subbundles have degrees equal to those of the subsheaves which generated them. They therefore coincide with these subsheaves.

Consequently, the sections of the subsheaf $\mathbf{E}^i$ span an r_i-dimensional subspace of the fiber of $(f \times 1)^* \mathbf{E}$ over every point in $X \times C$. It follows that $\mathbf{E}^i$ is a subbundle.

Finally, we claim that for any $x \in X \setminus V$, the HN filtration of $\mathbf{E}_x$ is the restriction of the $\mathbf{E}^i$. First, $\mathbf{E}^1_x$ is a $\mathbf{\Phi}_x$-invariant subbundle of rank and degree equal to that of the first term in the HN filtration. By (7.7) these two subbundles must coincide. Now pass to $\mathbf{E}/\mathbf{E}^1$ and use induction on the length of the HN filtration to do the rest. $\square$

Recall that $\mathbf{End}'$ refers to the two-term complex, defined in § 3, involving endomorphisms fixing a flag of subbundles. In what follows, this flag will always be the HN filtration.

(7.8) *There are deformation maps*

$$T_s S \to \mathbf{H}^1\,\mathbf{End}(\mathbf{E},\mathbf{\Phi})_s \quad \textit{and} \quad T_s S^\mu \to \mathbf{H}^1\,\mathbf{End}'(\mathbf{E},\mathbf{\Phi})_s$$

so that the following diagram commutes:

$$\begin{array}{ccc} T_s S^\mu & \longrightarrow & \mathbf{H}^1\,\mathbf{End}'(\mathbf{E},\mathbf{\Phi})_s \\ \downarrow & & \downarrow \\ T_s S & \longrightarrow & \mathbf{H}^1\,\mathbf{End}(\mathbf{E},\mathbf{\Phi})_s \end{array}$$

Proof. This follows immediately from (3.1), (3.3) and (7.6). $\square$

(7.9) *Let (E,ϕ) be a Higgs pair with values in L. For m large enough, $(E,\phi(m))$ belongs to a family, parametrized by a smooth base X, of Higgs pairs with values in $L(m)$ such that the deformation map $T_{(E,\phi(m))}X \to \mathbf{H}^1\,\mathbf{End}(E,\phi(m))$ is an isomorphism.*

Proof. It suffices to find a smooth X so that the deformation map is surjective, since then one may restrict to a smooth subvariety transverse to the kernel.

Choose k large enough that $E(k)$ is generated by its sections and has $H^1 = 0$. There is then a surjection $\mathcal{O}^\chi \to E(k)$, where χ is the Euler characteristic of $E(k)$. This represents a point q in the Quot scheme parametrizing quotients of $\mathcal{O}^\chi$ with fixed rank and degree. Let F be the kernel of the map $\mathcal{O}^\chi \to E(k)$. The tangent space to the Quot scheme at q is then $H^0\,\mathrm{Hom}(F,E(k))$, and the natural map to the deformation space of $E(k)$ is the connecting homomorphism in the long exact sequence of

$$0 \longrightarrow \mathrm{End}\,E \longrightarrow \mathrm{Hom}(\mathcal{O}^\chi, E(k)) \longrightarrow \mathrm{Hom}(F,E(k)) \longrightarrow 0.$$

This is surjective since $H^1\,\mathrm{Hom}(\mathcal{O}^\chi,E(k)) = \mathbb{C}^\chi \otimes H^1(E(k)) = 0$. For the same reason, $H^1\,\mathrm{Hom}(F,E(k)) = 0$, so the Quot scheme is smooth at q.

Now choose m large enough that $H^1(\mathrm{End}\,E \otimes L(m)) = 0$, and let X be the total space of $\pi_*(\mathrm{End}\,\mathbf{E} \otimes L(m))$, where $\mathbf{E}$ is the tautological quotient on Quot $\times\, C$, and π is projection to Quot. This is smooth near q since the push-forward is locally free there. Moreover, there is a tautological section $\mathbf{\Phi} \in H^0(X \times C, \mathrm{End}\,\mathbf{E} \otimes L(m))$. That the deformation map is surjective is easily seen from the diagram

$$\begin{array}{ccccccc} H^0(\mathrm{End}\,E \otimes L(m)) & \longrightarrow & T_{(E,\phi(m))}X & \longrightarrow & H^0\,\mathrm{Hom}(F,E(k)) & \longrightarrow & 0 \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(\mathrm{End}\,E \otimes L(m)) & \longrightarrow & \mathbf{H}^1\,\mathbf{End}(E,\phi(m)) & \longrightarrow & H^1\,\mathrm{End}\,E & \longrightarrow & 0 \end{array}$$

$\square$

Now fix a type μ, and let (E, ϕ) be a pair of this type, with values in L.

(7.10) *For m large enough, $(E, \phi(m))$ belongs to a family, parametrized by a smooth base Y, of Higgs pairs of type μ with values in $L(m)$ such that the deformation map $T_{(E,\phi(m))}Y \to \mathbf{H}^1\mathbf{End}'(E, \phi(m))$ is an isomorphism.*

Proof. The proof is parallel to that of the previous theorem. Let

$$0 = E^0 \subset \ldots \subset E^l = E$$

be the HN filtration of (E, ϕ), as usual, and let $E_i = E/E^i$. Choose k large enough that every $E_i(k)$ is generated by its sections and every $H^1(E^i(k)) = 0$. Then $H^1(E_i(k)) = 0$ as well, and $H^0(E(k)) \to H^0(E_i(k))$ is surjective. If

$$\chi = \dim H^0(E(k)),$$

a choice of basis for $H^0(E(k))$ then determines a sequence of quotients

$$\mathcal{O}^\chi \longrightarrow E(k) \xrightarrow{\psi_1} E_1(k) \xrightarrow{\psi_2} E_2(k) \xrightarrow{\psi_3} \ldots \xrightarrow{\psi_l} E_l(k)$$

determining a point q in the product of l different Quot schemes. Let R be the subspace of this product parametrizing flags of bundles; then T_qR is $\mathbf{H}^0(C^\bullet)$, where $C^\bullet$ is the third row in an exact sequence of two-term complexes

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\bigoplus_i \mathrm{Hom}(E_i, E_i) \arrow[r] \arrow[d] & \bigoplus_i \mathrm{Hom}(E_i, E_{i+1}) \arrow[d] \\
\bigoplus_i \mathrm{Hom}(\mathcal{O}^\chi, E_i) \arrow[r] \arrow[d] & \bigoplus_i \mathrm{Hom}(\mathcal{O}^\chi, E_{i+1}) \arrow[d] \\
\bigoplus_i \mathrm{Hom}(F_i, E_i) \arrow[r] \arrow[d] & \bigoplus_i \mathrm{Hom}(F_i, E_{i+1}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

and F_i is the kernel of the map $\mathcal{O}^\chi \to E_i$. Now the second row $B^\bullet$ is isomorphic to χ copies of $\bigoplus_i E_i \to \bigoplus_i E_{i+1}$, with the map given by $(b_i) \mapsto (\psi_{i+1} b_i - b_{i+1})$. Using the long exact sequence

$$\begin{aligned}
0 \longrightarrow \mathbf{H}^0(B^\bullet) &\longrightarrow \bigoplus_i H^0(E_i) \longrightarrow \bigoplus_i H^0(E_{i+1}) \\
\longrightarrow \mathbf{H}^1(B^\bullet) &\longrightarrow \bigoplus_i H^1(E_i) \longrightarrow \bigoplus_i H^1(E_{i+1}) \\
\longrightarrow \mathbf{H}^2(B^\bullet) &\longrightarrow 0
\end{aligned}$$

together with $H^1(E_i) = 0$ and the surjectivity of $H^0(E_i) \to H^0(E_{i+1})$, we see that a descending induction on i shows that $\mathbf{H}^1(B^\bullet) = \mathbf{H}^2(B^\bullet) = 0$. Finally, if $A^\bullet$ is the first row, from the short exact sequence

$$0 \longrightarrow \operatorname{End}' E \longrightarrow \bigoplus_i \operatorname{Hom}(E_i, E_i) \longrightarrow \bigoplus_i \operatorname{Hom}(E_i, E_{i+1}) \longrightarrow 0$$

we conclude that $\mathbf{H}^*(A^\bullet) = H^*(\operatorname{End}' E)$ and hence that $\mathbf{H}^0(C^\bullet) = T_q R$ surjects on $\mathbf{H}^1(A^\bullet) = H^1(\operatorname{End}' E)$. Moreover,

$$\mathbf{H}^2(A^\bullet) = H^2(\operatorname{End}' E) = 0.$$

Hence $\mathbf{H}^1(C^\bullet) = \mathbf{H}^2(C^\bullet) = 0$, so R is smooth at q.

Now choose m large enough that $H^1(\operatorname{End}' E \otimes L(m)) = 0$, and let Y be the total space of $\pi_*(\operatorname{End}' \mathbf{E} \otimes L(m))$, where $\mathbf{E}$ is the obvious tautological quotient on R, $\operatorname{End}' \mathbf{E} \subset \operatorname{End} \mathbf{E}$ is the subbundle of endomorphisms preserving the flag, and $\pi : R \times C \to R$ is the projection. Then Y is smooth in a neighborhood over q, contains a point representing $(E, \phi(m))$, and has a tautological Higgs field $\mathbf{\Phi} \in H^0(Y \times C, \operatorname{End}' \mathbf{E} \otimes L(m))$. The deformation map is surjective, as may be seen from the diagram

$$\begin{array}{ccccccc}
H^0(\operatorname{End}' E \otimes L(m)) & \longrightarrow & T_{(E,\phi(m))} Y & \longrightarrow & T_q R & \longrightarrow & 0 \\
{\scriptstyle =}\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
H^0(\operatorname{End}' E \otimes L(m)) & \longrightarrow & \mathbf{H}^1\, \mathbf{End}'(E, \phi(m)) & \longrightarrow & H^1 \operatorname{End}' E & \longrightarrow & 0
\end{array}$$

□

8. *The infinite-dimensional stratification*

Let $\mathcal{E}$ be a Hermitian vector bundle over C of rank r and degree d. A rigorous construction of $\mathcal{H}_n$ as an infinite-dimensional quotient involves connections and sections in Sobolev spaces associated to $\mathcal{E}$. So choose any $k \geqslant 2$; Atiyah and Bott prefer $k = 2$, but for us, as for them, any greater k will also do. Then, for any Hermitian bundle $\mathcal{V}$ over C, denote by $\Omega^{p,q}(\mathcal{V})$ the Banach space consisting of sections, of Sobolev class L^2_{k-p-q}, of the bundle of differential forms of types p and q with values in $\mathcal{V}$. Also let $\mathcal{A}$ be the space of holomorphic structures on $\mathcal{E}$ differing from a fixed C^∞ one by an element of the Sobolev space $\Omega^{0,1}(\operatorname{End} \mathcal{E})$. We hope the reader will pardon the unorthodox use of Ω to refer to a Sobolev completion, rather than just the space of smooth forms.

Define a map

$$\overline{\partial} : \mathcal{A} \times \Omega^{1,0}(\operatorname{End} \mathcal{E} \otimes K(n)) \longrightarrow \Omega^{1,1}(\operatorname{End} \mathcal{E} \otimes K(n))$$

by $\overline{\partial}(E, \phi) = \overline{\partial}_E \phi$, and let $\mathcal{B}_n = \overline{\partial}^{-1}(0)$. This parametrizes all pairs where ϕ is holomorphic.

Let $\mathcal{G}$ be the complex gauge group consisting of all complex automorphisms of $\mathcal{E}$ of Sobolev class L^2_k. Then $\mathcal{G}$ acts naturally and smoothly on $\mathcal{A}$ as shown by Atiyah and Bott, and likewise on $\Omega^{1,0}(\operatorname{End} \mathcal{E} \otimes K(n))$ since L^2_{k-1} is a topological L^2_k-module. The $\mathcal{G}$-action on the product of these spaces preserves $\mathcal{B}_n$.

(8.1) *Every $\mathcal{G}$-orbit in $\mathcal{B}_n$ has a C^∞ representative (E, ϕ), and any two are interchanged by a C^∞ gauge transformation. The stabilizer of (E, ϕ) is the group of holomorphic automorphisms of E preserving ϕ.*

Proof. According to Lemma 14.8 of Atiyah and Bott's paper, every $\mathcal{G}$-orbit in $\mathcal{A}$ contains a C^∞ representative. If $(E, \phi) \in \mathcal{B}_n$, so that ϕ satisfies the elliptic equation $\overline{\partial}_E \phi = 0$, it follows from elliptic regularity that ϕ is also C^∞. If (E, ϕ) and (E', ϕ') are pairs in the same $\mathcal{G}$-orbit, then by Lemma 14.9 of Atiyah and Bott, any gauge transformation interchanging E and E' is C^∞; hence the same is true for the pairs. Finally, if an element of $\mathcal{G}$ preserves (E, ϕ), this means precisely that it preserves $\overline{\partial}_E$, and hence is a holomorphic automorphism, and fixes ϕ. □

(8.2) *Let $(E, \phi) \in \mathcal{B}_n$ be a C^∞ pair. Then the normal space to the $\mathcal{G}$-orbit at (E, ϕ) is canonically isomorphic to $\mathbf{H}^1\, \mathbf{End}(E, \phi)$, and the cokernel of the derivative of $\overline{\partial}$ at (E, ϕ) is canonically isomorphic to $\mathbf{H}^2\, \mathbf{End}(E, \phi)$.*

Proof. The infinitesimal action of the Lie algebra of $\mathcal{G}$ is the map f, and the derivative of $\overline{\partial}$ is the map g, in the complex

$$\Omega^{0,0}(\operatorname{End}\mathcal{E}) \xrightarrow{f} \Omega^{0,1}(\operatorname{End}\mathcal{E}) \oplus \Omega^{0,0}(\operatorname{End}\mathcal{E} \otimes K(n)) \xrightarrow{g} \Omega^{0,1}(\operatorname{End}\mathcal{E} \otimes K(n)),$$

where $f(a) = (-\overline{\partial}_E(a), [a, \phi])$ and $g(b, c) = \overline{\partial}_E(c) + [b, \phi]$. The symbol sequence of this complex is the direct sum of those of the Dolbeault complexes of $\operatorname{End} E$ and $\operatorname{End} E \otimes K(n)$, so it is elliptic. By elliptic regularity the cohomology of the complex is then the same as its counterpart where the Sobolev spaces are replaced by spaces of smooth forms; this is precisely the Dolbeault hypercohomology of $\mathbf{End}(E, \phi)$. □

(8.3) *For each $(E, \phi) \in \mathcal{B}_n$, $(E, \phi(m))$ is a smooth point of $\mathcal{B}_{m+n}$ for sufficiently large m.*

Proof. Choose m large enough that $H^1(\operatorname{End} E \otimes K(m+n)) = 0$. Then from the long exact sequence

$$\ldots \longrightarrow H^1(\operatorname{End} E) \longrightarrow H^1(\operatorname{End} E \otimes K(m+n)) \longrightarrow \mathbf{H}^2\, \mathbf{End}(E, \phi(m)) \longrightarrow 0$$

associated to the hypercohomology of a two-term complex, $\mathbf{H}^2\, \mathbf{End}(E, \phi(m)) = 0$. Hence $\overline{\partial}$ is a submersion at $(E, \phi(m))$, and the implicit function theorem for Banach manifolds [**9**, A3] implies that $\mathcal{B}_{m+n} = \overline{\partial}^{-1}(0)$ is a smooth embedded Banach submanifold in a neighborhood of $(E, \phi(m))$. □

Let us now find a slice for the $\mathcal{G}$-action, using the results of the previous section.

(8.4) *Let (E, ϕ) be a C^∞ pair in $\mathcal{B}_n$. Then for m large enough, there is a $\mathcal{G}$-equivariant submersion $\mathcal{G} \times U \to \mathcal{B}_{m+n}$ onto a neighborhood of $(E, \phi(m))$, where U is an open neighborhood of x in the algebraic family of* (7.9).

Proof. Result (7.9) provides a family $(\mathbf{E}, \mathbf{\Phi})$ of pairs over some smooth $X \ni x$ such that $(\mathbf{E}, \mathbf{\Phi})_x = (E, \phi(m))$ and the natural map $T_x X \to \mathbf{H}^1\, \mathbf{End}(E, \phi(m))$ is an isomorphism. Choose a Hermitian metric on $\mathbf{E}$ extending the given one on E. This determines a smooth map $X \to \mathcal{B}_{m+n}$, which by (8.2) is transverse to the $\mathcal{G}$-orbit. It extends to a $\mathcal{G}$-equivariant map $\mathcal{G} \times X \to \mathcal{B}_{m+n}$ whose derivative is a surjection at $\mathcal{G} \times \{x\}$, and hence in some neighborhood $\mathcal{G} \times U$, thanks to the diagram

$$\begin{tikzcd} T_e\mathcal{G} \arrow[r] \arrow[d, "="'] & T_{(e,x)}(\mathcal{G} \times X) \arrow[r] \arrow[d] & T_xX \arrow[r] \arrow[d] & 0 \arrow[d] \\ \Omega^{0,0}(\operatorname{End} E) \arrow[r] & T\mathcal{B}_n \arrow[r] & \mathbf{H}^1\,\mathbf{End}(E,\phi) \arrow[r] & 0 \end{tikzcd}$$ □

Let $\mathcal{B}_n^\mu$ denote the union of all $\mathcal{G}$-orbits in $\mathcal{B}_n$ whose C^∞ representatives have type μ in the sense of § 7. In particular, let $\mathcal{B}_n^s$ denote the stable orbits.

(8.5) *For any μ, $\bigcup_{\nu \geqslant \mu} \mathcal{B}_n^\nu$ is closed in $\mathcal{B}_n$.*

Proof. This follows immediately from (8.4) and the corresponding fact for X, (7.4). □

Let $\overline{\mathcal{G}}$ be the quotient of $\mathcal{G}$ by the central subgroup $\mathbb{C}^\times$. Since by (4.1) the latter is the stabilizer of all stable pairs, each stable orbit is isomorphic to $\overline{\mathcal{G}}$.

(8.6) *The natural map $\mathcal{B}_n^s \to \mathcal{H}_n$ is a principal $\overline{\mathcal{G}}$-bundle.*

Proof. The submersions of (8.4) descend to maps $\overline{\mathcal{G}} \times U \to \mathcal{B}_n$, whose derivatives are isomorphisms on $\overline{\mathcal{G}} \times \{x\}$, and which can be made injective by shrinking U if necessary. By the inverse function theorem for Banach manifolds [**9**, A1] these are $\overline{\mathcal{G}}$-equivariant diffeomorphisms onto their images. They therefore constitute an atlas of local trivializations. □

Given a C^∞ pair $(E,\phi) \in \mathcal{B}_n$, define $\operatorname{End}'' E$ by the short exact sequence

$$0 \longrightarrow \operatorname{End}' E \longrightarrow \operatorname{End} E \longrightarrow \operatorname{End}'' E \longrightarrow 0,$$

where $\operatorname{End}' E$, as before, is the subsheaf of $\operatorname{End} E$ preserving the HN filtration of (E,ϕ). Also let $\mathbf{End}''(E,\phi)$ be the two-term complex on C defined analogously to $\mathbf{End}(E,\phi)$ and $\mathbf{End}'(E,\phi)$. There is then a short exact sequence of two-term complexes

$$\mathbf{0} \longrightarrow \mathbf{End}'(E,\phi) \longrightarrow \mathbf{End}(E,\phi) \longrightarrow \mathbf{End}''(E,\phi) \longrightarrow \mathbf{0}.$$

(8.7) *For any $(E,\phi) \in \mathcal{B}_n$ of type μ, and for m large enough, $\mathcal{B}_{m+n}^\mu$ is an embedded submanifold of $\mathcal{B}_{m+n}$ near $(E,\phi(m))$ with normal space canonically isomorphic to $\mathbf{H}^1\,\mathbf{End}''(E,\phi(m))$.*

Proof. By acting with an element of $\mathcal{G}$ if necessary we may assume that (E,ϕ) is C^∞.

For m large, there was constructed in (7.10) a family of pairs $(\mathbf{E},\mathbf{\Phi})$ of type μ over a smooth base $Y \ni y$, having $(\mathbf{E},\mathbf{\Phi})_y = (E,\phi(m))$ and $T_yY \to \mathbf{H}^1\,\mathbf{End}'(E,\phi(m))$ an isomorphism. Choose a metric on $\mathbf{E}$ extending the given one on E. This determines a smooth map $Y \to \mathcal{B}_{m+n}$, which by (8.2) is transverse to the $\mathcal{G}$-orbit.

On an open neighborhood V of y in Y, choose a lifting of this map to the domain of the submersion $\mathcal{G} \times U \to \mathcal{B}_{m+n}$ of (8.4). Projecting this lifting to U gives a map $V \to U$ whose image consists of pairs of type μ, and hence is contained in X^μ.

Its derivative at y is the natural map from $T_yY = \mathbf{H}^1\,\mathbf{End}'(E,\phi(m))$ to $T_xX = \mathbf{H}^1\,\mathbf{End}(E,\phi(m))$.

This derivative is injective. Indeed, the kernel is the image of $\mathbf{H}^0\,\mathbf{End}''(E,\phi(m))$. But if $0 = E^0 \subset \ldots \subset E^l = E$ is the HN filtration of (E,ϕ) as usual, then there is a short exact sequence

$$0 \longrightarrow \mathrm{End}''\,E^{l-1} \longrightarrow \mathrm{End}''\,E \longrightarrow \mathrm{Hom}(E^{l-1}, E/E^{l-1}) \longrightarrow 0$$

and hence a short exact sequence of the corresponding two-term complexes. The first hypercohomology $\mathbf{H}^0$ of both of the outer complexes vanishes by an induction on l using (4.1), so $\mathbf{H}^0\,\mathbf{End}''(E,\phi(m)) = 0$ too.

Hence, in a neighborhood of x, X^μ contains an embedded submanifold with tangent space $\mathbf{H}^1\,\mathbf{End}'(E,\phi(m)) \subset \mathbf{H}^1\,\mathbf{End}(E,\phi(m))$. But by (7.8), X^μ is a subscheme of X with Zariski tangent space contained in $\mathbf{H}^1\,\mathbf{End}'(E,\phi(m))$. It therefore must be smooth near x.

The inverse image of $\mathcal{B}^\mu_{m+n}$ under the submersion $\mathcal{G} \times U \to \mathcal{B}_{m+n}$ is $\mathcal{G} \times (U \cap X^\mu)$, so this immediately implies that $\mathcal{B}^\mu_{m+n}$ is a smoothly embedded submanifold in a neighborhood of the orbit of $(E,\phi(m))$. Its normal space at $(E,\phi(m))$ is the quotient of $\mathbf{H}^1\mathbf{End}(E,\phi(m))$ by $\mathbf{H}^1\,\mathbf{End}'(E,\phi(m))$; by choosing m large enough we may arrange as in (8.3) that $\mathbf{H}^2\,\mathbf{End}'(E,\phi(m)) = 0$, so that this quotient is nothing but $\mathbf{H}^1\,\mathbf{End}''(E,\phi(m))$. □

9. *The direct limit of Higgs spaces*

The inclusions $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$ make the set of all $\mathcal{B}_n$ into a directed set. Let $\mathcal{B}_\infty$ be the direct limit. It may be regarded as a set of pairs (E,ϕ) as before, but where ϕ may now have a pole of arbitrary finite order at p. Note that for each type μ, the direct limit of $\mathcal{B}^\mu_n$ is a subset $\mathcal{B}^\mu_\infty$ of $\mathcal{B}_\infty$. Note also that $\mathcal{G}$ acts naturally on $\mathcal{B}_\infty$ and that $\mathcal{B}^s_\infty/\mathcal{G}$ is $\mathcal{H}_\infty$, the direct limit of the $\mathcal{H}_n$. In another context, $\mathcal{H}_n$ has appeared in the work of Donagi and Markman [**7**].

Our aim in this section is to show that $\mathcal{B}^s_\infty$ is contractible. Essentially, the reason is that $\mathcal{B}_\infty$ is contractible, and the complement of the stable set has infinite codimension.

Recall that a subspace of a topological space is a *deformation neighborhood retract* (hereinafter DNR) if it is the image of a map defined on some open neighborhood of itself and homotopic to the identity. It is *equivariant* if the homotopy is equivariant for the action of some group.

(9.1) *As a subspace of $\mathcal{H}_\infty$, each $\mathcal{H}_n$ is a DNR, and the open sets which retract can be chosen to be nested.*

Proof. Since $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$ is an embedding of finite-dimensional manifolds, we may choose a tubular neighborhood U^1_n and a projection $U^1_n \to \mathcal{H}_n$. This tubular neighborhood in turn has a tubular neighborhood U^2_n in $\mathcal{H}_{n+2}$, namely its inverse image in U^1_{n+1}, and so on. The direct limits U^∞_n of these tubular neighborhoods are nested open subsets of $\mathcal{H}_\infty$. Each is homeomorphic to a vector bundle over $\mathcal{H}_n$ and hence deformation retracts onto it. Indeed, the deformation retraction preserves each U^i_n. □

(9.2) *As a subspace of* $\mathcal{B}_\infty$, *each* $\mathcal{B}^s_n$ *is a* $\mathcal{G}$*-equivariant DNR, and the* $\mathcal{G}$*-invariant open sets which retract can be chosen to be nested.*

Proof. Let $\pi : \mathcal{B}^s_\infty \to \mathcal{H}_\infty$ be the quotient map, and let U^i_n be the tubular neighborhoods of the previous proof. By the definition of direct limit, it suffices to construct a sequence of $\mathcal{G}$-equivariant deformation retractions of $\pi^{-1}(U^i_n)$ onto $\mathcal{B}^s_n = \pi^{-1}(\mathcal{H}_n)$ compatible with the inclusions $\pi^{-1}(U^i_n) \subset \pi^{-1}(U^{i+1}_n)$. Obviously we would like to lift the deformation retracts of the previous proof to the principal $\mathcal{G}$-bundle.

These liftings are guaranteed to exist by the first covering homotopy theorem. This asserts that if $F : Y \times [0,1] \to Z$ is a homotopy with any reasonable domain (including manifolds, but unfortunately not their direct limits), and if E is a fiber bundle over Z, then F^*E is isomorphic as a fiber bundle to $F^*E|_{Y\times 0} \times [0,1]$. Actually, a slight refinement of this result is needed, namely that the isomorphism can be chosen so as to extend a given one over a closed DNR $X \subset Y$. We then apply this refined result to the case where $X = U^i_n$, $Y = U^{i+1}_n$, and the homotopy is the retraction of U^i_n on $\mathcal{H}_n$ described above. It is easy to construct the desired deformation retraction of $\pi^{-1}(U^i_n)$ from the resulting isomorphism.

The slight refinement of the first covering homotopy theorem can be proved by the same argument as the theorem itself, given by Steenrod [**42**, § 11.3]. Just choose the atlas for F^*E so that its restriction to $X \times [0,1]$ is pulled back from an atlas on X using the given isomorphism. The existence of such an atlas follows easily from the fact that X is a closed DNR and the ordinary version of the theorem. □

(9.3) *The quotient map* $\pi : \mathcal{B}^s_\infty \to \mathcal{H}_\infty$ *is a principal* $\overline{\mathcal{G}}$*-bundle.*

Proof. This is immediate from (8.6) and (9.2). □

(9.4) *For all* $k \geqslant 0$, $\pi_k(\mathcal{B}^s_\infty) = 1$.

Proof. The open subsets of $\mathcal{B}_\infty$ provided by (9.2), which retract onto $\mathcal{B}^s_n$, form a nested open cover of $\mathcal{B}^s_\infty$. By compactness, any map $S^k \to \mathcal{B}^s_\infty$, and any homotopy of such maps, has image contained in one such neighborhood. Hence $\pi_k(\mathcal{B}^s_\infty) = \lim_{n\to\infty} \pi_k(\mathcal{B}^s_n)$.

So let $f : S^k \to \mathcal{B}^s_n$ be any map. Now $\mathcal{B}_n$ is contractible just by retracting it first on $\mathcal{A} \times \{0\}$, so f certainly extends to a continuous map $f : D^{k+1} \to \mathcal{B}_n$. Our task is to perturb this so that it misses the unstable locus.

For each $x \in D^{k+1}$, by (8.3) and (8.7) there is some integer $m_x \geqslant n$ such that for all $m \geqslant m_x$, $f(x)$ is a smooth point of $\mathcal{B}_m$, and the stratum $\mathcal{B}^\mu_m$ containing $f(x)$ is an embedded submanifold at $f(x)$. Passing to a finite subcover and taking $m = \max m_x$, we find that $f(D^{k+1})$ maps entirely into the smooth locus of $\mathcal{B}_m$, and that near its image each stratum $\mathcal{B}^\mu_m$ is an embedded submanifold of finite codimension.

Yet another application of compactness shows that $f(D^{k+1})$ intersects only a finite number of strata $\mathcal{B}^\mu_m$. By increasing m again if necessary we may assume by (8.7) that each of these strata has codimension greater than $k+1$. Then, starting with the stratum of highest codimension and working our way up, we may perturb

f so that it no longer touches that stratum, but so that its value on S^k remains unchanged. After these perturbations, f will have image entirely within $\mathcal{B}^s_m$.

Thus for any $f : S^k \to \mathcal{B}^s_n$, the homotopy class of f is killed by the inclusion in $\mathcal{B}^s_m$ for some $m \geqslant n$, so $\lim_{n\to\infty} \pi_k(\mathcal{B}^s_n) = 1$. □

For an alternate proof in the rank 2 case, see the first author's thesis [**16**, 7.5.1].

(9.5) *The space $\mathcal{B}^s_\infty$ is contractible.*

Proof. A theorem of Whitehead [**43**, 10.28] asserts that if X is a CW-space (that is, a space homotopy equivalent to a CW-complex) whose homotopy groups all vanish, then it is contractible. In the light of (9.4) above, it therefore suffices to show that $\mathcal{B}^s_\infty$ is a CW-space.

Consider the fiber bundle

$$\mathcal{B}^s_\infty \longrightarrow \frac{\mathcal{B}^s_\infty \times E\mathcal{G}}{\mathcal{G}} \longrightarrow B\mathcal{G}. \tag{9.6}$$

We will show that this is a fibration whose total space and base space are CW-spaces. It then follows from Corollary 13 of Stasheff [**14**] that the fiber is a CW-space.

Note that $\mathcal{G}$ acts on $\mathcal{B}^s_\infty$ with stabilizer $\mathbb{C}^\times$ and quotient $\mathcal{H}_\infty$. There is therefore a fiber bundle

$$B\mathbb{C}^\times \longrightarrow \frac{\mathcal{B}^s_\infty \times E\mathcal{G}}{\mathcal{G}} \longrightarrow \mathcal{H}_\infty,$$

which is just the associated bundle to the principal $\overline{\mathcal{G}}$-bundle $\mathcal{B}^s_\infty \to \mathcal{H}_\infty$. The base of this fiber bundle, being a direct limit of manifolds, is metrizable and hence paracompact by Stone's theorem [**31**, 6-4.3]; hence the fiber bundle is a fibration [**47**, Chapter I, 7.13]. Moreover, the base is a CW-space, as is the fiber. Proposition 0 of Stasheff [**41**] then implies that the total space is a CW-space.

According to Proposition 2.4 of Atiyah and Bott [**1**], $B\mathcal{G}$ is a component of the space of maps from C to BG, with the compact-open topology. Since the domain is a compact metric space and the range is a CW-complex, by Corollary 2 of Milnor [**29**] the space of maps is a CW-space. Moreover, since BG is metrizable and C is compact, the space of maps $B\mathcal{G}$ is metrizable, and hence paracompact. The fiber bundle (9.6) is therefore a fibration. □

(9.7) *The space $\mathcal{H}_\infty$ is homotopy equivalent to $B\overline{\mathcal{G}}$.*

Proof. By (9.3) and (9.5), there is a principal $\overline{\mathcal{G}}$-bundle on $\mathcal{H}_\infty$ with contractible total space. □

Those who dislike the appearance of infinite-dimensional, gauge-theoretic methods in the last two sections may wish to reflect that it is no doubt possible to replace every infinite-dimensional construction by a finite-dimensional, algebraic approximation, in the style of Bifet *et al.* [**4**] or Kirwan [**25**]. In any case, algebraic geometry will reappear on the scene shortly.

10. *Surjectivity of the restriction on cohomology*

Let $\mathcal{E}$ be as in § 8. The pull-back of $\mathcal{E} \to C$ to the product $\mathcal{B}^s_\infty \times C$ is acted on by $\mathcal{G}$, so the projective bundle $\mathbb{P}\mathcal{E}$ is acted on by $\overline{\mathcal{G}}$. It therefore descends to a $\mathbb{P}^r$-bundle $\mathbb{P}\mathbf{E}$ over $\mathcal{H}_\infty \times C$, whose characteristic classes can be decomposed as usual into Künneth components:

$$\overline{c}_i = \alpha_i \sigma + \sum_{j=1}^{2g} \psi_{i,j} e_j + \beta_i.$$

Likewise, the natural determinant maps $\mathcal{H}_n \to \mathrm{Jac}^d C$ given by $(E, \phi) \mapsto \Lambda^r E$ extend to $\mathcal{H}_\infty \to \mathrm{Jac}^d C$. Let $\varepsilon_1, \ldots, \varepsilon_{2g}$ be the pull-backs of the standard generators of $H^1(\mathrm{Jac}^d C)$.

It is straightforward to check that the universal classes α_i, β_i, $\psi_{i,j}$, ε_j thus defined restrict to their counterparts on $\mathcal{H}_n$ for $n \geqslant 0$.

(10.1) *The rational cohomology ring $H^*(\mathcal{H}_\infty)$ is generated by these universal classes.*

Proof. According to Atiyah and Bott, $H^*(B\mathcal{G})$ is generated by universal classes, which means the following. First of all, $B\mathcal{G}$ can be identified with the component of the space of maps $C \to B\mathrm{U}(r)$ such that the pull-back of the universal bundle over $B\mathrm{U}(r)$ is isomorphic to $\mathcal{E}$. Atiyah and Bott call this component $\mathrm{Map}_{\mathcal{E}}(C, B\mathrm{U}(r))$. Then, the pull-back of the universal bundle by the canonical map $\mathrm{Map}_{\mathcal{E}}(C, B\mathrm{U}(r)) \times C \to B\mathrm{U}(r)$ is a bundle whose Chern classes can be decomposed into Künneth components as usual. Atiyah and Bott prove [**1**, 2.20] that these generate the ring $H^*(B\mathcal{G})$.

On the other hand, by (9.7), $B\overline{\mathcal{G}}$ can also be identified with $\mathcal{H}_\infty$. Hence $B\mathcal{G}$ is a bundle over $\mathcal{H}_\infty$ with fiber $B\mathbb{C}^\times = \mathbb{CP}^\infty$. As explained by Atiyah and Bott, the rational cohomology of this bundle splits: $H^*(B\mathcal{G}) = H^*(B\overline{\mathcal{G}})[h]$. By restricting to a single $\mathbb{CP}^\infty$ fiber, it can be checked that $\beta_1 = rh$ modulo elements of $H^2(B\overline{\mathcal{G}})$; it may therefore be discarded since we seek only generators of $H^*(B\overline{\mathcal{G}})$. Also $\alpha_1 \in H^0(B\mathcal{G})$ may be discarded since, by (9.4), $B\mathcal{G}$ is connected. Finally, it can be checked that $\psi_{1,j} = \varepsilon_j$, and that for $i > 1$, the classes α_i, β_i and $\psi_{i,j}$ of Atiyah and Bott agree with those defined above. (Strictly speaking, they may differ by some lower order terms, since the characteristic classes of a projective bundle are evaluated by formally twisting so that $c_1 = 0$.) □

Now by (9.1), $H_*(\mathcal{H}_\infty)$ is the direct limit of $H_*(\mathcal{H}_n)$, and hence $H^*(\mathcal{H}_\infty)$ is the inverse limit of $H^*(\mathcal{H}_n)$. Consequently, the surjectivity of the restriction map $H^*(\mathcal{H}_\infty) \to H^*(\mathcal{H}_n)$ for all k, and hence the generation theorem (6.1), is implied by the following result, whose proof occupies the remainder of this section.

(10.2) *When $r = 2$, the restriction $H^*(\mathcal{H}_{n+1}) \to H^*(\mathcal{H}_n)$ is surjective.*

Let $T = \mathbb{C}^\times$ act on each $\mathcal{H}_n$ by $\lambda \cdot (E, \phi) = (E, \lambda\phi)$. This action is compatible with the inclusion $\mathcal{H}_n \hookrightarrow \mathcal{H}_{n+1}$. Furthermore, since this is an algebraic action on a smooth quasi-projective variety, the U(1)-part of the action is Hamiltonian, and the $\mathbf{R}^\times$ part of the action is the Morse flow of the moment map.

(10.3) *For any $(E, \phi) \in \mathcal{H}_n$, there exists a limit* $\lim_{\lambda \to 0}(E, \lambda\phi) \in \mathcal{H}_n$.

Proof. We may regard this as a limit of the downward Morse flow in $\mathcal{H}_n$. Note that it need not be simply $(E,0)$ as this may be unstable. Nevertheless, a stable limit always exists; this may be seen in two ways.

First, one can regard $\mathcal{H}_n$ as a space of solutions (A,ϕ) to the self-duality equations, as Hitchin [**20**] regards $\mathcal{H}$; the moment map is then $(A,\phi)\mapsto\|\phi\|^2$, and what we need to know is that this is proper and bounded below. The boundedness is obvious, and the properness is proved by following Hitchin's argument for $\mathcal{H}$ [**20**, 7.1(i)].

Alternatively and more algebraically, one can observe, as does Simpson [**37**, Proposition 3], that the Hitchin map defined by Nitsure [**34** 6.1], taking $\mathcal{H}_n$ holomorphically to a vector space, is proper and intertwines the T-action on $\mathcal{H}$ with a linear action on the vector space having positive weights. A limit must therefore exist in the zero fiber of the Hitchin map. □

Now any U(1) moment map whose Morse flows have lower limits is a *perfect* Bott–Morse function: see, for example, Kirwan [**24**, 9.1]. This means that its Morse inequalities are equalities. More explicitly, it means the following. Let $y_0,\ldots,y_k$ be the critical values of the moment map $\mu : X\to\mathbf{R}$, and F_i the corresponding critical submanifolds. Choose real numbers x_i so that $x_0<y_0<x_1<y_1<\ldots<y_k<x_{k+1}$. If $X_i=\mu^{-1}(x_0,x_i)$, then, as for any Bott–Morse function, there is a homotopy equivalence of pairs $(X_{i+1},X_i)\simeq(D_i,S_i)$, where D_i is the disc bundle, and S_i the sphere bundle, associated to the *negative normal bundle* of F_i, that is, the bundle of downward Morse flows; cf. Milnor [**30**]. For the function to be *perfect* means that, moreover, the connecting homomorphism vanishes in each long exact sequence

$$\ldots\longrightarrow H^*(X_{i+1},X_i)\longrightarrow H^*(X_{i+1})\longrightarrow H^*(X_i)\longrightarrow\ldots,$$

breaking it up into short exact sequences.

Suppose now that X contains a T-invariant submanifold Y on which the moment map is again perfect. Then by induction on i, $H^*(X)$ surjects on $H^*(Y)$ if and only if $H^*(X_{i+1},X_i)$ surjects on $H^*(Y_{i+1},Y_i)$ for all i.

We find ourselves in this situation, with $X=\mathcal{H}_{n+1}$ and $Y=\mathcal{H}_n$. To prove (10.2), it therefore suffices to show that the relative cohomology of the disc bundle for the downward flow from each critical submanifold in $\mathcal{H}_{n+1}$ surjects on that of its intersection with $\mathcal{H}_n$. This will be true in the case $r=2$. Indeed, the Thom isomorphism identifies this relative cohomology with the ordinary cohomology of the critical submanifold itself. We will prove, first, that this identification is compatible with the restriction to $\mathcal{H}_n$, and second, that the latter restriction is surjective. Both will follow from the description below of the critical set (cf. Hitchin [**20**, 7.1]).

(10.4) *The critical submanifolds of the moment map on $\mathcal{H}_n$ are a disjoint union*

$$\mathcal{F}_n=\coprod_{j=0}^{g+[(n-1)/2]}F_n^j,$$

where:

(a) *for $j=0$, the absolute minimum F_n^0 of the moment map is the moduli space of stable bundles of rank 2 and degree d, parametrizing Higgs bundles (E,ϕ) with $\phi=0$;*

(b) *for* $j > 0$, $F_n^j = \mathrm{Jac}^{(d+1)/2-j}C \times \mathrm{Sym}^{2g+n-1-2j}C$, *parametrizing Higgs bundles* (E, ϕ) *with* $E = L \oplus M$, $\deg L = \frac{1}{2}(d+1) - j$, *and*

$$\phi = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix},$$

where $s \in H^0(KML^{-1}(n))$ *vanishes on an effective divisor of degree* $2g + n - 1 - 2j$.

Proof. The critical points for the moment map of the action of $\mathrm{U}(1) \subset T$ are exactly the fixed points of T. For any $(E, \phi) \in \mathcal{H}_n$ fixed by T, by (4.4), T acts by automorphisms on the universal bundle restricted to $\{(E, \phi)\} \times C$, which is nothing but E itself, and $\lambda \in T$ takes ϕ to $\lambda\phi$. If the weights of the action are distinct, this splits E as a sum of line bundles $L \oplus M$, and ϕ is forced to be of the stated form. If the weights are not distinct, then T acts by scalars, so ϕ is invariant and hence must be 0. □

(10.5) *The downward flow from* $F_n^j = F_{n+1}^j \cap \mathcal{H}_n$ *in* $\mathcal{H}_{n+1}$ *is wholly contained in* $\mathcal{H}_n$.

Proof. The statement is vacuous for $j = 0$, since at the absolute minimum there is no downward flow. Consider then a point $(E, \phi) \in F_n^j$ for $j > 0$, as described in (10.4)(b). According to (3.1), the tangent space to $\mathcal{H}_{n+1}$ at (E, ϕ) is the hypercohomology $\mathbf{H}^1\, \mathbf{End}(E, \phi)$, where $\mathbf{End}(E, \phi)$ is the two-term complex

$$\mathrm{End}\, E \xrightarrow{[\ ,\phi]} \mathrm{End}\, E \otimes K(n+1).$$

Since $E = L \oplus M$ and ϕ is strictly lower-triangular, this breaks up as a direct sum of complexes, which are the weight spaces for the T-action. The downward flow corresponds to $\mathbf{H}^1$ of the complex

$$\mathrm{Hom}(L, M) \longrightarrow 0,$$

which is of course just $H^1(ML^{-1})$. This is independent of n, and so the downward flow in $\mathcal{H}_{n+1}$ is wholly contained in $\mathcal{H}_n$, as desired. □

Consequently, we may use the Thom isomorphisms to identify the map of relative cohomology on the disc bundles with the ordinary restriction map $H^*(\mathcal{F}_{n+1}) \to H^*(\mathcal{F}_n)$. To prove (10.2), then, it remains to prove the following statement.

(10.6) *The restriction* $H^*(\mathcal{F}_{n+1}) \to H^*(\mathcal{F}_n)$ *is surjective.*

Proof. Again, this is vacuous for $j = 0$, since $F_{n+1}^0 = F_n^0$. It is not much harder for $j > 0$, for F_n^j is isomorphic to a product of a Jacobian and a symmetric product, and the embedding $F_n^j \hookrightarrow F_{n+1}^j$ corresponds to the identity on the first factor and a map of effective divisors of the form $D \mapsto D + p$ on the second. The latter map is easily seen (for example, from the description of Macdonald [**26**]) to induce a surjection on cohomology. □

This completes the proof of (10.2), and hence of the generation theorem (6.1).

Note, by the way, that the theorem is *false* for the moduli space $\mathcal{M}_0$ consisting of pairs with fixed determinant line bundle $\Lambda^n E$ and trace-free ϕ. One can see this already from Hitchin's description [**20**, 7.6] of its cohomology: the universal classes are all invariant under the natural action of $\Sigma = \mathbb{Z}_2^{2g}$, but there also exist classes which are not Σ-invariant. (See also our companion paper [**19**, § 4].)

In the cases of ranks 2 and 3, the first author has found an alternative proof of the generation theorem, which will appear in a forthcoming paper [**18**]. It uses the vector bundles over $\mathcal{H}_n$ whose Chern classes above their ranks furnish the so-called 'Mumford relations'; these are well known to have the appropriate dimension in the $\mathrm{GL}(r)$ case, but not in the $\mathrm{SL}(r)$ case.

The generation theorem does, however, tell us the following about the cohomology ring of the fixed-determinant moduli space.

(10.7) *Let Ξ be a fixed line bundle of odd degree, and let $\mathcal{M}_n$ be the moduli space of Higgs bundles $(E, \phi) \in \mathcal{H}_n$ with $\Lambda^2 E \cong \Xi$ and $\operatorname{tr}\phi = 0$. Then $\Sigma \cong \mathbb{Z}_2^{2g}$ acts naturally on $\mathcal{M}_n$, and*

$$H^*(\mathcal{H}_n) = H^*(\operatorname{Jac} C) \otimes H^*(\mathcal{M}_n)^{\Sigma},$$

where $H^*(\operatorname{Jac} C)$ is generated by the $\varepsilon_1, \ldots, \varepsilon_{2g}$, and $H^*(\mathcal{M}_n)^{\Sigma}$ by the remaining universal classes.

Proof. In fact $\mathcal{H}_n$ is the quotient of $T^* \operatorname{Jac} C \times \mathcal{M}_n$ by the free action of Σ; the quotient map is $(L, \psi) \times (E, \phi) \mapsto (L \otimes E, \psi\, \mathrm{id} + \phi)$. A theorem of Grothendieck referred to earlier [**12**] asserts that the rational cohomology of a quotient by a finite group is the invariant part of the rational cohomology. Hence

$$H^*(\mathcal{H}_n) = H^*(T^* \operatorname{Jac} C \times \mathcal{M}_n)^{\Sigma} = H^*(\operatorname{Jac} C) \otimes H^*(\mathcal{M}_n)^{\Sigma},$$

since Σ acts on $T^* \operatorname{Jac} C$ only by translations. The determinant map $\mathcal{H}_n \to \operatorname{Jac}^d C$ lifts to the map $T^* \operatorname{Jac} C \times \mathcal{M}_n \to \operatorname{Jac}^d C$ given by projection on the first factor followed by an isogeny of order r; hence $\varepsilon_1, \ldots, \varepsilon_{2g}$ generate the rational cohomology of the first factor. A universal pair on $\mathcal{H}_n$ pulls back to the tensor product of the Poincaré line bundle on $\operatorname{Jac} C \times C$ with a universal pair on $\mathcal{M}_n \times C$; hence its projectivization, and consequently the remaining universal classes, are pulled back from the second factor. □

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