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ON A GALLAI-TYPE PROBLEM FOR LATTICES

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1. Introduction

Motivated by the well-known Helly-theorem [2], Gallai [1] raised the following problem in the Euclidean plane $\mathbf{E}^2$. Let $\mathcal{D}$ denote a finite collection of closed disks in $\mathbf{E}^2$ such that every two disks of $\mathcal{D}$ intersect. Find the minimum integer n with the property that for an arbitrary $\mathcal{D}$ there are n points in $\mathbf{E}^2$ such that every disk of $\mathcal{D}$ contains at least one of the points. Independently from each other, Danzer (unpublished) and Stachó [3] proved that $n \leqq 4$ i.e. any $\mathcal{D}$ can be pinned down by 4 needles. An analogous problem arises if the needles can be chosen from a rather regular subset of $\mathbf{E}^2$ only. Let $\mathbf{L}$ be the lattice of $\mathbf{E}^2$, i.e. the set of points of $\mathbf{E}^2$ which have integer coordinates.

It is easy to prove the following Helly-type theorem (see [4]). If $\mathcal{F}$ is a finite collection of convex sets in $\mathbf{E}^2$ such that any four of the sets of $\mathcal{F}$ have a lattice point in common, then there exists a lattice point common to every set of $\mathcal{F}$. Moreover this theorem can be extended to the d-dimensional Euclidean space $\mathbf{E}^d$ replacing 4 by 2^d. Thus it is very natural to ask the following Gallai-type problem for planar lattices. Let $\mathcal{F}$ denote a finite collection of convex sets in $\mathbf{E}^2$ such that any three of the sets of $\mathcal{F}$ have a lattice point in common. Find the least integer n such that for an arbitrary $\mathcal{F}$ there exist n lattice points (i.e. n needles positioned at the lattice points) with the property that every set of $\mathcal{F}$ contains (i.e. is pinned down) by at least one of the n lattice points (i.e. needles).

We prove the following

THEOREM 1. *If $\mathcal{F}$ is a finite family of convex sets in $\mathbf{E}^2$ such that any three of them have a lattice point in common, then there exist two lattice points which pin down $\mathcal{F}$.*

REMARK. It is easy to see that 2, i.e. the number of needles cannot be reduced to 1. Moreover, if we replace 3 (the number which guarantees that so many convex sets always intersect in a common lattice point) by 2, then the problem has a trivial negative answer.

2. Proof of Theorem 1

First we introduce some simple notations. The points of the plane will be denoted by $A, B, \ldots$. The segment with endpoints A and B is denoted by AB, and the line passing through the points A and B is denoted by $\overline{AB}$. We fix a so-called negative orientation of the plane. A convex polygon will be described with the sequence of its vertices according to the given negative orientation.

The line $\overline{AB}$ splits the plane into two open half-planes $\mathbf{F}_{A,B}$ and $\mathbf{F}_{B,A}$. In this notation the order of the subscripts is important, namely, for any point C (D, resp.) of $\mathbf{F}_{A,B}$ ($\mathbf{F}_{B,A}$, resp.) the sequence ABC (BAD, resp.) determines the negative orientation of the plane. For the closed half-plane determined by the open half-plane $\mathbf{F}_{A,B}$ we use the notation $\overline{\mathbf{F}}_{A,B}$ (i.e. $\overline{F}_{A,B} = \mathbf{F}_{A,B} \cup \overline{AB}$).

To each convex pentagon $ABCDE$ we assign the convex pentagon

$$\overline{ABCDE} = \overline{\mathbf{F}}_{A,C} \cap \overline{\mathbf{F}}_{B,D} \cap \overline{\mathbf{F}}_{C,E} \cap \overline{\mathbf{F}}_{D,A} \cap \overline{\mathbf{F}}_{E,B}.$$

(In other words $\overline{ABCDE}$ is enclosed by the diagonals of $ABCDE$.) The following two concepts are basically important for our proof.

DEFINITION 1. Let $\mathbf{L}$ be the set of points of $\mathbf{E}^2$ which have integer coordinates. A point of $\mathbf{L}$ is called lattice point. A lattice point P is called a fixed lattice point (shortly an fl-point) if there are three sets of $\mathcal{F}$ the intersection of which contains P as the only lattice point.

DEFINITION 2. We define the following fixed lattice-point algorithm (FLP-algorithm). For each $\mathbf{K} \in \mathcal{F}$ we proceed as follows. Let $\mathbf{K}^{(1)}$ be the convex hull of the lattice points which are points in common of $\mathbf{K}$ with two more sets of $\mathcal{F}$. Note that $\mathbf{K}^{(1)}$ is a convex lattice-polygon. Let $\mathcal{F}^{(1)}$ be the family arising from $\mathcal{F}$ when we replace $\mathbf{K}$ in it by $\mathbf{K}^{(1)}$. In general, suppose that $\mathbf{K}^{(i)}$ as well as $\mathcal{F}^{(i)}$ have already been defined. Then take a vertex of $\mathbf{K}^{(i)}$ which is not an fl-point with respect to a triplet of $\mathcal{F}^{(i)}$ containing $\mathbf{K}^{(i)}$. Remove this vertex from the vertices of $\mathbf{K}^{(i)}$. Obviously, this algorithm terminates after finitely many steps, say n. Then it is easy to see that every vertex of $\mathbf{K}^{(n)}$ is an fl-point with respect to a triplet of $\mathcal{F}^{(n)}$ containing $\mathbf{K}^{(n)}$. Observe that $\mathcal{F}^{(n)}$ satisfies the conditions of the theorem.

After this for the next $\mathbf{K}$ we use $\mathcal{F}^{(n)}$ instead of $\mathcal{F}$. Finally (after finitely many steps), the above FLP-algorithm yields a "new" $\mathcal{F}$ such that every vertex of any $\mathbf{K}$ of $\mathcal{F}$ is an fl-point with respect to a triplet of $\mathcal{F}$ containing $\mathbf{K}$. Then we say that $\mathcal{F}$ is <u>fixed</u>.

We shall make use of the following

LEMMA 1. *If $ABCDE$ is a convex lattice-pentagon, then $\overline{ABCDE}$ contains a lattice point.*

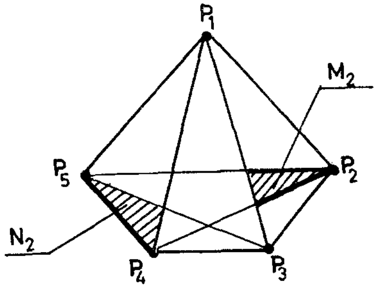

Fig. 1

PROOF. (Indirect.) Let $P_1P_2P_3P_4P_5$ be the convex lattice-pentagon with minimum number of lattice points for which the claim is false. Let $\mathbf{M}_2$ denote the region $\overline{\mathbf{F}}_{5,2} \cap \overline{\mathbf{F}}_{2,4} \cap \mathbf{F}_{3,1}$ (see Fig. 1).

Similarly we get $\mathbf{M}_1, \mathbf{M}_3, \mathbf{M}_4$ and $\mathbf{M}_5$. Furthermore, let $\mathbf{N}_2$ be the region $\mathbf{F}_{5,3} \cap \mathbf{F}_{1,4} \cap \overline{\mathbf{F}}_{4,5}$. In the same way we define the regions $\mathbf{N}_1, \mathbf{N}_3, \mathbf{N}_4$ and $\mathbf{N}_5$. It is easy to see that the convex lattice-pentagon $P_1P_2P_3P_4P_5$ contains a lattice point different from its vertices. Let P_6 be one of these lattice-points. By assumption, $P_6 \notin \overline{P_1P_2P_3P_4P_5}$. Suppose that $P_6 \in \mathbf{M}_2$. Then for the convex lattice-pentagon $P_1P_6P_3P_4P_5$ we have $\overline{P_1P_6P_3P_4P_5} \subset \subset \overline{P_1P_2P_3P_4P_5}$, a contradiction by the indirect assumption. This implies that the regions $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$ and $\mathbf{M}_5$ do not contain a lattice point different from P_1, P_2, P_3, P_4 and P_5. Thus we may suppose that $P_6 \in \mathbf{N}_i$ for some $i \in \{1,2,3,4,5\}$. Let $i = 2$. As the convex lattice-pentagon $P_1P_2P_3P_6P_5$ contains less lattice points than $P_1P_2P_3P_4P_5$ the indirect assumption implies the existence of a lattice-point $P_7 \in \overline{P_1P_2P_3P_6P_5}$. Then it is easy to prove that either $P_7 \in \mathbf{M}_5$ or $P_7 \in \overline{P_1P_2P_3P_4P_5}$. In both cases we get a contradiction. This completes the proof of Lemma 1. Q.E.D.

THEOREM 2. *Consider five convex sets in $\mathbf{E}^2$ such that any three of them have a point of $\mathbf{L}$ in common. Then for each convex set there are three others such that the intersection of these four sets contains a point of $\mathbf{L}$.*

PROOF. Let the five convex sets be denoted by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. We are going to prove our claim for the set $\mathbf{K}_1$. We shall make use of the following special notation. $P_{i_1,i_2,\dots,i_k}(\overline{P}_{i_1,i_2,\dots,i_k}$ resp.) stands for a lattice-point in $\mathbf{K}_1 \cap \mathbf{K}_{i_1} \cap \ldots \cap \mathbf{K}_{i_k}$ $((\mathbf{E}^2 \setminus \mathbf{K}_1) \cap \mathbf{K}_{i_1} \cap \ldots \cap \mathbf{K}_{i_k}$ resp.), $2 \leqq i_1 < < i_2 < \ldots < i_k \leqq 5$.

The following rather technical lemma reduces the number of cases we have to investigate in the proofs of many statements.

LEMMA 2. *Let P_{23} be a fixed lattice point with respect to the convex sets $\mathbf{K}_1, \mathbf{K}_2$, and $\mathbf{K}_3$, and let $P_{23}P_2P_3P_{2'}$ be a convex lattice-quadrangle where P_2 and $P_{2'}$ are distinct lattice-points in $\mathbf{K}_1 \cap \mathbf{K}_2$. Then $P_{23}^* \in \overline{\mathbf{F}}_{2,23} \cap \overline{F}_{23,2'} \cap \cap \mathbf{F}_{3,2'} \cap \mathbf{F}_{2,3}$.*

PROOF. If $P_{23}^* \in \overline{\mathbf{F}}_{2,23} \cap \overline{\mathbf{F}}_{3,2}$ then $P_2 \in P_3P_{23}P_{23}^*$ i.e. $P_2 \in \mathbf{K}_3$, but $P_2 \not\equiv \not\equiv P_{23}$ in contradiction with the fl-point property of P_{23}. Similarly, we get a contradiction if $P_{23}^* \in \overline{\mathbf{F}}_{32,2'} \cap \overline{\mathbf{F}}_{2',3}$ (Fig. 2.).

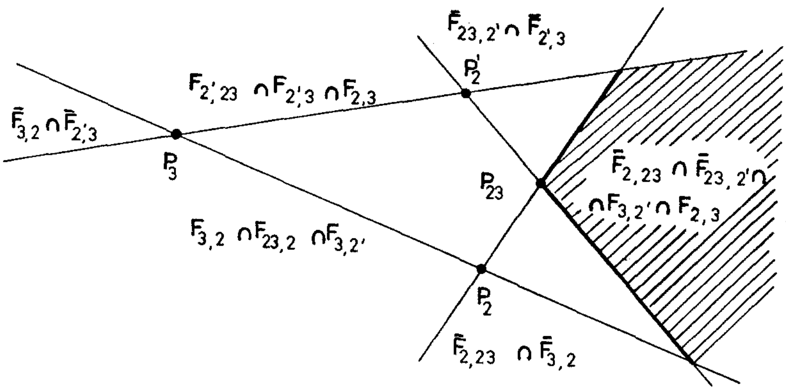

Fig. 2

If $P_{23}^* \in \mathbf{F}_{3,2} \cap \mathbf{F}_{23,2} \cap \mathbf{F}_{3,2'}$, then $P_{23}P_2P_{23}^*P_3P_{2'}$ is a convex lattice pentagon. By Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P_2P_{23}^*P_3P_{2'}} \subset P_{23}P_{23}^*P_{2'} \cap P_{23}P_{23}^*P_3 \cap P_{23}P_2P_3 \subset \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_1,$$

but $A \not\equiv P_{23}$ in contradiction with the fl-point property of P_{23}. The case $P_{23}^* \in \mathbf{F}_{2',3} \cap \mathbf{F}_{2',23} \cap \mathbf{F}_{2,3}$ can be disproved similarly.

If $P_{23}^* \in \overline{\mathbf{F}}_{3,2} \cap \overline{\mathbf{F}}_{2',3}$ then $P_3 \in P_{2'}P_2P_{23}^* \subset \mathbf{K}_2$ but $P_3 \not\equiv P_{23}$, a contradiction.

If $P_{23}^* \in \mathbf{F}_{2',23} \cap \mathbf{F}_{2,3} \cap \mathbf{F}_{2,23}$, then $P_{23}P_{23}^*P_2P_3P_{2'}$ is a convex lattice pentagon. By Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P_{23}^*P_2P_3P_{2'}} \subset P_{23}P_2P_{2'} \cap P_{23}P_{23}^*P_3 \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3,$$

but $A \not\equiv P_{23}$, a contradiction. Similarly, we get a contradiction if $P_{23}^* \in \in \mathbf{F}_{23,2'} \cap \mathbf{F}_{23,2} \cap \mathbf{F}_{3,2}$. □

Let $\mathcal{C} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$ and apply the FLP-algorithm to $\mathcal{C}$. Then we take $\mathbf{K}_1$ which is convex lattice-polygon with the property that each vertex is an fl-point P_{ij} for some i and j with respect to $\mathbf{K}_1$, furthermore we take $\mathbf{K}_i$ and $\mathbf{K}_j$. Obviously, two vertices cannot have the same "name" P_{ij}. As the number of sides of $\mathbf{K}_1$ is at most 6 we distinguish 5 cases. Each of them has some further subcases depending on the positions of the P_{ij}'s. We prove Theorem 2 as well as the fact that $\mathbf{K}_1$ is either a triangle or a point. The rough idea of the proof is the following: we take a point P^*_{ijk} and show that independently from its position the above claim is true. However, there are some cases where we have to consider the positions of two P^*_{ijk}'s.

I. $\mathbf{K}_1$ *is a convex hexagon.* The vertices of $\mathbf{K}_1$ are the points P_{ij}. Suppose that a vertex of $\mathbf{K}_1$, say P_{23}, belongs to more than three convex sets, say $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap K_3 \cap \mathbf{K}_4$. But then P_{24} is not an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_4$, a contradiction. Thus every vertex of $\mathbf{K}_1$ belongs to exactly three convex sets. Next we prove that any two opposite vertices of $\mathbf{K}_1$ cannot be covered by $\mathbf{K}_i$, where $i > 1$. Namely, assume that $\mathbf{K}_1 = A_1A_2A_3A_4A_5A_6$ with $A_1 = P_{23}$ and $A_4 = P_{24}$. Without loss of generality we may assume that $A_3 = P_{25}$. First we consider the case $A_2 = P_{34}$. As $P_{23}P_{34}P_{25}P_{24}P_{45}$ is a convex pentagon, Lemma 1 implies that there exists a lattice point B such that

$$B \in \overline{P_{23}P_{34}P_{25}P_{24}P_{45}} \subset P_{34}P_{24}P_{45} \cap P_{23}P_{25}P_{24} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4.$$

Finally, $B \not\equiv P_{24}$, a contradiction since P_{24} must be an fl-point.

Now assume that $A_2 = P_{35}$. Since $P_{23}P_{35}P_{25}P_{24}P_{45}$ is a convex lattice pentagon, hence there exists a lattice point B such that

$$B \in \overline{P_{23}P_{35}P_{25}P_{24}P_{45}} \subset P_{23}P_{25}P_{24} \cap P_{35}P_{25}P_{45} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

but $B \not\equiv P_{24}$ so we get a contradiction since P_{24} is an fl-point. Finally, if $A_2 \equiv P_{45}$, then a similar argument yields a contradiction.

Thus it is sufficient to consider the convex hexagon $P_{23}P_{25}P_{35}P_{45}P_{34}P_{24}$ (see Fig. 3).

If P_{345} exists, then $P_{345} \not\equiv P_{34}$ which we proved above, and this is contradiction since P_{34} is an fl-point. Hence P^*_{345} exists. As P_{35} is an fl-point and $P_{35}P_{45}P_{23}P_{25}$ is a convex quadrangle, by Lemma 2 we get $P^*_{345} \in \in \mathbf{F}_{45,23}$. On the other hand P_{34} is an fl-point and $P_{34}P_{24}P_{23}P_{45}$ is a convex quadrangle so by Lemma 2 we get $P^*_{345} \in \mathbf{F}_{23,45}$, a contradiction. □

II. K_1 *is a convex pentagon.* We may assume that the vertices of $\mathbf{K}_1$ are $P_{23}, P_{24}, P_{25}, P_{34}$ and P_{35}. It is easy to prove that we have to investigate four cases only.

(a) $\mathbf{K}_1$ *is the pentagon* $P_{23}P_{35}P_{25}P_{34}P_{24}$. By Lemma 1 there is a lattice point A such that

$$A \in \overline{P_{23}P_{35}P_{25}P_{34}P_{24}} \subset P_{23}P_{25}P_{24} \cap P_{23}P_{35}P_{34} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

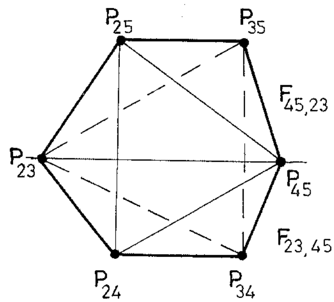

Fig. 3

Since $A \not\equiv P_{23}$, this contradicts the fl-point property of P_{23}.

(b) $\mathbf{K}_1$ *is the pentagon* $P_{25}P_{35}P_{23}P_{34}P_{24}$. If $P_{45} \in P_{23}P_{24}P_{25}$, then $P_{45} \in \mathbf{K}_2$, but $P_{45} \not\equiv P_{24}$, a contradiction.

If $P_{45} \in P_{23}P_{34}P_{24}$, then $P_{23}P_{45}P_{24}P_{25}P_{35}$ is a convex pentagon, so by Lemma 1 we have a lattice point A, such that

$$A \in \overline{P_{23}P_{45}P_{24}P_{25}P_{35}} \subset P_{23}P_{24}P_{25} \cap P_{45}P_{25}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

but $A \not\equiv P_{25}$, a contradiction. Similarly we get a contradiction if $P_{45} \in \in P_{25}P_{35}P_{23}$.

Notice that if $\mathbf{K}_1$ is a $P_{25}P_{34}P_{23}P_{35}P_{24}$ pentagon we can proceed similarly.

(c) $\mathbf{K}_1$ *is the pentagon* $P_{34}P_{35}P_{23}P_{25}P_{24}$. We may assume that $P_{45} \in \in P_{23}P_{24}P_{34}$ (Fig. 4). Namely, if $P_{45} \in P_{23}P_{34}P_{35}$, then $P_{45} \in \mathbf{K}_3$. As $P_{45} \not\equiv P_{34}$, this contradicts the fl-point property of P_{34}.

Since P_{25} is an fl-point, $P_{25}P_{24}P_{45}P_{23}$ is a convex quadrangle. Then Lemma 2 implies that $P^*_{235} \in \mathbf{F}_{45,23}$. If P_{235} exists, then P_{23} and P_{25} are fl-points. As P_{35} is an fl-point and $P_{45}P_{34}P_{35}P_{23}$ is a convex quadrangle by Lemma 2 we get $P^*_{235} \in \mathbf{F}_{23,45}$, a contradiction.

(d) $\mathbf{K}_1$ *is the pentagon* $P_{34}P_{23}P_{25}P_{24}P_{35}$. As P_{34} and P_{35} are fl-points, P^*_{345} does exist (Fig. 5). Since P_{34} is an fl-point and $P_{34}P_{23}P_{24}P_{35}$ is a convex quadrangle, we get by Lemma 2 that $P^*_{345} \in \mathbf{F}_{24,35} \cap \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{23,34} \cap \mathbf{F}_{23,24}$.

If $P^*_{345} \in \overline{\mathbf{F}}_{25,34} \cap \overline{\mathbf{F}}_{34,35}$, then $P_{34} \in P^*_{345}P_{25}P_{35} \subset \mathbf{K}_5$, which contradicts the fl-point property of P_{35}. Hence we may suppose that $P^*_{345} \in \overline{\mathbf{F}}_{23,34} \cap \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{24,35}$.

If P_{235} exists, then we get a contradiction since P_{23} and P_{25} are fl-points. Thus P^*_{235} exists.

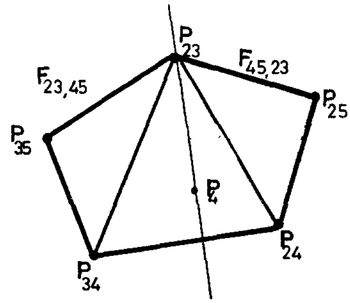

Fig. 4

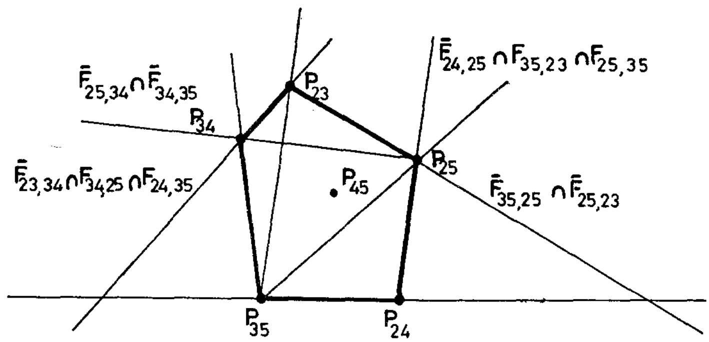

Fig. 5

Since P_{25} is an fl-point, $P_{25}P_{24}P_{35}P_{23}$ is a convex quadrangle thus Lemma 2 implies that

$$P^*_{235} \in \mathbf{F}_{35,23} \cap \overline{\mathbf{F}}_{25,23} \cap \overline{\mathbf{F}}_{24,25} \cap \mathbf{F}_{24,35}.$$

If $P^*_{235} \in \overline{\mathbf{F}}_{35,25} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{25} \in P^*_{235}P_{35}P_{23} \subset \mathbf{K}_3$ which contradicts the fl-point property of P_{23}. Hence we may assume that $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{35,23} \cap \cap \mathbf{F}_{25,35}$.

Since $P^*_{345} \in \mathbf{F}_{25,34} \cap \mathbf{F}_{235^*,25} \cap \mathbf{F}_{235^*,35}$ we get that $P^*_{345}P^*_{235}P_{25}P_{35}$ is a convex quadrangle. As P_{25} is an fl-point $P_{23} \in P^*_{345}P^*_{235}P_{25}P_{35} \subset \mathbf{K}_5$ cannot occur. Thus $P_{23} \notin P^*_{345}P^*_{235}P_{25}P_{35}$ so

$$P_{23} \in \mathbf{F}_{235^*,25} \cap \mathbf{F}_{35,345^*} \cap \mathbf{F}_{235^*,345^*}.$$

It follows from the foregoing that $P_{23}P_{235}P_{25}P_{35}P^*_{345}$ is a convex pentagon. Hence by Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P^*_{235}P_{15}P_{35}P^*_{345}} \subset P_{23}P_{25}P_{35} \cap P^*_{345}P^*_{235}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5.$$

Since $A \not\equiv P_{35}$ and P_{35} is an fl-point, this is a contradiction. □

III. $\mathbf{K}_1$ *is a quadrangle.* It is easy to prove that we have to investigate four cases only.

(a) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{24}P_{45}P_{35}$. If $P_{34} \in P_{23}P_{24}P_{25} \subset \mathbf{K}_2$ or $P_{34} \in P_{25}P_{35}P_{45} \subset \mathbf{K}_5$, then this contradicts the fl-point property of P_{23} and P_{24} or P_{35} and P_{45}. Thus we may assume that $P_{34} \in P_{24}P_{45}P_{25}$ (Fig. 6).

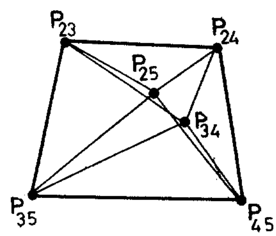

Fig. 6

Similarly we may assume that $P_{25} \in P_{23}P_{24}P_{34}$. Then $P_{23}P_{25}P_{34}P_{45}P_{35}$ is a convex pentagon, and according to Lemma 1 there exists a lattice point A such that

$$A \in \overline{P_{23}P_{25}P_{34}P_{45}P_{35}} \subset P_{23}P_{34}P_{35} \cap P_{25}P_{45}P_{35} \subset \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5$$

but $A \not\equiv P_{35}$, a contradiction.

(b) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{24}P_{35}P_{45}$. If $P_{25} \in P_{23}P_{35}$, then $P_{25} \in \mathbf{K}_3$, but this contradicts the fl-point property of P_{23} and P_{35} (Fig. 7).

If $P_{25} \in P_{23}P_{35}P_{45}$ then $P_{23}P_{24}P_{35}P_{25}$ is a convex quadrangle and since P_{23} is an fl-point, applying Lemma 2 we get that $P^*_{234} \in \overline{\mathbf{F}}_{23,25} \cap \overline{\mathbf{F}}_{24,23}$. (If P_{234} exists we get a contradiction since P_{23} and P_{24} are fl-points.) Then $P_{23} \in P^*_{234}P_{24}P_{45} \subset \mathbf{K}_4$, but this contradicts the fl-point property of P_{23} and P_{24}. Similarly, we get a contradiction if $P_{25} \in P_{35}P_{23}P_{24}$.

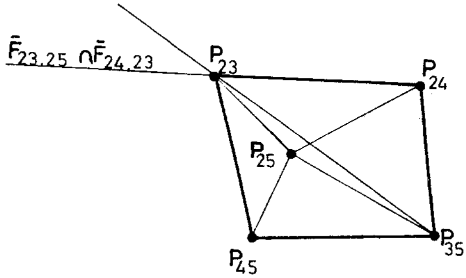

Fig. 7

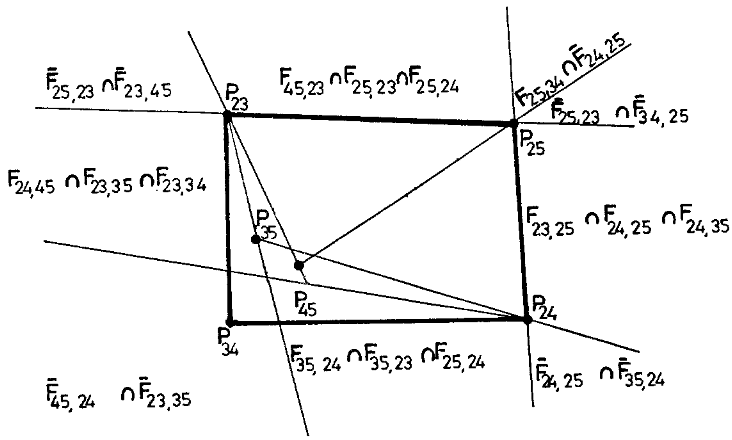

Fig. 8

(c) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{25}P_{24}P_{34}$. If P_{35} or $P_{45} \in P_{23}P_{25}P_{24} \subset \subset \mathbf{K}_2$, then we have a contradiction since P_{23} and P_{25} or P_{24} and P_{25} are fl-points. Thus we may assume that P_{35} and $P_{45} \in P_{23}P_{24}P_{34}$ (Fig. 8).

If P_{235} exists, then we get a contradiction as P_{23} and P_{25} are fl-points. Thus we may suppose that P^*_{235} exists.

If $P^*_{235} \in \overline{\mathbf{F}}_{25,23} \cap \overline{\mathbf{F}}_{34,25}$, then $P_{25} \in P^*_{235}P_{34}P_{23} \subset \mathbf{K}_3$, but $P_{25} \not\equiv P_{23}$, a contradiction. Our proof is similar if $P^*_{235} \in \overline{\mathbf{F}}_{25,34} \cap \overline{\mathbf{F}}_{24,25}$.

If $P^*_{235} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{24,25} \cap \mathbf{F}_{24,35}$, then $P^*_{235}P_{24}P_{35}P_{23}P_{25}$ is a convex pentagon. Applying Lemma 2 we have a lattice point A for which

$$A \in \overline{P^*_{235}P_{24}P_{35}P_{23}P_{25}} \subset P_{24}P_{23}P_{25} \cap P^*_{235}P_{35}P_{23} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

As $A \not\equiv P_{23}$ this is a contradiction. We can settle the case $P^*_{235} \in \mathbf{F}_{45,23} \cap \cap \mathbf{F}_{25,23} \cap \mathbf{F}_{25,24}$ similarly.

If $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{35,24}$, then $P_{24} \in P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$, a contradiction. If $P^*_{235} \in \overline{\mathbf{F}}_{25,35} \cap \overline{\mathbf{F}}_{35,24}$, then the reasoning is similar.

If $P^*_{235} \in \mathbf{F}_{35,24} \cap \mathbf{F}_{35,23}\mathbf{F}_{25,24}$, then $P^*_{235}P_{35}P_{23}P_{25}P_{24}$ is a convex pentagon thus according to Lemma 1 we have a lattice point A such that

$$A \in \overline{P^*_{235}P_{35}P_{23}P_{25}P_{24}} \subset P^*_{235}P_{35}P_{25} \cap P^*_{235}P_{23}P_{25} \cap P_{35}P_{25}P_{24} \in \\ \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

As $A \not\equiv P_{25}$ we get a contradiction. The reasoning in the case $P^*_{235} \in \mathbf{F}_{24,25} \cap \cap \mathbf{F}_{23,35} \cap \mathbf{F}_{23,34}$ follows word for word the previous reasoning.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,25} \cap \overline{\mathbf{F}}_{23,35}$, then P_{35} or $P_{45} \in P^*_{235}P_{23}P_{24} \subset \mathbf{K}_2$, but this is a contradiction since P_{23} and P_{25} or P_{24} and P_{25} are fl-points.

(d) $\mathbf{K}_1$ *is the quadrangle* $P_{23}P_{34}P_{25}P_{24}$. If P_{35} or $P_{45} \in P_{25}P_{24}P_{23}$, then we get a contradiction as in the case (c). Hence we may assume that P_{35} and $P_{45} \in P_{34}P_{25}P_{23}$ (Fig. 9).

P^*_{235} does exist. (The proof is the same as in the case (c).)

If $P^*_{235} \in \mathbf{F}_{45,25} \cap \mathbf{F}_{25,35} \cap \mathbf{F}_{34,23}$, then $P_{35} \in P^*_{235}P_{25}P_{23} \subset \mathbf{K}_2$, but this contradicts the fl-point property of P_{23} and P_{25}.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,23} \cap \overline{\mathbf{F}}_{25,45}$, then $P_{45} \in P^*_{235}P_{25}P_{23} \subset K_2$, but this is a contradiction since P_{24} and P_{25} are fl-points.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,34}$, then $P_{45} \in P^*_{235}P_{23}P_{34} \subset \mathbf{K}_3$. This is possible only in case $P_{45} \equiv P_{34}$. But then this vertex is a P_{35} vertex and changing $\mathbf{K}_4$ and $\mathbf{K}_5$ we get case (c). (Notice that we have not utilized the fl-point property of P_{34} in the reasoning of case (c).) Hence we get a contradiction just like in case (c).

If $P^*_{235} \in \mathbf{F}_{24,25} \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{24,23}$, then $P^*_{235}P_{24}P_{23}P_{25}$ is a convex pentagon. According to Lemma 1 we have a lattice point A such that

$$A \in \overline{P^*_{235}P_{24}P_{23}P_{34}P_{25}} \subset P^*_{235}P_{23}P_{34} \cap P_{24}P_{23}P_{25} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3.$$

Since $A \not\equiv P_{23}$ this is a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,34} \cap \overline{\mathbf{F}}_{24,35}$, then $P_{24} \in P^*_{235}P_{23}P_{25} \subset \mathbf{K}_2$. As $P_{24} \not\equiv P_{34}$ this is a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{35,24}$, then $P_{24} \in P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$. Since $P_{24} \not\equiv P_{25}$ this is a contradiction.

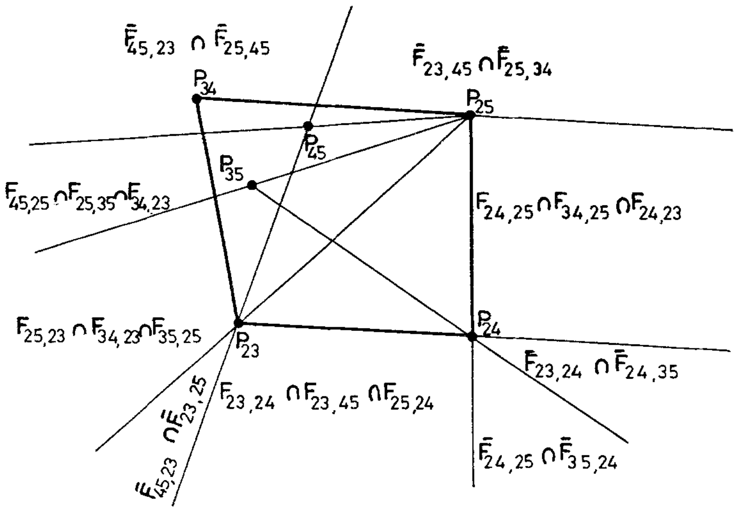

Fig. 9

If $P^*_{235} \in \mathbf{F}_{23,24} \cap \mathbf{F}_{23,45} \cap \mathbf{F}_{25,24}$, then $P^*_{235}P_{23}P_{45}P_{25}P_{24}$ is a convex pentagon, hence by Lemma 1 we get a lattice point A such that

$$A \in \overline{P^*_{235}P_{23}P_{45}P_{25}P_{24}} \subset P_{24}P_{23}P_{25} \cap P^*_{235}P_{45}P_{25} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

As $A \not\equiv P_{25}$ we get a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{45,23} \cap \overline{\mathbf{F}}_{23,25}$, then $P_{23} \in P^*_{235}P_{45}P_{25} \subset \mathbf{K}_5$. Since $P_{25} \not\equiv P_{23}$ we get a contradiction.

Thus we may suppose that $P^*_{235} \in \mathbf{F}_{25,23} \cap \mathbf{F}_{34,23} \cap \mathbf{F}_{35,25}$.

If P_{245} exists, then we have a contradiction as P_{24} and P_{25} are fl-points. Hence we may assume that P^*_{245} exists.

Since P_{24} is an fl-point and $P_{23}P_{34}P_{25}P_{24}$ is a convex quadrangle hence applying Lemma 2 we get that

$$P^*_{245} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{23,24} \cap \mathbf{F}_{34,25} \cap \mathbf{F}_{23,34}.$$

Since $P^*_{245} \in \mathbf{F}_{35,25} \cap \mathbf{F}_{235^*,35} \cap \mathbf{F}_{235^*,25}$, $P^*_{245}P^*_{235}P_{35}P_{25}$ is a convex quadrangle.

If $P_{24} \in P^*_{245}P^*_{235}P_{35}P_{25} \subset \mathbf{K}_5$, then since $P_{24} \not\equiv P_{25}$ we get a contradiction.

If $P_{24} \notin P^*_{245}P^*_{235}P_{35}P_{25}$, then $P_{24} \in \mathbf{F}_{235^*,35} \cap \mathbf{F}_{235^*,245^*} \cap \mathbf{F}_{25,245^*}$, thus $P_{24}P^*_{235}P_{35}P_{25}P_{245}$ is a convex pentagon. By Lemma 1 we get a lattice point A such that

$$A \in \overline{P_{24}P^*_{235}P_{35}P_{25}P^*_{245}} \subset P_{34}P_{35}P_{25} \cap P_{24}P^*_{235}P_{25} \cap P^*_{235}P_{25}P^*_{245} \subset \\ \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5.$$

But this is a contradiction since $A \not\equiv P_{25}$. □

IV. $\mathbf{K}_1$ *is a triangle*. It is easy to prove that we have to investigate three cases only.

(a) $\mathbf{K}_1$ *is the triangle* $P_{23}P_{24}P_{25}$. Then $P_{34} \in \mathbf{K}_1 \cap \mathbf{K}_2$, which is a contradiction since P_{23} and P_{24} are fl-points.

(b) $\mathbf{K}_1$ *is the triangle* $P_{23}P_{24}P_{34}$. If P_{234} exists, then we get a contradiction as P_{23} and P_{24} are fl-points. Thus we may suppose that P^*_{234} exists (Fig. 10).

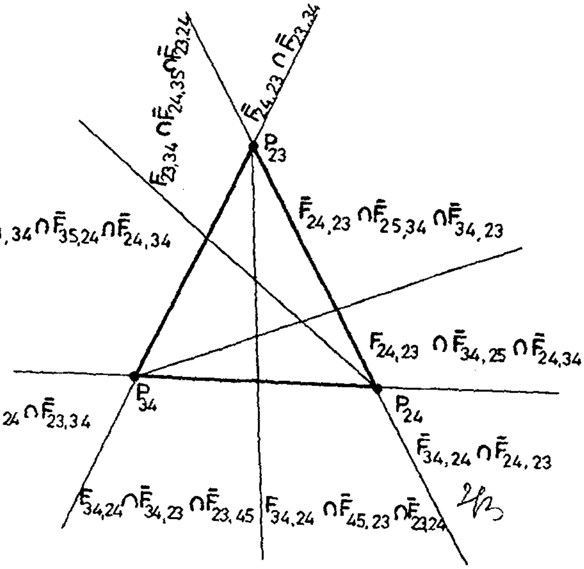

Fig. 10

It is easy to prove that we have to investigate the following two cases.

If $P^*_{234} \in \overline{\mathbf{F}}_{34,24} \cap \overline{\mathbf{F}}_{24,23}$, then $P_{24} \in P^*_{234}P_{34}P_{23} \subset \mathbf{K}_3$ but this contradicts the fl-point property of P_{23}.

If $P^*_{234} \in \mathbf{F}_{24,23} \cap \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{24,34}$, then $P_{24} \in P^*_{234}P_{34}P_{23} \subset \mathbf{K}_3$ i.e. $P_{25} \in \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$. Thus in this case Theorem 2 is true.

(c) $\mathbf{K}_1$ *is the triangle* $P_{24}P_{25}P_{34}$. If $P_{45} \in P_{23}P_{24}P_{25} \subset \mathbf{K}_2$, then $P_{45} \in \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$ which proves Theorem 2 in this case.

If $P_{23} \in P_{45}P_{34}P_{24} \subset \mathbf{K}_4$, then $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$. Hence we may assume that $P_{45} \in \mathbf{F}_{24,23}$.

If $P_{45} \in P_{23}P_{34} \subset \mathbf{K}_3$, then $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_4 \cap \mathbf{K}_5$ and we are done. We may assume that $P_{45} \notin P_{23}P_{34}$.

It follows from the foregoing that we have to investigate the following two cases:

(α) $P_{23}P_{45}P_{34}P_{24}$ *is a convex quadrangle.* If P_{234} exists, then we get a contradiction since P_{24} and P_{34} are fl-points. Thus P^*_{234} exists (Fig. 11).

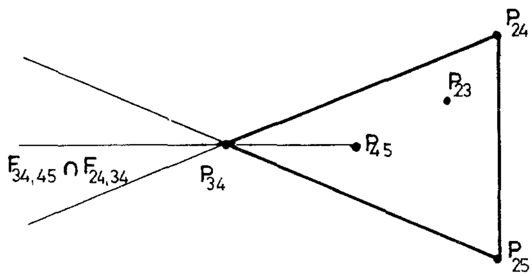

Fig. 11

Since P_{34} is an fl-point and $P_{34}P_{24}P_{23}P_{45}$ is a convex quadrangle, applying Lemma 2 we get that $P^*_{234} \in \overline{\mathbf{F}}_{34,45} \cap \overline{\mathbf{F}}_{24,34}$. Then we have that $P_{34} \in P^*_{234}P_{24}P_{25} \subset \mathbf{K}_2$ thus $P_{34} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$ which is our claim.

(β) $P_{34}P_{45}P_{23}P_{25}$ *is a convex quadrangle.* If P_{245} exists we get a contradiction since P_{24} and P_{25} are fl-points. Thus P^*_{245} exists (Fig. 12).

If $P^*_{245} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{23} \in P^*_{245}P_{25}P_{45} \subset \mathbf{K}_5$. Thus $P_{23} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{25,24} \cap \mathbf{F}_{25,34} \cap \mathbf{F}_{45,23}$, then $P_{23} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_4$.

If $P^*_{245} \in \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{25,24}$, then $P_{25} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{25} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{34,24}$, then we have two cases since $\mathbf{F}_{23,25} \cap \mathbf{F}_{34,24} = = (\mathbf{F}_{23,25} \cap \mathbf{F}_{35,24}) \cup (\mathbf{F}_{24,35} \cap \mathbf{F}_{34,24})$ wherever P_{35} is. If $P^*_{245} \in \mathbf{F}_{23,25} \cap \mathbf{F}_{35,24}$,

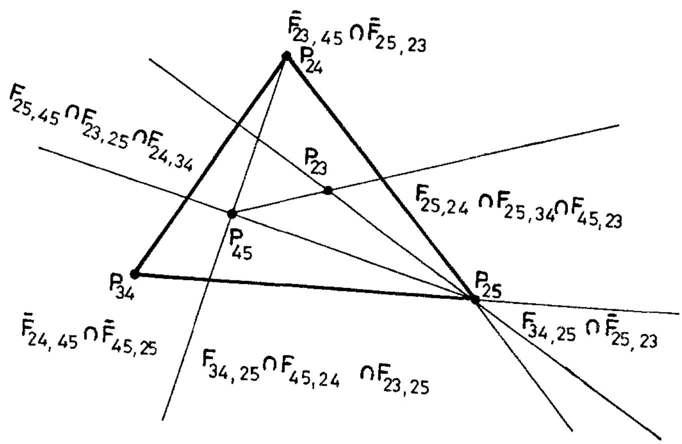

Fig. 12

then $P_{35} \in P^*_{245}P_{34}P_{24} \subset \mathbf{K}_4$. Thus $P_{35} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_5$. If $P^*_{245} \in \mathbf{F}_{24,35} \cap \cap \mathbf{F}_{34,24}$, then $P_{35} \in P^*_{245}P_{24}P_{25} \subset \mathbf{K}_2$. Thus $P_{35} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{245} \in \mathbf{F}_{24,45} \cap \mathbf{F}_{45,25}$, then $P_{45} \in P^*_{245}P_{24}P_{25} \subset \mathbf{K}_2$. Thus $P_{45} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

Thus we may assume that $P^*_{245} \in \mathbf{F}_{25,45} \cap \mathbf{F}_{23,25} \cap \mathbf{F}_{24,34}$.

If P_{235} exists then Theorem 2 is true. Hence we may suppose that P^*_{235} exists.

If $P^*_{235} \in \overline{\mathbf{F}}_{24,25} \cap \overline{\mathbf{F}}_{45,25}$, then the proof is similar to the previous one.

If $P^*_{235} \in \mathbf{F}_{25,45} \cap \mathbf{F}_{23,25} \cap \mathbf{F}_{24,34}$, then $P_{45} \in P^*_{235}P_{23}P_{34} \subset \mathbf{K}_3$. Thus $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P^*_{235} \in \overline{\mathbf{F}}_{23,45} \cap \overline{\mathbf{F}}_{25,23}$, then the proof is similar to the proof of the case P^*_{245}.

If $P^*_{235} \in \mathbf{F}_{25,24} \cap \mathbf{F}_{25,34} \cap \mathbf{F}_{45,23}$, then $P^*_{235}P_{25}P_{45}P^*_{245}$ is a convex quadrangle. Namely, $P^*_{245} \in \mathbf{F}_{235,45} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{25,45}$.

If $P_{23} \in P^*_{235}P_{25}P_{45}P^*_{245}$, then $P_{23} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P_{23} \notin P^*_{235}P_{25}P_{45}P^*_{245}$, then $P_{23} \in \mathbf{F}_{25,235} \cap \mathbf{F}_{45,245} \cap \mathbf{F}_{235,245}$. Thus $P_{23}P^*_{235}P_{25}P_{45}P^*_{245}$ is a convex pentagon. By Lemma 1 we have a lattice point A such that

$$A \in \overline{P_{23}P^*_{235}P_{25}P_{45}P^*_{245}} \subset P_{23}P_{25}P_{45} \cap P^*_{235}P_{25}P^*_{245} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

a contradiction.

If $P^*_{235} \in \overline{\mathbf{F}}_{34,25} \cap \overline{\mathbf{F}}_{25,23}$, then $P_{25} \in P^*_{235}P_{34}P_{23} \subset \mathbf{K}_3$. Thus $P_{25} \in \mathbf{K}_1 \cap \cap \mathbf{K}_2 \cap \mathbf{K}_3 \cap \mathbf{K}_5$.

If $P^*_{235} \in \mathbf{F}_{34,25} \cap \mathbf{F}_{45,24} \cap \mathbf{F}_{23,25}$, then $P^*_{245}P_{23}P_{25}P^*_{235}$ is a convex quadrangle. Namely, $P^*_{245} \in \mathbf{F}_{23,35} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{23,235}$.

If $P_{45} \in P^*_{245}P_{23}P_{25}P^*_{235}$, then $P_{45} \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{K}_5$.

If $P_{45} \notin P^*_{245}P_{23}P_{25}P^*_{235}$, then $P_{45} \in \mathbf{F}_{245,23} \cap \mathbf{F}_{25,235} \cap \mathbf{F}_{245,235}$. Thus $P_{45}P^*_{245}P_{23}P_{25}P^*_{235}$ is a convex pentagon. By Lemma 1 we have a lattice point A such that

$$A \in \overline{P_{45}P^*_{245}P_{23}P_{25}P^*_{235}} \subset P_{45}P_{23}P_{25} \cap P^*_{245}P_{25}P^*_{235} \subset \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_5,$$

a contradiction. □

V. $\mathbf{K}_1$ *is a segment.* Then $\mathbf{K}_i \cap \mathbf{K}_j \cap \mathbf{K}_1$ contains a lattice point in common. Thus applying Helly's theorem to the segment $\mathbf{K}_i \cap \mathbf{K}_1$ we get that they have a lattice point in common. Hence, we have proved that in this case the convex sets have a lattice point in common, which proves Theorem 2.

In fact, we have proved more. Namely, we have shown that the fixed system of five convex sets of Theorem 2 either have a lattice point in common or each of them is a triangle. □

Now we are able to prove Theorem 1, though we still need a few definitions and several lemmas to do so.

We need the following

DEFINITION 3. Let $\mathcal{F}$ be a fixed system of at least four sets such that any three of them have a lattice point in common. We say that $\mathcal{F}$ is good if the convex hull of $\mathcal{F}$ possesses a vertex S which belongs to exactly three sets. Let us denote these sets by $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ and call them the main configurations of $\mathcal{F}$. If a set of $\mathcal{F}$ is not a main configuration then we call it an ordinary configuration.

THEOREM 3. *Let $\mathcal{F}$ be a good system of convex sets. Then one of the three main configurations of $\mathcal{F}$ is such that removing it from $\mathcal{F}$ the remaining convex sets have a lattice point in common.*

In the following proof step by step we discover more. We are going to characterize the good systems of convex sets. Notice that applying the FLP-algorithm we get lattice-polygons.

LEMMA 3. *Each vertex of a main configuration is included in another one.*

PROOF. Let A be a vertex of $\mathbf{K}_1$. Suppose that $A \notin \mathbf{K}_2$ and $A \notin \mathbf{K}_3$. This entails a contradiction. As A is a vertex of $\mathbf{K}_1$ we can find $\mathbf{K}_4$ and $\mathbf{K}_5$ such that A is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_4$ and $\mathbf{K}_5$. It follows from the foregoing that $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_2$; $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_5$ groups of four sets do not contain a lattice point in common. So we cannot choose further three sets from $\mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$ to $\mathbf{K}_1$ such that this four sets have a lattice point in common. Thus it is a contradiction with Theorem 2. □

Let us denote the convex hull of $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ by $\mathbf{M}$. Let $\mathbf{M}$ be the convex lattice-polygon $A_1A_2 \dots A_kS$, where S is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$. A_i is naturally a vertex of some main configuration of $\mathcal{F}$. Hence according to Lemma 3 it is included in another one, too. Then we say A_i is a type B_{12} vertex, if $A_i \notin \mathbf{K}_3$ and $A_i \in \mathbf{K}_1 \cap \mathbf{K}_2$. We define type B_{13} and type B_{23} vertices similarly.

LEMMA 4. *$\mathbf{M}$ has got type B_{12}, B_{13} and B_{23} vertices.*

PROOF. Assume that there is no type B_{12} vertex. Then $A_i \in \mathbf{K}_3$ for each i. Since $S \in \mathbf{K}_3$ we get that $\mathbf{M} \subset \mathbf{K}_3$. But $\mathbf{K}_3 \subset \mathbf{M}$ thus $\mathbf{K}_3 \equiv \mathbf{M}$. We show that there is only one lattice point in $\mathbf{K}_1 \cap \mathbf{K}_2$. Suppose that there is a lattice point S_1 such that $S_1 \not\equiv S$ and $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2$. In this way we get that $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2 \subset \mathbf{K}_3$, that is, $S_1 \in \mathbf{K}_1 \cap \mathbf{K}_2 \cap \mathbf{K}_3$ which contradicts the fl-point property of S. Thus the only lattice point of $\mathbf{K}_1 \cap \mathbf{K}_2$ is S. Since any three sets of $\mathcal{F}$ have a lattice point in common, hence any set of $\mathcal{F}$ contains S, which is a contradiction. □

LEMMA 5. *$\mathbf{M}$ has got exactly one type B_{12}, B_{13} and B_{23} vertex.*

PROOF. (Indirect.) Let n be the least number with the following property: There exists a system $\mathcal{C}$ of n convex sets such that any three sets of $\mathcal{C}$ have a lattice point in common, moreover the claim is false for $\mathcal{C}$. Let us consider such a $\mathcal{C}$. Then we may assume that there are two type B_{12} vertices, say A_1 and A_2.

It is trivial that $n \geqq 5$. We show that $n \geqq 6$. Namely, if $n = 5$ then among the vertices of $\mathbf{K}_1$ we have S, A_1, A_2 and a type B_{13} vertex. But that is impossible since we have already proved that $\mathbf{K}_1$ is a triangle or a point. Thus $n \geqq 6$.

We need the following

LEMMA 6. *There exists at most one ordinary configuration of $\mathcal{C}$ with the following property: Removing this configuration from $\mathcal{C}$ then A_1 will not be an fl-point with respect to any triplet of $\mathcal{C}$ containing a main configuration.*

PROOF. Suppose that this statement is false. Then there are two sets $\mathbf{K}_4$ and $\mathbf{K}_5$ with the previous property. It is easy to see that A_1 is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_4$ and $\mathbf{K}_5$; and similarly with respect to $\mathbf{K}_2, \mathbf{K}_4$ and $\mathbf{K}_5$. Then the sets of groups $\mathbf{K}_1, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_2, \mathbf{K}_4, \mathbf{K}_5$ and $\mathbf{K}_3$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$; $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_5$ do not contain a lattice point in common. But this contradicts Theorem 2. □

If there exists a convex set of $\mathcal{C}$ that satisfies the conditions of Lemma 6 then let us call it $\mathbf{K}_4$. Similarly we define $\mathbf{K}_5$ by replacing A_1 by A_2. Since $n \geqq 6$ there exists a convex set of $\mathcal{C}$, say $\mathbf{K}_i$, different from $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Removing $\mathbf{K}_i$ from $\mathcal{C}$ we get a convex set system $\mathcal{C}'$, containing

$n-1$ sets. Let us apply the FLP-algorithm to $\mathcal{C}'$. Notice that $\mathcal{C}'$ is good with respect to S. We prove that the claim is false for $\mathcal{C}'$. By Lemma 6 we get a triplet of $\mathcal{C}'$ containing $\mathbf{K}_1$, in which A_1 is an fl-point with respect to it. According to Lemma 6 we have that A_1 or A_2 is an fl-point with respect to a triplet of $\mathcal{C}$ containing $\mathbf{K}_1$ or $\mathbf{K}_2$ (all the variations are allowed).

In this way, applying the FLP-Algorithm we cannot eliminate A_1 or A_2 from neither $\mathbf{K}_1$ nor $\mathbf{K}_2$. Thus for $\mathcal{C}'$ the claim is false, a contradiction. □

In the following part of our proof we will describe all the good $\mathcal{C}$ systems containing five sets.

Let the five sets be denoted by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Let $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ be the main configuration of $\mathcal{C}$ with respect to S.

Let $\mathbf{M}'$ be the convex hull of $\mathcal{C}$. Then $\mathbf{M} \equiv \mathbf{M}'$. Namely, each triplet of $\mathcal{C}$ contains a main configuration. Let A_1, A_2 and A_3 be the type B_{23}, B_{13} and B_{12} vertex of $\mathbf{M}$, resp. Let $\mathbf{M}$ be the convex quadrangle $SA_1A_2A_3$. As each set of $\mathcal{C}$ is a triangle, $\mathbf{K}_1$ is the triangle SA_2A_3, $\mathbf{K}_2$ the triangle SA_1A_3 and $\mathbf{K}_3$ the triangle SA_1A_2. We prove that $A_1A_2A_3$ is a member of $\mathcal{C}$.

If each of the points A_1, A_2 and A_3 is covered by four sets of $\mathcal{C}$, then $\mathbf{K}_4$ and $\mathbf{K}_5$ will contain A_1, A_2 and A_3. Since $\mathbf{K}_4$ and $\mathbf{K}_5$ are triangles we get that $A_1A_2A_3 \equiv \mathbf{K}_4 \equiv \mathbf{K}_5$.

If some A_i is covered by exactly three sets of $\mathcal{C}$, then $\mathcal{C}$ will also be good with respect to A_i. Thus it follows from this that $A_1A_2A_3$ is a member of $\mathcal{C}$. Let us call it $\mathbf{K}_4$.

We show that SA_2 and A_1A_3 do not contain any lattice point except the endpoints.

Let N be the intersection of the diagonals of $\mathbf{M}$. Notice that any three sets of $\mathcal{C}$ have a point in common, hence it follows from the Helly-theorem that there exists a point common to every set of $\mathcal{C}$. As the intersection of $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ and $\mathbf{K}_4$ is a point N we get that $N \in \mathbf{K}_5$.

Let D be one of S, A_1, A_2 and A_3. If DN contains a lattice point different from D, say E, then E is covered by all sets $\mathbf{K}_i$ covering D. But D is an fl-point with respect to some triplet of $\mathcal{C}$, thus we are led to a contradiction. Hence the diagonals of $\mathbf{M}$ do not contain a lattice point except the endpoints. Since $\mathbf{K}_1 \cap \mathbf{K}_3 \cap \mathbf{L} \equiv S \cup A_2$ and $\mathbf{K}_2 \cap \mathbf{K}_4 \cap \mathbf{L} \equiv A_1 \cup A_3$, $\mathbf{K}_5$ contains two neighbouring vertices of $\mathbf{M}$. Let these two neighbouring vertices be A_1 and A_2. As $\mathbf{K}_5$ is a triangle, its third vertex is A_5 where $A_5 \in \mathbf{K}_1 \cap \mathbf{K}_2$. This way we described all good $\mathcal{C}$ containing five sets (see Fig. 13). □

Let $\mathcal{C}$ be a good system of convex sets, and let A_1, A_2 and A_3 be the type B_{23}, B_{13} and B_{12} vertex of $\mathbf{M}$, resp.

LEMMA 7. *There exists an ordinary configuration of $\mathcal{C}$, $\mathbf{K}_j$ such that $A_2 \in \mathbf{K}_j$ and A_2 is an fl-point with respect to $\mathbf{K}_1$, $\mathbf{K}_3$ and $\mathbf{K}_j$.*

PROOF. Suppose that the claim is false. A_2 is an fl-point with respect to a triplet containing $\mathbf{K}_1$. Let this triplet be $\mathbf{K}_1$, $\mathbf{K}_4$ and $\mathbf{K}_5$. Let us consider $\mathcal{G} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$. Apply the FLP-algorithm to $\mathcal{G}$ as follows: Let us consider $\mathbf{K}_3$. A_2 is not an fl-point with respect to a triplet containing $\mathbf{K}_3$,

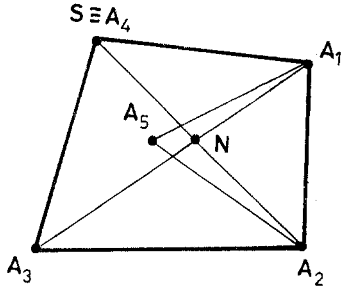

Fig. 13

otherwise A_2 would be an fl-point with respect to $\mathbf{K}_3, \mathbf{K}_4$ and $\mathbf{K}_5$. Then we could get a contradiction in the same way as in the proof of Lemma 6. Thus applying the FLP-algorithm we can remove A_2 from $\mathbf{K}_3$. Hence we get a good $\mathcal{G}'$ with the property that one of the main configurations of $\mathcal{G}'$, $\mathbf{K}_1$, has got a vertex A_2 which is not included in another main configuration, and this contradicts Lemma 3. □

LEMMA 8. *A_2 is covered by all the ordinary configurations of $\mathcal{C}$.*

PROOF. According to Lemma 7 there exists an ordinary configuration of $\mathcal{C}$; $\mathbf{K}_4$ such that A_2 is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_3$ and $\mathbf{K}_4$. Assume that there exists an ordinary configuration $\mathbf{K}_5$ not containing A_2. Let $\mathcal{G} = \{\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_5\}$. Applying the FLP-algorithm to $\mathcal{G}$ we get a good $\mathcal{G}'$. Let $\mathbf{M}$ be the convex hull of $\mathcal{G}'$. Obviously, A_2 and S are vertices of $\mathbf{M}$. Let A_3' be a type B_{12} vertex and A_1' be a type B_{23} vertex of $\mathbf{M}$. We prove that $\mathbf{M}$ is the quadrangle $SA_1'A_2A_3'$. Consider $\mathcal{C}$. If H is a type B_{23} lattice point, then $H \in \overline{\mathbf{F}}_{SA}$; otherwise we get a contradiction since S is an fl-point with respect to $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$. Similarly if G is a type B_{12} lattice point of $\mathbf{M}$, then $G \in \overline{\mathbf{F}}_{AS}$. Thus it follows that $\mathbf{M}$ is the quadrangle $SA_1'A_2A_3'$. Notice that A_2 is not covered by any set of $\mathcal{C}$ different from $\mathbf{K}_1, \mathbf{K}_3$ and $\mathbf{K}_4$. Thus $\mathcal{G}$ has got two opposite vertices S and A_2 with the following property: S and A_2 are included in exactly three sets of $\mathcal{C}$. But this is impossible. Thus we get a contradiction. □

Notice that Theorem 3 follows from Lemma 8. □

Let us consider a convex set system $\mathcal{F}$ satisfying the conditions of Theorem 1. Applying the FLP-algorithm to $\mathcal{F}$ we get a fixed $\mathcal{F}'$. Let $\mathbf{M}$ be the convex hull of $\mathcal{F}'$. Let R be one of its vertices. Obviously R is an fl-point. Suppose that R is an fl-point with respect to $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$. Removing all sets of $\mathcal{F}'$ containing R and different from $\mathbf{K}_1, \mathbf{K}_2$ and $\mathbf{K}_3$ we get a convex set system $\mathcal{C}$. Applying the FLP-algorithm to $\mathcal{C}$ we get $\mathcal{C}'$. Obviously $\mathcal{C}'$ is good. According to Theorem 3 there exists a lattice point J covered by all ordinary configurations of $\mathcal{C}'$. It is easy to see that J and R pin down $\mathcal{F}$. The proof of Theorem 1 is complete. □

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