The moment map, toric varieties and mixed volumes *

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*dissertation

Contents

1 Introduction

This dissertation consists of two main parts. The first part (sections 2-7) concerns symplectic geometry. The second part (sections 8-11) is about toric geometry. The two parts are linked by the following beautiful theorem of Kushnirenko [12].

Theorem 1.1 (Kushnirenko) *Let S be a finite subset of $\mathbf{Z}^n$. $N(S)$ denotes the number of solutions of the system of equations*

$$\sum_{\alpha \in S} c_\alpha^j z^\alpha = 0 \quad j = 1, ..., n$$

where z^α denotes the monomial $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}$, where the z_i's are nonzero complex numbers and the coefficients c_α^j are assumed to be 'general' complex numbers. Then

$$N(S) = n! V(C(S))$$

where $S \subset \mathbf{Z}^n$ considered to be a subset of $\mathbf{R}^n$, C stands for the convex hull operation and V for the standard Euclidean volume in $\mathbf{R}^n$.

We will give two proofs of this theorem. The first one in section 7 approaches from symplectic geometry. The second proof in subsection 11.3 uses ideas from toric geometry.

Our aim in this dissertation is to show the power of new ideas in modern geometry by applying them to prove theorems which can be stated without using the terminology of higher geometries.

2 Symplectic geometry

2.1 Introduction

Our approach in the first half of this dissertation will be group-theoretical: we will consider group actions on various spaces. In this context there are three basic objects: the space, the group and the action. In our case the space will be a symplectic manifold, the group a Lie-group and later a torus, the action symplectic and later Hamiltonian.

Symplectic geometry is a natural framework for Hamiltonian mechanics. The need to introduce symplectic geometry -from a purely mathematical point of view- comes from the naturality of the construction, the great number of canonical examples and the usefulness of the subject in other branches of mathematics.

In the first three sections we will follow the approach sketched above. The second section will be on symplectic manifolds, the third on Lie-groups and smooth Lie-group actions in general, and the fourth on symplectic actions. Here we will introduce the notion of moment map. The fifth section will contain the description of some properties of the moment map. In the sixth one we will describe the convexity theorem of Atiyah and Guillemin-Sternberg. In the final section of the first part, using symplectic geometry, we prove a theorem of Kushnirenko.

The first part of this dissertation is based on the book of Audin [3]. We also used material here from Weinstein [14], Atiyah [1], [2] and the lecture notes and useful comments of Hitchin [11].

2.2 Symplectic vector spaces

We will first examine geometrical structures on finite dimensional real or complex vector spaces.

Let V be an n-dimensional real vector space. In order to introduce a geometry, we need something to measure: e.g. distances and angles. The notion of Euclidean scalar product then naturally arises. This is a positive definite bilinear form $\langle,\rangle_V$ on the vector space V. Now we have the space V with a Euclidean vector space structure.

We need to introduce more geometrical concepts on a vector space. Let $W_{\mathbf{C}}$ be a complex vector space of dimension n. Here we have the hermitian inner product $\langle,\rangle_W$. This is a positive definite sesquilinear form on $W_{\mathbf{C}}$. It gives the structured space $(W_{\mathbf{C}};\langle,\rangle_W)$.

Now consider W as a real vector space $W_{\mathbf{R}}$. What kind of structures does $W_{\mathbf{R}}$ inherit from $(W;\langle,\rangle_W)$? First look at the complex structure on $W_{\mathbf{C}}$. We get a real linear mapping I on $W_{\mathbf{R}}$ satisfying $I^2 = -1_{W_{\mathbf{R}}}$ (where $1_{W_{\mathbf{R}}}$ stands for the identity in $W_{\mathbf{R}}$) corresponding to the multiplication by i on $W_{\mathbf{C}}$. Writing down the real and imaginary part of $\langle,\rangle_W$ we get

$$\langle,\rangle_W = \mathrm{Re}\langle,\rangle_W + i\mathrm{Im}\langle,\rangle_W,$$

where $\mathrm{Re}\langle,\rangle_W$ is a positive definite bilinear form on $W_{\mathbf{R}}$ called a Euclidean form, while $\mathrm{Im}\langle,\rangle_W$ is an antisymmetric, non-degenerate bilinear form on $W_{\mathbf{R}}$ called a symplectic form.

Definition 2.1 *Let V be a real n-dimensional vector space.*

A positive definite bilinear form is called a Euclidean form.

An antisymmetric non-degenerate bilinear form is called a symplectic form.

A linear transformation I of V with $I^2 = -1_V$ is called a complex structure *on V.*

Furthermore, if we are given a Euclidean form g, symplectic form ω and a complex structure I on V such that $\omega(IX, IY) = \omega(X, Y)$ and $g(X, Y) = \omega(IX, Y)$ then V is equipped with a calibrated complex structure.

Thus $W_{\mathbf{R}}$ is equipped with a Euclidean, symplectic, complex and calibrated complex structure as well. In this section we will examine the relation between these structures. We will see that any symplectic (complex, calibrated complex) stucture on a real vector space V comes from a complex vector space in the way described above. Thus any two symplectic (complex, calibrated complex) vector spaces of the same dimension turn out to be isomorphic. We already know this for Euclidean vector spaces.

Now consider the following canonical example of a symplectic vector space. V is a real vector space of dimension n. Consider $V \oplus V^*$. We define a symplectic form ω_V on $V \oplus V^*$ by setting

$$\omega_V(v_1 \oplus v_1^*, v_2 \oplus v_2^*) = v_1^*(v_2) - v_2^*(v_1).$$

Any symplectic vector space W is equivalent to one of these. Here to be equivalent means that there exists a symplectic structure preserving isomorphism between the two spaces, or symplectomorphism for short.

Proposition 2.1 *For any symplectic vector space (W, ω) there is a symplectomorphism $\beta = (W, \omega) \to (V \oplus V^*, \omega_V)$.*

We need a few definitions to prove this proposition.

Definition 2.2 *Let* (W, ω) *be a symplectic vector space. If* $V \subseteq W$ *then*

$$V^\perp = \{ w \in W ; \omega(v, w) = 0 \ \forall \ v \in V \}.$$

V is called isotropic *if* $V \subseteq V^\perp$, coisotropic *if* $V \supseteq V^\perp$, lagrangian *if* $V = V^\perp$ *and* symplectic *if* $V \cap V^\perp = 0$.

Note that ω restricted to V is zero iff V is isotropic and is symplectic iff V is symplectic. Since ω is a non-degenerate bilinear form $\dim V + \dim V^\perp = \dim W$.

Proof of Proposition. We show that $W = V \oplus U$, where V and U are lagrangian. First consider a highest element V in the lattice of isotropic subspaces. If V is not lagrangian then we would be able to enlarge V by an element from $V^\perp \setminus V$ to get a larger isotropic subspace. Therefore V is lagrangian. (Note that this immediately yields that $\dim W$ is even, say $2n$.) Similarly, consider that sublattice of the isotropic subspaces which contains the isotropic subspaces having trivial intersection with V. This is clearly a non-empty lattice. Consider a highest element U in this lattice. We show that U is lagrangian. Since V is lagrangian and $V \cap U = 0$ we get that $V + U^\perp = W$. Thus if $V + U \neq W$ then we would enlarge U by an element from $U^\perp \setminus (V + U)$ getting a larger element in our sublattice. Thus $V + U = W$ hence U is langrangian and $V \oplus U = W$.

Now let $\alpha : U \to V^*$ be given by $\alpha(u)(v) = \omega(u, v)$, where $u \in U$ and $v \in V$. Since U is lagrangian and thus coisotropic, the kernel of α is trivial. $\dim U = \dim(V) = n$, hence α is an isomorphism.

Let $\beta = 1 \oplus \alpha : V \oplus U \to V \oplus V^*$. Then β is an isomorphism. Moreover, it is a symplectomorphism between the symplectic vector spaces (W, ω) and $(V \oplus V^*, \omega_V)$. To see this, let $v_1 \oplus u_1$ and $v_2 \oplus u_2$ be elements in $V \oplus U = W$. Then

$$
\begin{aligned}
\omega_V \big(\beta(v_1 \oplus u_1), \beta(v_2 \oplus u_2) \big) &= \omega_V \big(v_1 \oplus \alpha(u_1), v_2 \oplus \alpha(u_2) \big) \\
&= \alpha(u_1)(v_2) - \alpha(u_2)(v_1) \\
&= \omega(u_1, v_2) - \omega(u_2, v_1) \\
&= \omega \big(v_1 \oplus u_1, v_2 \oplus u_2 \big)
\end{aligned}
$$

since $\omega(v_1, v_2) = \omega(u_1, u_2) = 0$, as V and U are lagrangian.

Corollary 2.1 *One has a standard basis* $v_1, \dots, v_n, u_1, \dots, u_n$ *for a* $2n$-*dimensional* (W, ω) *symplectic vector space with* $\omega(v_i, v_j) = \omega(u_i, u_j) = 0$ *and* $\omega(v_i, u_j) = \delta_{i,j}$ *where* $i = 1, \dots, n$ *and* $j = 1, \dots, n$. *This is called a* symplectic basis.

Proof. Let $v_1, \dots, v_n$ be a basis for V, and let $v_1^*, \dots, v_n^*$ be the corresponding basis for the dual space V^*. Then the vectors $v_1, \dots, v_n$ and $u_1 = \beta^{-1}(v_1^*), \dots, u_n = \beta^{-1}(v_n^*)$ will do the job.

Theorem 2.1 *Let W be a real vector space equipped with a symplectic (complex, calibrated complex) structure. Then there exists a complex hermitian vector space $W_{\mathbf{C}}$ and a symplectomorphism (complex isomorphism, calibrated complex isomorphism) between W and $W_{\mathbf{R}}$ with the inherited symplectic (complex, calibrated complex) structure from $W_{\mathbf{C}}$.*

Proof. We know that the symplectic vector space W has even dimension say, $2n$. Let $W_{\mathbf{C}}$ be an n-dimensional complex vector space and $\langle , \rangle_W$ be a hermitian inner product on $W_{\mathbf{C}}$. Let $e_1, \ldots, e_n$ be a unitary basis. Set $f_j = -ie_j$, then

$$\operatorname{Im}\langle e_j, f_k \rangle_W = -\operatorname{Im}\langle e_j, ie_k \rangle_W = -\operatorname{Im}(-i\langle e_j, e_k \rangle_W) = \delta_{j,k},$$

and similarly $\operatorname{Im}\langle e_j, e_k \rangle_W = \operatorname{Im}\langle f_j, f_k \rangle_W = 0$. Now it is easy to construct a symplectomorphism between W and $W_{\mathbf{R}}$.

For the complex case consider W with complex structure I. Then the definition $(\alpha + i\beta)v = \alpha v + \beta I v$; $\alpha, \beta \in \mathbf{R}$, $v \in V$ makes V a complex vector space so that the theorem follows.

In the complex calibrated case for a lagrangian V we can choose $U = IV$ such that U is lagrangian and $V \oplus U = W$ since I is symplectomorphism and for $0 \neq w \in W$ $\omega(Iw, w) = g(w, w) \neq 0$. Then the above two isomorphisms fit together, giving the required isomorphism in this case. Note that $g + I\omega$ is a hermitian metric on $W_{\mathbf{C}}$.

2.3 Symplectic manifolds

Let M be a finite dimensional smooth connected manifold. In this case we say that M is a manifold. We want to introduce a geometry on M, and generalize our geometrical structures on vector spaces. The very first natural idea is to give a smoothly varying geometrical structure on the tangent bundle TM.

Definition 2.3 *A smoothly varying Euclidean (symplectic, complex, calibrated complex) structure on the tangent bundle TM is called an* almost Euclidean *(almost symplectic, almost complex, almost Kaehlerian) structure on M.*

To give canonical examples consider $\mathbf{R}^m$ with the standard structures.

In the Euclidean case this is the standard Euclidean inner product $\langle , \rangle$.

In the remaining cases $m = 2n$ and the standard structures are derived from the standard hermitian form on $\mathbf{C}^n$. This vector space is canonically isomorphic to $\mathbf{R}^{2n}$.

Now the tangent bundle of $\mathbf{R}^m$ is trivial so we can choose the constant standard structure on every tangent space, which is clearly smoothly varying. These structured spaces are called the flat models for the corresponding geometries.

What is a manifold? A topological space which looks like an open subset of the standard m-dimensional real vector space $\mathbf{R}^m$ ('the flat model for the real vector space geometry') in a neighbourhood of every point of M. Thus the following definitions arise naturally.

Definition 2.4 *An almost Euclidean (almost symplectic, almost complex, almost Kaehlerian) structure on a manifold M is said to be* flat Euclidean *(*flat symplectic, flat complex, flat Kaehlerian) if for every point p of M there exists a neighbourhood U_p and a local diffeomorphism between U_p and the corresponding flat model preserving the almost geometrical structure, where U_p inherits the almost geometrical structure from M.*

Thus an almost geometrical structure on a manifold M is flat if it looks like an open subset of the flat model of the corresponding geometry in a neighbourhood of every point of M.

Definition 2.5 *An almost Euclidean structure on the tangent bundle TM is called a* Riemannian structure *on M.*

It is known that the manifold M is flat-Riemannian iff the curvature of the Levi-Civita connection vanishes everywhere on M.

Definition 2.6 *A flat symplectic structure on a manifold M is called a* symplectic structure *for short.*

In the following theorem of Darboux we give the necessary and sufficient condition for an almost symplectic manifold to be a symplectic manifold. Here note that an almost symplectic structure on M is just an everywhere non-degenerate 2-form $\omega \in \bigwedge^2(T^*M)$.

Definition 2.7 *A flat complex structure on a manifold M is called a* complex structure *for short.*

It is easy to see that our definition of a complex manifold coincides with the old one.

Definition 2.8 *An almost Kaehlerian stucture is called a* Kaehlerian structure *iff it is complex and a symplectic structure as well.*

Note that this is equivalent to say that a manifold M is Kaehlerian iff the manifold M is complex with a hermitian metric on it, whose imaginary part defines a symplectic structure. Notice that the real part of the hermitian structure automatically becomes a Riemannian structure.

We remark that an almost Kaehlerian manifold is Kaehlerian iff $\nabla I = 0$, where ∇ is the Levi-Civita connection corresponding to g.

A simple observation is that a Kaehlerian structure is flat Kaehlerian iff it is flat Riemannian.

Theorem 2.2 (Darboux) *Let the almost symplectic structure on M given by the non-degenerate 2-form $\omega \in \bigwedge^2(T^*M)$. Then M is symplectic iff $d\omega = 0$.*

Proof. Suppose that M is a symplectic manifold. Since the condition to be closed is local, it suffices to show that for the standard symplectic manifold $\mathbf{R}^{2n}$ the standard symplectic 2-form $\omega_{\mathbf{R}^{2n}}$ is closed. Let $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ be a symplectic basis. Then from Proposition 2.1 and Corollary 2.1 we can deduce that $\omega_{\mathbf{R}^{2n}} = \sum_{i=1}^{n} dx_i \wedge d\xi_i$. This is obviously a closed form.

Now let ω be a closed form. Let $p \in M$ and U a neighbourhood of p with a diffeomorphism $\phi : U \to \mathbf{R}^{2n}$. Let ω_0 be the restriction of ω to U and $\omega_1 = \phi^* \omega_{\mathbf{R}^{2n}}$ where $\omega_{\mathbf{R}^{2n}}$ is the standard symplectic 2-form on $\mathbf{R}^{2n}$. Therefore ω_0 and ω_1 are closed 2-forms on U. Choosing ϕ more precisely (i.e. composing with a suitable linear automorphism of $\mathbf{R}^{2n}$) we can assume that ω_0 and ω_1 coincide at p. We will show that there exists a neighbourhood V of p and a local diffeomorphism $\psi : V \to M$ fixing p such that $\psi^* \omega_1 = \omega_0$.

Consider the form $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. This is closed for all real t and coincides at p with ω_0 that is non-degenerate at p and so it is in a neighbourhood of p. From the compactness of the interval $[-3\epsilon, 1 + 3\epsilon]$ ($\epsilon > 0$) we have a neighbourhood of p, say U_1 diffeomorphic to $\mathbf{R}^{2n}$, where ω_t is non-degenerate whenever $t \in [-3\epsilon, 1 + 3\epsilon]$.

Now $\omega_0 - \omega_1$ is a closed 2-form on U_1, therefore from Poincare's lemma this is an exact form as well, i.e. there is a 1-form $\varphi \in T^* U_1$ on U_1 such that $d\varphi = \omega_0 - \omega_1$. Adding a suitable constant 1-form to φ we can suppose that φ vanishes at p.

Since ω_t is non-degenerate the formula $i(X_t)\omega_t = \varphi$ defines a unique vector field X_t on U_1 vanishing at p. Moreover, we have that

$$\mathcal{L}_{X_t}\omega_t = d(i(X_t)\omega_t) + i(X_t)d\omega_t = d(i(X_t)\omega_t) = d\varphi = \omega_0 - \omega_1.$$

Since $T(M \times (-2\epsilon, 1 + 2\epsilon)) = T(M) \oplus \mathbf{R}$ the definition $X(p, t) = (X_t(p), t)$ gives a local vector field on $M \times (-2\epsilon, 1 + 2\epsilon)$.

Since $[-\epsilon, 1+\epsilon]$ is compact we can uniformly integrate this vector field on $V \times (-\epsilon, 1+\epsilon)$ where V is a neighbourhood of p. This means that there exists a $\delta > 0$ and a local one-parameter group of local diffeomorphisms

$$\Phi : (-\delta, +\delta) \times V \times (-\epsilon, 1 + \epsilon) \to M \times (-2\epsilon, 1 + 2\epsilon)$$

such that

$$\frac{d}{dv}(\Phi_v(m, t))|_{v=0} = X(m, t).$$

In particular $\Phi_v(p, 0) = p$. Now it is clear that for the local diffeomorphisms $f_v(m) = \Phi_v(m, 0)$ one has the formula $\frac{d}{dv}f_v(m)|_{v=t} = X_t(f_t(m))$ and obviously $f_v(p) = p$. In the next calculation we use Lemma 2.1 (proved at the end of this section).

$$\frac{d}{dt}(f_t^* \omega_t) = f_t^* \left[\frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t \right] = f_t^*(\omega_1 - \omega_0 + \omega_0 - \omega_1) = 0.$$

Therefore $f_t^* \omega_t$ is constant, thus $f_t^* \omega_t = f_0^* \omega_0 = \omega_0$. Hence for $\psi = f_1$, which is a local diffeomorphism on V, we have that $\psi^* \omega_1 = \omega_0$.

Thus, around p we have $\omega = (\psi\phi)^* \omega_{\mathbf{R}^{2n}}$ with the local diffeomorphism $\psi\phi : V \to \mathbf{R}^{2n}$.

Corollary 2.2 *Let p be a point in a symplectic manifold (M, ω). There exists a system of local coordinates $(x_1, \ldots, x_n, \xi_i, \ldots \xi_n)$ centered at p such that $\omega = \sum_{i=1}^{n} dx_i \wedge d\xi_i$.*

Example. Now we show that the manifold T^*M has a canonical symplectic structure for every manifold M (cf. Proposition 2.1). First we find a canonical 1-form $\alpha \in T^*(T^*M)$ on T^*M called a Liouville form. Set

$$\alpha_{(p,\varphi)}(X) = \varphi(T_p\pi^*(X)),$$

where $p \in M$, $\varphi \in T_p^*M$, $X \in T_{(p,\varphi)}(T^*M)$ and π^* is the bundle projection $\pi^* : T^*M \to M$. We show that the 2-form $\omega = -d\alpha$ can be written locally as $\sum_{i=1}^n dx_i \wedge d\xi_i$, and therefore ω is a symplectic form.

Let $(x_1, \ldots, x_n) : U \to \mathbf{R}^n$ be a coordinate system on an open subset of M. Then we have the basis sections dx_i in T^*M that give us a coordinate system $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ on $\pi^{*-1}(U)$. Clearly $x_i(\varphi) = x_i(\pi^*\varphi)$ and $\xi_i(\varphi) = \varphi(\frac{\partial}{\partial x_i})$, where $\varphi \in \pi^{*-1}(U)$.

Now $\alpha_\varphi(\frac{\partial}{\partial x_i}) = \varphi(\frac{\partial}{\partial x_i}) = \xi(\varphi)$ and $\alpha_\varphi(\frac{\partial}{\partial \xi_i}) = \varphi(0) = 0$ since $T\pi^*(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ and $T\pi^*(\frac{\partial}{\partial \xi_i}) = 0$. Thus $\alpha = \sum_{i=1}^n \xi_i dx_i$ hence $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$ indeed.

Lemma 2.1 *Let $f_t : M \to M$ be a smooth one-parameter family of diffeomorphisms. Let $X_{t_0}(f_{t_0}(m)) = \frac{d}{dt}f_t(m) \mid_{t=t_0}$ be vector fields on M and let σ_t be a smooth one-parameter family of k-forms on M. Then*

$$\frac{d}{dt}(f_t^*\sigma_t) \mid_{t=t_0} = f_{t_0}^* \left[\frac{d\sigma_t}{dt} \mid_{t=t_0} + \mathcal{L}_{X_{t_0}} \sigma_{t_0} \right].$$

Proof. First we prove the formula when $k = 0$, i.e. when the forms $\sigma_t = g_t \in \mathcal{C}^\infty(M)$ are functions.

$$
\begin{aligned}
\frac{d}{dt}(f_t^* g_{t_0}(m)) \mid_{t=t_0} &= i(\frac{d}{dt}f_t(m) \mid_{t=t_0})dg_{t_0} \\
&= i(X_{t_0}(f_{t_0}(m)))dg_{t_0} \\
&= f_{t_0}^* \left[\mathcal{L}_{X_{t_0}} g_{t_0} \right](m),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{d}{dt}(f_t^* g_t(m)) \mid_{t=t_0} &= \frac{d}{dt}(f_{t_0}^* g_t(m)) \mid_{t=t_0} + \frac{d}{dt}(f_t^* g_{t_0}(m)) \mid_{t=t_0} \\
&= f_{t_0}^* \left[\frac{dg_t}{dt} \mid_{t=t_0} \right](m) + f_{t_0}^* \left[\mathcal{L}_{X_{t_0}} g_{t_0} \right](m).
\end{aligned}
$$

Now if we are given k-forms σ_t $(k > 0)$ then if we evalute the k-forms in both sides of our equation with arbitrary vector fields $Y_1, \ldots, Y_k$, then what remains to be proven is exactly the case $k = 0$ with functions $\sigma_t(Y_1, \ldots, Y_k)$.

3 Smooth actions

In this section we list some properties of smooth Lie-group actions for later reference. At the end of this section we will show a very important construction how to make symplectic manifolds.

Definition 3.1 *Let G be a Lie-group with Lie algebra $\mathbf{g}$.*

A smooth action (or simply an action) of a manifold M is a smooth map $\phi : G \times M \to M$ such that for fixed $g \in G$ $\phi_g : M \to M$ is a diffeomorphism of M and the map $g \mapsto \phi_g$ is a homomorphism from G to the group of diffeomorphisms of M. We will denote the element $\phi_g(m)$ by gm.

If m is a point in M its orbit *is denoted by Gm and its* stabilizer *by G_m.*

The map $f_m : G \to M$ with the definition $f_m(g) = gm$ is called an orbit *map.*

If $\mathbf{g}$ is a Lie-algebra then a Lie-algebra homomorphism $\nu : \mathbf{g} \to \Gamma(TM)$ is called a smooth infinitesimal action *(or simply an infinitesimal action).*

We can associate an infinitesimal action ν_ϕ with the smooth Lie-group action ϕ by the following definition. Let $\nu_\phi : \mathbf{g} \to \Gamma(TM)$ given by $\nu_\phi(X)_m = T_e f_m(X)$ where $X \in \mathbf{g}$ and $\mathbf{g}$ is identified with $T_e G$. $\nu_\phi(X)$ is a fundamental vector field. The term 'infinitesimal' refers to the fact that $\exp(tX)$ is a flow of $\nu_\phi(X)$. This easily yields that ν_ϕ is in fact a Lie-algebra homomorphism.

The infinitesimal action contains a lot of information about the group action. We now list some of these. ν_ϕ is injective iff the action is effective, i.e. only the unit element of G acts as an identity of M. $\nu_\phi(X)$ vanishes at m iff the one-parameter subgroup $\exp(tX)$ fixes the point p. Hence the Lie-algebra $\mathbf{g}_p$ of the Lie-group G_p consists of the vectors X for which $\nu_\phi(X)$ vanishes at p. The vector fields in the image of ν_ϕ are tangent to the orbits. Moreover $\nu_\phi(X)_m, X \in \mathbf{g}$ spans the tangent space of the orbit Gm at m.

Theorem 3.1 *If G is compact, then every orbit is a submanifold of M.*

Proof. Let $m \in M$. G_m is a closed subgroup of G hence G/G_m is a compact manifold. We show that Gm is a submanifold of M diffeomorphic to G/G_m. Trivially, we can lift the orbit map to get $f_m : G/G_m \to M$. Thus it suffices to prove that f_m is an embedding. It is clearly injective.

f_m is an immersion. To see this let us examine the kernel of the linear map $T_g f_m : T_g G \to T_{gm} M$. Now from a remark above

$$0 = T_g f_m(X) = T_e f_{gm}(Tg^{-1}(X)) = \nu_\phi(Tg^{-1}(X))_{gm} \Leftrightarrow Tg^{-1}(X) \in \mathbf{g}_{gm},$$

so $\ker(T_g f_m) = Tg(\mathbf{g}_{gm})$. Therefore $f_m : G/G_m \to M$ is an immersion indeed.

Since the manifold G/G_m is compact f_m is closed and so is an embedding.

We give a useful description of $T(Gm)$. Consider the trivial bundle $Gm \times \mathbf{g}$ over Gm. Then it is easy to see that the subbundle over Gm which corresponds the subspace $\mathbf{g}_{gm}$ to gm is smooth and the tangent bundle of Gm is the quotient of $Gm \times \mathbf{g}$ by this subbundle.

Since we need it later and the proof is rather complicated we cite the following result without proof from Audin [3].

Theorem 3.2 *The set of fixed points of G is a submanifold of M.*

In the last part of this section we will study canonical smooth Lie-group actions, namely, the adjoint and coadjoint actions.

Definition 3.2 *Let* $\varphi_g : G \to G$ *be given by* $\varphi_g(h) = ghg^{-1}$ *and* $\mathrm{Ad}_g = T_e(\varphi_g) : \mathbf{g} \to \mathbf{g}$.

Clearly, φ *is an action on the Lie-group* G *and* Ad *is an action on the Lie-algebra* $\mathbf{g}$. *The latter is called the* adjoint action.

The transpose of the adjoint action Ad *is the* coadjoint action Ad^* *on* $\mathbf{g}^*$ *which is therefore defined by*

$$\langle \mathrm{Ad}_g^* \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle,$$

where X *denotes an element in* $\mathbf{g}$, g *in* G, ξ *in* $\mathbf{g}^*$ *and* $\langle , \rangle$ *stands for the evaluation of a vector in* $\mathbf{g}$ *with a function in* $\mathbf{g}^*$.

Remark. One has the following formula for the Lie bracket on $\mathbf{g}$

$$[X, Y] = \frac{d}{dt} \mathrm{Ad}_{\exp(tX)} Y|_{t=0}.$$

Lemma 3.1 *In the situation described above the following holds*

$$\nu_{\mathrm{Ad}}(X)_Y = [X, Y],$$

i.e. $\nu_{\mathrm{Ad}} = \mathrm{ad}$ *and*

$$\langle \nu_{\mathrm{Ad}^*}(X)_\xi, Y \rangle = \langle \xi, [Y, X] \rangle = \langle \xi, -\nu_{\mathrm{Ad}}(X)_Y \rangle,$$

i.e. ν_{Ad^*} *is the transpose of* ν_{Ad}.

Proof. The first statement comes simply from the infinitesimality of ν combined with the above remark.

The second equation follows simply from the fact that

$$\nu_{\mathrm{Ad}^*}(X)_\xi = T_e f_\xi(X) = \frac{d}{dt} \mathrm{Ad}^*_{\exp(tX)} \xi|_{t=0},$$

where $f_\xi(g) = \mathrm{Ad}_g^*(\xi)$ is the orbit map.

Now we give a very important canonical example for symplectic manifolds. (Although we will only prove that they are almost symplectic.) Namely, the orbits of the coadjoint action of a compact Lie-group.

Definition 3.3 *For any* $\xi \in \mathbf{g}^*$ *we define an alternating bilinear form* ω_ξ *on* $\mathbf{g}$ *by*

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle = -\langle \nu_{\mathrm{Ad}^*}(X)_\xi, Y \rangle.$$

Now we show that this ω defines an almost symplectic structure on $G\xi$. The kernel of ω_ξ is $\mathbf{g}_\xi$ since

$$X \in \ker(\omega_\xi) \Leftrightarrow \omega_\xi(X, Y) = 0 \ \ \forall(Y \in \mathbf{g}) \Leftrightarrow \nu_{\mathrm{Ad}^*}(X)_\xi = 0 \Leftrightarrow X \in \mathbf{g}_\xi.$$

Thus using the descripton of $T(G\xi)$ we gave above, we have a non-degenerate alternating 2-form $\omega \in \bigwedge^2(T(G\xi))$ on G. Thus we defined an almost symplectic structure on $G\xi$.

It turns out that ω is closed, i.e. it defines a symplectic structure on $G\xi$.

4   Symplectic actions

Clearly, for the definition of a symplectic Lie group action $\phi : G \times M \to M$ we demand ϕ_g to be a symplectomorphism. Turning to the inifinitesimal level one gets the result that the flow of the vector field $\nu_\phi(X)$ the one-parameter group of symplectomorphisms $\exp(\mathrm{tX})$, leaves ω invariant, i.e. $\mathcal{L}_{\nu_\phi(X)}\omega = 0$.

Definition 4.1 *Let G be a Lie group and (M,ω) be a symplectic manifold. A smooth Lie-group action $\phi : G \times M \to M$ is called symplectic if ϕ_g is a symplectomorphism for every $g \in G$.*

Let $\mathbf{g}$ be a Lie-algebra. A smooth infinitesimal action $\nu : \mathbf{g} \to \Gamma(TM)$ is called symplectic if $\mathcal{L}_{\nu(X)}\omega = 0$ for every $X \in \mathbf{g}$.

Let us examine the vector fields $X \in \Gamma(TM)$ for which $\mathcal{L}_X\omega = 0$. Thus

$$0 = \mathcal{L}_X\omega = d(i(X)\omega) + i(X)d\omega = d(i(X)\omega),$$

i.e. $\mathcal{L}_{\nu(X)}\omega = 0$ iff $i(X)\omega$ is closed. From Poincare's lemma every closed form is locally exact so locally we have a $\mathcal{C}^\infty$-function f for which $df = i(X)\omega$. If we have this property globally, then we say that the vector field $X \in \Gamma(TM)$ is Hamiltonian and is associated with the $\mathcal{C}^\infty$-function $f \in \mathcal{C}^\infty(M)$.

Definition 4.2 *The vector field $X \in \Gamma(T(M))$ is called* Hamiltonian *if the 1-form $i(X)\omega$ is exact and called* locally Hamiltonian *if it is closed.*

Notice that if X_f is a Hamiltonian vector field associated with a $\mathcal{C}^\infty$-function f, then $X_f f = df(X_f) = \omega(X_f, X_f) = 0$, which means that the flow generated by X_f leaves f invariant, i.e. the vector field X_f is tangent to the level surfaces of f.

Starting with an $f \in \mathcal{C}^\infty(M)$, the condition $i(X_f)\omega = df$ gives a unique Hamiltonian vector field X_f by ω being non-degenerate. This defines a map α from $\mathcal{C}^\infty(M)$ to $\mathcal{H}(M)$, the set of Hamiltonian vector fields on M. Now examine this set $\mathcal{H}(M)$. It is clearly a sub-vector space of $\Gamma(TM)$. Moreover, we claim that this is a Lie subalgebra of $\Gamma(TM)$. We show more, namely that the larger vector space $\mathcal{H}_{loc}(M)$, the vector space of locally Hamiltonian vector fields on M, is a Lie subalgebra of $\Gamma(TM)$ whose centralizer is in $\mathcal{H}(M)$. To do this, let $X, Y \in \mathcal{H}_{loc}(M)$. Then

$$
\begin{aligned}
i([X,Y])\omega &= \mathcal{L}_X i(Y)\omega - i(Y)\mathcal{L}_X\omega \\
&= (di(X) + i(X)d)i(Y)\omega - i(Y)(di(X) + i(X)d)\omega \\
&= di(X)(i(Y)\omega)
\end{aligned}
$$

since 'Lie differentiation is a derivation of any naturally defined bilinear operation on tensors' [1] and from the assumption for X and Y, $i(X)\omega$ and $i(Y)\omega$ as well as ω are closed forms. Thus indeed $[X,Y] \in \mathcal{H}(M)$ and the spaces $\mathcal{H}(M) \subset \mathcal{H}_{loc}(M)$ are Lie subalgebras of $\Gamma(TM)$.

Now we give a Lie-algebra structure on $\mathcal{C}^\infty(M)$ which makes α a Lie-algebra homomorphism.

---

[1] see Weinstein[**3**], page 17

Definition 4.3 *The Poisson bracket of two $\mathcal{C}^\infty$-functions f and g is defined by*

$$\{f, g\} = \omega(X_g, X_f).$$

From the calculation above one has that

$$X_{\{f,g\}} = X_{\omega(X_g, X_f)} = [X_f, X_g].$$

Thus after the next theorem we will see that α is a Lie-algebra homomorphism indeed.

Theorem 4.1 *The Poisson bracket defines a Lie-algebra structure on $\mathcal{C}^\infty(M)$.*

Proof. Using that ω is closed and the previous formula $X_{\{f,g\}} = [X_f, X_g]$ the calculation is straightforward.

It is a nice way to summarize what we learned so far in a diagram :

$$
\begin{array}{ccccccc}
\mathcal{C}^\infty(M) & \xleftarrow{\tilde{\mu}} & & \mathbf{g} & & & \\
\alpha \downarrow & & & \nu \downarrow & & & \\
0 \longrightarrow & \mathcal{H}(M) & \xrightarrow{\iota} & \mathcal{H}_{loc}(M) & \xrightarrow{\beta} & H^1(M, \mathbf{R}) \longrightarrow 0 &
\end{array}
$$

Give the abelian Lie-algebra structure on $H^1(M, \mathbf{R})$ in order to have a diagram of Lie-algebras. Now examine the maps. Let $\beta(X) = [i(X)\omega]$. Then clearly α, ν and ι (the embedding) are Lie-algebra homeomorpisms. Then so is β, as we saw that

$$[\mathcal{H}_{loc}(M), \mathcal{H}_{loc}(M)] \subset \mathcal{H}(M) = \ker(\beta).$$

Furthermore, the horizontal row is clearly a short exact sequence of Lie-algebras.

Examining the diagram carefully, we notice a new map $\tilde{\mu}$.

Definition 4.4 *Let ν be a symplectic infinitesimal action of $\mathbf{g}$. If there is a Lie-algebra homomorphism $\tilde{\mu} : \mathbf{g} \to \mathcal{C}^\infty(M)$ which makes the diagram commute, then we say that the infinitesimal action is* Hamiltonian.

Moreover, a symplectic Lie-group action ϕ is called Hamiltonian *if its infinitesimal action ν_ϕ is Hamiltonian.*

The name of $\tilde{\mu}$ is comoment map, *while the associated map $\mu : M \to \mathbf{g}^*$ with the definiton $\langle \mu(m), X \rangle = \tilde{\mu}_X(m)$ is the* moment map.

5 Properties of the moment map

First we examine whether we can find a comoment map making the action Hamiltonian. The first problem is to find a lift $\tilde{\mu} : \mathbf{g} \to \mathcal{C}^\infty(M)$ of ν which is a linear map (not necessarily a Lie-algebra homomorphism) making the diagram commute. In other words, we need a function $\tilde{\mu}(X) \in \mathcal{C}^\infty(M)$ with $d\tilde{\mu}(X) = i(\nu(X))\omega$ for each $X \in \mathbf{g}$. Therefore a sufficient condition for the existence of such a $\tilde{\mu}$ is that any closed form is an exact form i.e. $H^1(M, \mathbf{R}) = 0$.

The next step is to find out whether such a $\tilde{\mu}$ is a Lie-algebra homomorphism. As we saw $d\{\tilde{\mu}(X), \tilde{\mu}(Y)\} = d\tilde{\mu}([X,Y])$ so $\{\tilde{\mu}(X), \tilde{\mu}(Y)\} - \tilde{\mu}([X,Y])$ is locally constant. Thus the necessary condition now is that this constant vanishes for every $X, Y \in \mathbf{g}$.

In our most important case the following theorem holds.

Theorem 5.1 *If G is commutative and the manifold M is compact then any lift $\tilde{\mu}$ is a Lie-algebra homomorphism.*

Proof. In this case $\mathbf{g}$ is abelian so from the foregoing $\{\tilde{\mu}(X), \tilde{\mu}(Y)\}$ is locally constant. But M is compact hence $\tilde{\mu}(Y)$ has a critical point m i.e., with $d\tilde{\mu}(Y)_m = 0$. At this point $\{\tilde{\mu}(X), \tilde{\mu}(Y)\}(m) = \omega(\nu(Y), \nu(X)) = d\tilde{\mu}(X)_m(\nu(Y)) = 0$. Therefore $\{\tilde{\mu}(X), \tilde{\mu}(Y)\}$ vanishes i.e. the result follows.

We have the following simple but very important theorem:

Theorem 5.2 *If μ is the moment map of an infinitesimal action $\nu : \mathbf{g} \to \Gamma(M)$ and $\iota : \mathbf{h} \to \mathbf{g}$ is a Lie algebra homomorphism then $\iota^*\mu : M \to \mathbf{h}^*$ is a moment map for the induced infinitesimal action $\nu\iota$ of $\mathbf{h}$ on M.*

Examples. 1. Suppose that we are given a symplectic action of $\mathbf{R}$ on the manifold M that is a one-parameter subgroup of symplectomorphisms. Let X be a generator of the Lie algebra of $\mathbf{R}$. Then the action is Hamiltonian if and only if $\nu(X)$ is Hamiltonian. Let $f \in \mathcal{C}^\infty(M)$ such that $df = \nu(X)$. Then we can identify f with a moment map $\mu : M \to \mathbf{R}^*$.

2. As we saw T^*M was a symplectic manifold. Suppose that we are given a smooth action $\phi : G \times M \to M$. This naturally defines a smooth G-action $\tilde{\phi}$ on the cotangent bundle, which, as we will see turns out to be Hamiltonian. Thus let

$$\tilde{g}(m, \varphi) = (gm, \varphi(T_{gm}g^{-1})),$$

where $g \in G$, $m \in M$ and $\varphi \in T_m^*M$. If $g, h \in G$ then

$$
\begin{aligned}
\tilde{h}\tilde{g}(m, \varphi)) &= \tilde{h}(gm, \varphi(T_{gm}g^{-1})) \\
&= (hgm, \varphi(T_{gm}g^{-1}T_{ghm}h^{-1})) \\
&= (hgm, \varphi(T_{ghm}(hg)^{-1})) \\
&= \tilde{h}\tilde{g}(m, \varphi).
\end{aligned}
$$

Thus we get indeed a G-action on T_M^*. Now we show that α is G-invariant, i.e. $g^*\alpha = \alpha$ for all $g \in G$. Let $X \in T_{(m_0, \varphi_0)}(T^*M)$, and let $\xi_t = (m_t, \varphi_t) \in T^*M$ such that $\frac{d}{dt}\xi_t \mid_{t=0} = X$. Then

$$
\begin{aligned}
\alpha_{\tilde{g}(m_0, \varphi_0)}(T\tilde{g}X) &= \alpha_{\tilde{g}(m_0, \varphi_0)}(T\tilde{g}\frac{d}{dt}\xi_t \mid_{t=0}) \\
&= \alpha_{\tilde{g}(m_0, \varphi_0)}(\frac{d}{dt}\tilde{g}(\xi_t) \mid_{t=0})
\end{aligned}
$$

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$$
\begin{aligned}
&= \alpha_{\tilde{g}(m_0,\varphi_0)}(\frac{d}{dt}(g(m_t),\varphi_t T_{gm_t}g^{-1})) \\
&= \varphi_t(T_{gm_0}g^{-1}T_{gm_0}\pi^*(\frac{d}{dt}(gm_t,\varphi_t T_{gm_t}g^{-1})\mid_{t=0})) \\
&= \varphi_0(T_{gm_0}g^{-1}(\frac{d}{dt}gm_t\mid_{t=0})) \\
&= \varphi_0(\frac{d}{dt}m_t\mid_{t=0}) \\
&= \alpha_{(m_0,\varphi_0)}(X).
\end{aligned}
$$

Hence $\omega = -d\alpha$ is also G-invariant, because d commutes with the pull-back. Therefore ϕ is symplectic. Now we show that it is in fact a Hamiltonian action with comoment map $\tilde{\mu}(X) = -i(\nu_{\tilde{\phi}}(X))\alpha = -\alpha(\nu_{\tilde{\phi}}(X))$. First, it is a lift of $\nu_{\tilde{\phi}}$ since $d\tilde{\mu}(X) = i(\nu_{\tilde{\phi}}(X))\omega$. Second, it is a Lie-algebra homomorphism since

$$
\begin{aligned}
\tilde{\mu}([X,Y]) &= i([\nu_{\tilde{\phi}}(Y),\nu_{\tilde{\phi}}(X)])\alpha \\
&= d(i(\nu_{\tilde{\phi}}(X))\alpha)\nu_{\tilde{\phi}}(Y) - d(i(\nu_{\tilde{\phi}}(Y))\alpha)\nu_{\tilde{\phi}}(X) - d\alpha(\nu_{\tilde{\phi}}(Y),\nu_{\tilde{\phi}}(X)) \\
&= 2\{\tilde{\mu}(X),\tilde{\mu}(Y)\} - \omega(\nu_{\tilde{\phi}}(X),\nu_{\tilde{\phi}}(Y)) \\
&= \{\tilde{\mu}(X),\tilde{\mu}(Y)\}.
\end{aligned}
$$

Let us describe now the tangent map $T_m\mu : T_mM \to T_{\mu(m)}\mathbf{g}^* = \mathbf{g}^*$ of the moment map μ. Let $Z = \frac{d}{dt}\varphi_t\mid_{t=0}, \varphi_t \in M$ and $Y \in \mathbf{g}$ then

$$
\begin{aligned}
\langle T_m\mu(Z),Y \rangle &= \langle T_m\mu(\frac{d}{dt}\varphi_t\mid_{t=0}),Y \rangle \\
&= \frac{d}{dt}\langle \mu(\varphi_t),Y \rangle \mid_{t=0} \\
&= \frac{d}{dt}\tilde{\mu}_Y(\varphi_t)\mid_{t=0} \\
&= \omega_m(\nu(Y),Z).
\end{aligned}
$$

Hence we conclude the following lemma.

Lemma 5.1 *Let $Z \in T_mM$ and $Y \in \mathbf{g}$ then*

$$
\langle T_m\mu(Z),Y \rangle = \omega_m(\nu(Y),Z).
$$

At the end of Section 3 we defined a canonical G-action on $\mathbf{g}^*$ the coadjoint action. In the next theorem we show that μ is an equivariant map.

Theorem 5.3 *The moment map is equivariant, i.e. $\mu g = g\mu$ for all $g \in G$.*

Proof. First we show the infinitesimal version of this theorem, namely that

$$
T_m\mu(\nu_\phi(X)) = \nu_{Ad^*}X_{\mu(m)}.
$$

Using the lemma,

$$
\begin{aligned}
\langle T_m\mu(\nu_\varphi(X)), Y \rangle &= \omega_m(\nu_\varphi(X), \nu_\varphi(Y)) \\
&= -\tilde{\mu}_{[X,Y]}(m) \\
&= \langle \mu(m), [Y, X] \rangle \\
&= \langle \nu_{Ad^*}(X)_{\mu(m)}, Y \rangle.
\end{aligned}
$$

Since G is connected we can write any $g \in G$ in the form $g = \exp X_1 \ldots \exp X_k$ with some $X_1, \ldots, X_k \in \mathbf{g}$. Therefore it suffices to prove that μ commutes with any g_1 in the form $g_1 = \exp(X)$. Let $g_t = \exp(tX)$ and $m \in M$. Note that $g_t(m)$ is an integral curve of $\nu_\phi(X)$ while $g_t\mu(m)$ is an integral curve of $\nu_{Ad^*}(X)$. From the infinitesimal equivariance one gets that $T_{g_tm}\mu(\nu_\phi(X)) = \nu_{Ad^*}X_{\mu(g_tm)}$. Thus both $g_t\mu(m)$ and $\mu(g_tm)$ are integral curves of $\nu_{Ad^*}(X)$ with $g_0\mu(m) = \mu(g_0m) = \mu(m)$, hence from the uniqueness $g_t\mu(m) = \mu(g_tm)$ and in particular $g_1\mu(m) = \mu(g_1m)$. The result follows.

6   The convexity theorem

In this section we examine Hamiltonian torus actions and prove the following theorem.

Theorem 6.1 *Let T be a torus and let $\phi : T \times M \to M$ be a Hamiltonian action with moment map $\mu : M \to \mathbf{t}^*$. Then $\mu(M)$ is convex.*

Furthermore, if $m_1, \ldots, m_k$ are the fix points of ϕ then $\mu(M)$ is the convex hull of the points $\mu(m_1), \ldots, \mu(m_k)$.

Let us investigate the fixed points of $\mathbf{T}$. Let $X \in \mathbf{t}$ such that the closure of $\exp(tX)$ is the whole torus $\mathbf{T}$. Then the associated Hamiltonian $\iota^* : M \to (\mathbf{R}X)^*$ has the fixed points of $\mathbf{T}$ as critical points.

Definition 6.1 *A function $h : M \to \mathbf{R}$ is an* almost periodic Hamiltonian function *if the flow of the associated Hamiltonian vector field X_h generates a subgroup of the group of all symplectomorphisms of M the closure of which is a torus.*

In this case we say that X_h is an almost periodic Hamiltonian vector field.

Remarks.   Since the closure of the one-parameter subgroup generated by any vector field is a commutative group the condition above is equivalent to require that the subgroup generated by X_h is relatively compact in the group of symplectomorphisms of M.

Thus for any $X \in \mathbf{t}$, $\tilde{\mu}(X)$ is an almost periodic Hamiltonian function and $\nu(X)$ is an almost periodic Hamiltonian vector field.

Now we describe the behaviour of an almost periodic function h at one of its critical points z. Let the corresponding symplectic torus action be $\phi : T \times M \to M$ with $\tilde{X}_h \in \mathbf{t}$ such that $\nu(\tilde{X}_h) = X_h$.

We know from Theorem 3.2 that a connected component of the fixed points of a smooth action is a submanifold M. Therefore the connected component Z containing z in the set of all critical points of h is a submanifold of M.

Theorem 6.2 *Z is a symplectic submanifold of M.*

In particular Z has even dimension, say $2r$.

Theorem 6.3 *There exists a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of $T_z M$ where $T_z Z$ is spanned by $u_1, \ldots, u_r, v_1, \ldots, v_r$ for which the Hessian of h is the quadratic form in the form*

$$d^2 h(x) = \sum_{i=r+1}^{n} \lambda_i (x_{u_i}^2 + x_{v_i}^2),$$

where $x = \sum_{i=1}^{n} (x_{u_i} u_i + x_{v_i} v_i)$ and $\lambda_i \neq 0$.

Before we prove the theorems we need the following lemma:

Lemma 6.1 *Let $\phi : G \times M \to M$ be a symplectic action on the symplectic manifold M. Then there exists an almost complex structure $I \in \Gamma(\text{End}(TM))$ which makes (M, ω) an almost Kaehlerian manifold, that is to say that I preserves ω and the symmetric bilinear form $\omega(IX, Y)$ is positive definite at each point. Furthermore G preserves I.*

Sketch Proof. It is possible to find a Riemannian metric $\tilde{g}$ on M which is preserved by G. This defines a skew-symmetric $A \in \Gamma(\text{End}(TM))$ with $\tilde{g}(X, AY) = \omega(X, Y)$ for all $X, Y \in \Gamma(T(M))$.

We know that A has polar decomposition, i.e. a $A = BI$ where $B \in \Gamma(\text{End}(TM))$ is symmetric positive definite and $I \in \Gamma(\text{End}(TM))$ is an isometry. We used the fact that $A \mapsto B = (AA^*)^{\frac{1}{2}}$ is a smooth correspondence. Furthermore G preserves B and I.

Now an easy calculation shows that I is an almost complex structure preserving ω and $g(X, Y) = \tilde{g}(X, BY)$ is a Riemmanian metric.

Proofs of the Theorems. From Lemma 6.1 we know that we can find a calibrated almost complex structure I on M and hermitian metric $\langle, \rangle$ whose imaginary part is ω, and for which **T** preserves I and $\langle, \rangle$.

Since z is a fixed point **T** acts on the complex vector space T_z preserving the hermitian form, i.e. as a subgroup of $U(n)$. Notice that any element in **T** commutes with I. Since every element of $\mathbf{u}(n)$ (the skew-hermitian matrices) is diagonisable (i.e. there exists a basis from eigenvectors) and $\exp : \mathbf{t} \to T$ is surjective we see that every element of **T** is diagonisable (as complex transformations). Recall that if two complex transformations A and B commute, then B leaves invariant any eigensubspace of A. $T_z Z$ is clearly the intersection of the eigensubspaces of each element of **T** corresponding to the eigenvalue 1. Using the previous three remarks one can easily find a complex basis $e_1, \ldots, e_n$ of $T_z M$ such that every e_i is an eigenvector of every element of **T**, furthermore $T_z Z$ is spanned by $e_1, \ldots, e_r$.

Now Theorem 6.2 follows from the fact that I commutes with every element of **T**, therefore leaves $T_z Z$ invariant, i.e. ω restricted to the submanifold Z is non-degenerate.

Let $V_k = \text{span}(e_k)$. On each V_j $\exp \tilde{X}_k$ acts as multiplication by some scalar $\exp(i\lambda_j)$ where λ_j is real since $\tilde{X}_h$ is skew symmetric as an element of $\mathbf{u}(n)$ and is 0 exactly when

$j \leq r$ as X_h generates the whole torus **T**. Now from the action of X_h on $T_z M$ we have just determined we can deduce Theorem 6.3 if we choose a real basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ corresponding to the complex basis $e_1, \ldots, e_n$.

Thus an almost periodic Hamiltonian function has very special properties.

Definition 6.2 *A function $f \in \mathcal{C}^\infty(M)$ is a* Morse function in the sense of Bott *if its critical points form a submanifold of M, and if at every critical point the Hessian is non-degenarate on the normal bundle of the critical submanifold.*

The connected components of the critical submanifold are the critical manifolds.

The negative normal bundle $NN(Z)$ *of a critical manifold Z is the maximal subbundle of its normal bundle $N(Z)$ on which the Hessian is negative definite.*

The dimension of the negative normal bundle of a critical manifold Z is the index $\lambda(Z)$ *of the critical manifold.*

Bott in [6] proved the following theorem:

Theorem 6.4 *Let $h \in \mathcal{C}^\infty(M)$ be a Morse function in the sense of Bott. For a real number a let $M_a = \{m \in M : h(m) \leq a\}$. If a is a regular value then M_a is a manifold with boundary. Suppose that the interval $[a, b]$ contains only one critical value c which is different from the endpoints. Let Z be the critical manifold corresponding to the critical value c. Then M_b is homotopically equivalent to the space*

$$M_a \cup_\varphi \mathcal{B}(NN(Z))$$

where $\mathcal{B}(NN(Z))$ is the ball bundle of the negative normal bundle and φ is a gluing map $\varphi : \mathcal{S}(NN(Z)) \to \partial M_a$ where $\mathcal{S}(NN(Z))$ is the sphere bundle of the negative normal bundle.

With these notions Theorems 6.2 and 6.3 lead to the corollary:

Corollary 6.1 *An almost periodic Hamiltonian function is in fact a Morse function in the sense of Bott with critical manifolds of even index.*

Now we can prove the central theorem of this section.

Theorem 6.5 *Let (M, ω) be a compact connected manifold, and consider k almost periodic Hamiltonian functions $f_1, \ldots, f_k$ such that any two of them Poisson commute. Then $f^{-1}(t)$ is empty or connected for every $t \in \mathbf{R}^k$.*

First we need the following two lemmas.

Lemma 6.2 *Under the conditions described in Theorem 6.5, any linear combination of the f_i's is an almost periodic Hamiltonian function.*

Proof. All we have to notice is that if two functions Poisson commute then their Hamiltonian vector fields Lie commute, and therefore their flows commute. Now the lemma follows from the fact that two commuting compact subgroups of the group of symplectomorphisms of M generate a compact subgroup.

Lemma 6.3 *Let $h : M \to \mathbf{R}$ be a Morse function in the sense of Bott on the compact manifold M and assume that neither h nor $-h$ has a critical manifold of index 1. Then $h^{-1}(a)$ is connected or empty for every $a \in \mathbf{R}$.*

Proof. First we show that h has a unique local minimum and a unique local maximum. Suppose h has a local minimum c different from the global one. We use Theorem 6.4. Homotopically M_a change when we cross the critical level c by attaching the ball bundle of the negative normal bundle of the corresponding critical manifold, i.e. in this case adding a new component. Now M_b is not connected. However M is. Hence we must connect the pieces later on. It is only possible when we attach along a non-connected sphere bundle of the negative normal bundle, which necessitates that the index of the corresponding critical manifold is 1. But in our case we do not have critical manifold with index 1, therefore there exists no local minimum different from the global one. For the local maximum the proof is similar, except h is replaced with $-h$.

By continuity, it suffices to prove the lemma only for regular values. Let a be a regular value of h. Notice that the foregoing proves that M_a is connected for every $a > \min(h)$. Now $h^{-1}(a)$ is the boundary of the manifold M_a, so that if h^{-1} is not connected we get a non-trivial $(2n-1)$-cycle for M_a from a boundary component.

We show that this is impossible, namely $H_{2n-1}(M_a) = 0$ if $\min(h) < a < \max(h)$. The first observation is that $H_{2n-1}(\mathcal{B}(NN(M_{\min(h)}))) = 0$ whenever h is not constant. In fact if h is not constant then $\dim(M_{\min(h)}) \leq 2n - 2$ using that $M_{\min(h)}$ is a symplectic manifold, thus $H_{2n-1}(M_{\min(h)}) = 0$. But $\mathcal{B}(NN(M_{\min(h)}))$ is homotopically equivalent to $M_{\min(h)}$ which proves our observation.

If we denote the negative normal bundle of a critical manifold Z for the function h by $NN_+(Z)$ and for $-h$ by $NN_-(Z)$, then easily $N(Z) = NN_+(Z) \oplus NN_-(Z)$. Thus if $\min(h) < a < \max(h)$ then from the foregoing $\lambda_-(Z) \geq 2$, therefore the total dimension of $\mathcal{B}(NN_+(Z))$ is $\leq 2n - 2$. Now starting at $\min(h)$ we can never produce a non-trivial $(2n-1)$-cycle in M_a.

Proof of Theorem 6.5. We proceed by induction on k. When $k = 1$ the function f_1 is an almost periodic Hamiltonian which from Corollary 6.1 has critical manifolds only with even index, therefore we can use Lemma 6.3 which establishes the proof in this case.

Suppose now that the theorem is true for $k \geq 1$ and prove for $k + 1$.

Let $f_1, \ldots, f_{k+1}$ be functions on M satisfying the hypotheses of the theorem. We have to show that if $a = (a_1, \ldots, a_{k+1}) \in \mathbf{R}^{k+1}$ then

$$f^{-1}(a) = f_1^{-1}(a_1) \cap \ldots \cap f_{k+1}^{-1}(a_{k+1})$$

is empty or connected. We may suppose that $f_1, \ldots, f_{k+1}$ are linearly independent, otherwise we can drop one of them and use the induction hypothesis. Furthermore, by continu-

ity it suffices to consider only regular values a of f, i.e. values so that the df_i are linearly independent for all $x \in f^{-1}(a)$. With this assumption

$$N = f_1^{-1}(a_1) \cap \ldots \cap f_k^{-1}(a_k)$$

is a submanifold of M and by the induction hypothesis it is connected. Since

$$f^{-1}(a) = N \cap f_{k+1}^{-1} = (f_{k+1} \mid_N)^{-1}(a_{k+1})$$

it suffices to show that $f_{k+1} \mid_N$ satisfies the conditions of Lemma 6.3 and then the lemma will yield the theorem.

Hence it is sufficient to prove that $f_{k+1} \mid_N$ has critical manifolds in N with even index only.

Let $z \in N$ be a critical point of $f_{k+1} \mid_N$. It means that at z

$$df_{k+1} + \sum_{i=1}^{k} \lambda_i df_i = 0,$$

for some constants λ_i. If we choose

$$h = f_{k+1} + \sum_{i=1}^{k} \lambda_i f_i$$

then z will be a critical point of h on M. From Lemma 6.1 h is an almost periodic Hamiltonian function. Let Z be the critical manifold of h containing Z.

We show that Z and N intersect transversally, or equivalently that $df_1, \ldots, df_k$ are still independent when restricted to Z at z. Let $X_1, \ldots, X_k$ be the Hamiltonian vector fields associated to $f_1, \ldots, f_k$. Since the Poisson brackets $\{f_i, h\}$ vanish, the independent vectors $X_1(z), \ldots, X_k(z)$ lie in $T_z Z$. From Theorem 6.2 Z is non-degenarate relative to the symplectic structure ω of M. Hence for any constant $\alpha = (\alpha_i, \ldots, \alpha_k) \neq 0$ there exists a tangent vector $Y \in T_z Z$ such that

$$0 \neq \omega(\sum_{i=1}^{k} \alpha_i X_i(z), Y) = \{\sum_{i=1}^{k} \alpha_i df_i(z)\}(Y),$$

which proves our assertion.

Now the index of $f_{k+1} \mid_N$ at z concides with that of $h \mid_N$ at z since on N they differ only by the constant $\sum_{i=1}^{k} \lambda_i a_i$. The index of $h \mid_N$ at z is the dimension of the maximal subspace where the Hessian is negative definite. But $d^2 h$ vanishes on Z, which is transversal to N, therefore the index of $h \mid_N$ at z coincides with the index of h at z. This is even from Corollary 6.1.

Corollary 6.2 *Let $f_1, \ldots, f_{k+1}$ be functions on M satisfying the hypotheses of Theorem 6.5. Then $f(M)$ is convex.*

Moreover if $Z_1, \ldots, Z_n$ are the connected components of the set of common critical points of the f_i's then $f(Z_i)$ is a point c_i and $f(W)$ is the convex hull of the points c_i.

Proof. For the convexity it suffices to show that $f(M)$ intersects every affine line l of $\mathbf{R}^{k+1}$ in a convex set, i.e. the intersection is empty or connected.

Let $\pi : \mathbf{R}^{k+1} \to \mathbf{R}^k$ be a linear map such that $\pi^{-1}(a) = l$ with a $t \in \mathbf{R}^k$. Let $f\pi = g = (g_1, \ldots, g_k)$. Trivially, the g_i's Poisson commute and are almost periodic Hamiltonian functions from Lemma 6.1. Applying Theorem 6.5 we get that $g^{-1}(a)$ is empty or connected and therefore

$$f(M) \cap \pi^{-1}(a) = f(g^{-1}(a))$$

is empty or connected as required.

The set of common critical points of $f_1, \ldots, f_{k+1}$ is also the set of fixed point of the torus $\mathbf{T}$ generated by the Hamiltonian fields $X_1, \ldots, X_{k+1}$. From Theorem 3.2 Z is a disjoint union of the submanifolds Z_i. On each Z_i we have $df_i = 0$ for all i and so each f_i is constant. Thus $f(Z_j) = c_j$ is a single point in $\mathbf{R}^{k+1}$. Moreover, if $h = \sum \lambda_i f_i$ is a generic linear combination, so that the corresponding Hamiltonian vector field generates $\mathbf{T}$, then the critical set of h is precisely Z, and in particular h takes its maximum on Z. Hence the linear form $\sum \lambda_i x_i$ considered as a function on $f(M) \subset \mathbf{R}^{k+1}$, takes its maximum at one of the points $c_1, \ldots, c_{k+1}$. Since this holds for almost all $(\lambda_1, \ldots, \lambda_{k+1})$ it follows that $f(M)$ lies in the convex hull of $c_1, \ldots, c_{k+1}$. But we have shown that $f(M)$ is convex hence the result follows.

Proof of Theorem 6.1. Choose a basis for $\mathbf{t}^*$ to get coordinate functions $\mu_1, \ldots, \mu_k$ of μ. Then μ_i is an almost periodic Hamiltonian function and any two of the μ_i's Poisson commute since $\tilde{\mu}$ is a Lie-algebra homomorphism. Now use Corollary 6.2 and that the common critical points of the μ_i's are exactly the fixed points of the torus action. The result follows.

7 An application of the convexity theorem

As an application in this section we sketch the proof of Theorem 1.1. We will not prove every assertion we use. Our aim is just to show how it is possible to prove the theorem of Kushnirenko with the help of the convexity theorem.

First of all we cite two standard theorems from Hitchin [11] that we will need later. Then we define the so called Fubiny-Study (or standard Kaehlerian) metric on $P^{N-1}(\mathbf{C})$ and examine the diagonal action of T^N on $P^{N-1}(\mathbf{C})$.

Theorem 7.1 *Let $N \subset M$ be a submanifold of a Kaehler manifold M such that at each point $m \in N$ I maps $T_m N$ into itself. Then the induced structure is a Kaehlerian structure.*

Theorem 7.2 *Let G be a compact Lie group of isometries of a Kaehler manifold preserving I and therefore the symplectic form ω.*

Suppose $\mu : M \to \mathbf{g}^*$ is a moment map for G and suppose G acts freely on $\mu^{-1}(0)$. Then the induced structure on $\mu^{-1}(0)/G$ is a Kaehlerian structure.

Example. Let V be a complex vector space of dimension N with suitable complex coordinates $z_1, \ldots, z_N$ where $z_i = x_i + iy_i$. Then V can be considered a Kaehlerian manifold. Then $\omega = \sum_{j=1}^n dy_j \wedge dx_j$. Now consider the action of S^1 on V by multiplication. Then the vector field X associated with the unit generator of the Lie algebra of S^1 by the infinitesimal action can be written in the form:

$$X = \sum_{j=1}^n \left(-y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j} \right).$$

This is clearly a symplectic action. Moreover, it is a Hamiltonian action with a moment map

$$\mu(x_1, \ldots, x_N, y_1, \ldots, y_N) = -1 - \sum_{j=1}^n (x_j^2 + y_j^2).$$

Now S^1 acts freely on $\mu^{-1}(0)$. Note that $\mu^{-1}(0)/S^1$ is exactly $P(V)$. Thus from Theorem 7.2. we get a Kaehlerian structure on $P(V)$. This metric is called the Fubini-Study metric.

The following is a simple corollary:

Corollary 7.1 *Any complex submanifold of $P(V)$ is a Kaehlerian manifold.*

Consider the following T^N-action ϕ (which we call the diagonal action) on $P(V)$:

$$t[v_1, \ldots, v_N] = [t_1 v_1, \ldots, t_n v_n],$$

where $t = (t_1, \ldots, t_N) \in T^N = S^1 \times \cdots \times S^1$. This action is clearly symplectic. Moreover it is Hamiltonian with moment map

$$\mu([v_1, \ldots, v_N]) = -\frac{1}{\sum_{j=1}^N |v^j|^2} (|v_1|^2, \ldots, |v_N|^2).$$

Note that the image of this moment map is the standard $(N-1)$-symplex.

Now we prove the theorem of Kushnirenko. The idea of the proof will be the following. First we define an n-dimensional torus action on $P^{N-1}(\mathbb{C})$ which will turn out to be Hamiltonian. Then using the convexity theorem we notice that the image of the moment map μ of this action is the convex hull of S. After that we find an algebraic submanifold of $P^{N-1}(\mathbb{C})$ for which we use a theorem of Atiyah, which will tell us that the image of the algebraic submanifold under μ is still the convex hull of S. Then we consider volumes, and find that the volume of $C(S)$ is the volume of the complex algebraic submanifold regarding the standard Kaehlerian metric on $P^{N-1}(\mathbb{C})$. Then we use theorems from the theory of complex algebraic varieties and get that the volume of our complex algebraic variety is its degree divided by $n!$ and the degree will be equal to the solution of the given system of equations.

Sketch Proof of Theorem 1.1   Let the multiindices $\alpha_1, \ldots, \alpha_N$ be the elements of S. We denote by T_c^n the complex torus, the group of n-tuples of non-zero complex numbers. We define an action $\tilde{\psi}_c$ of T_c^n on an N-dimensional complex vector space V by the formula

$$t(z_1, \ldots, z_N) = (t^{\alpha_1} z_1, \ldots, t^{\alpha_N} z_N),$$

where $(z_1, \ldots, z_n)$ are the coordinates of a point z in V with respect to the fixed basis $v_{\alpha_1}, \ldots, v_{\alpha_N}$ of V and $t \in T_c^n$. Considering the inclusion of the compact torus $T^n \subset T_c^n$ we get a $\mathbf{T}^n$-action $\tilde{\psi}$ on V. The hermitian metric corresponding to the fixed basis of V is clearly $\mathbf{T}^n$-invariant, since $\tilde{\psi}(t)$ is a unitary transformation of V for every $t \in T^n$.

Now we pass our attention to the projective space $P(V)$. From the actions on V we get a T_c^n-action ψ_c, a $\mathbf{T}^n$-action ψ and a $\mathbf{T}^n$-invariant Kaehlerian form ω on $P(V)$ in the way described above.

Let us examine the action ψ. Clearly, this is just the action corresponding to the representation $S : T^n \to T^N$ given by $S(t) = (t^{\alpha_1}, \ldots, t^{\alpha_N})$ and the standard diagonal action ϕ of T^N on $P(V)$. Therefore, our action ψ is Hamiltonian and its moment map $\mu_\psi : P(V) \to \mathbf{t}_n^*$ from Theorem 5.2 is the composition of the moment map $\mu_\phi : P(V) \to \mathbf{t}_N^*$ and the dual map $s^* : t_N^* \to \mathbf{t}_n^*$ of $s : \mathbf{t}_n \to \mathbf{t}_N$ corresponding to S. As we saw above, the image of μ_k was the standard $(N-1)$-simplex. The vertices of the simplex are the generators of the integer lattice $L^* \subset \mathbf{t}_n^*$ dual to the lattice $L \subset \mathbf{t}_n$ which is the kernel of the exponential map $\mathbf{t}_n \to T^n$. Under s^* these vertices become the points of $S \subset \mathbf{t}_n^*$. It follows from the convexity theorem that the image $\mu_\psi(P(V))$ is the convex hull of S.

In fact, Atiyah [1] proved that the image under μ_ψ of a generic T_c^n-orbit X is the whole interior of $C(S)$, and that $\mu_\psi : X \to \text{int}(C(S))$ is a fibration with fibre $\mathbf{T}^n$.

Let X be the orbit of the point $[1, \ldots, 1] \in P(V)$ at the T_c^n action ψ_c. Now from Atiyah's result we get that

$$V(T^n) V(C(S)) = V(X).$$

If we normalize so that $V(T^n) = 1$ then this determines the normalization of the Euclidean volume on $\mathbf{t}^*$.

Consider now the closure $\bar{X}$ of X.

$$V(C(S)) = V(X) = V(\bar{X})$$

and $\bar{X}$ is an algebraic submanifold of dimension n. Now from Wirtinger's theorem (see [9])

$$V(\bar{X}) = \frac{1}{n!} \int_{\bar{X}} \omega^n.$$

An other theorem (see [13]) tells us that

$$\int_{\bar{X}} \omega^n = \deg(\bar{X}).$$

This is the intersection number of $\bar{X}$ and a generic $P(W)$ of codimension n. Since $\dim(\bar{X} - X) < n$ it follows that

$$n! V(C(S)) = \deg(\bar{X}) = \deg(X)$$

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is the intersection number of X and a generic $P(W)$ of codimension n.

If the differences $\alpha - \beta$ $(\alpha, \beta \in S)$ generate a lattice of rank $< n$ then the system of equations we are investigating has no solution and the volume of the convex hull of S vanishes as well.

If the differences $\alpha - \beta$ $(\alpha, \beta \in S)$ generate a lattice L of rank n but not the whole $\mathbf{Z}^n$, then changing $\mathbf{Z}^n$ to L and choosing a basis for L to identify this lattice with $\mathbf{Z}^n$, we get a $\tilde{S} \subset L$ with $V(C(\tilde{S})) \det L = V(C(S))$ and clearly the system of equations with respect to S has $\det L$ times more solutions than it does when we consider $\tilde{S}$ instead of S. Therefore, it suffices to investigate only the following case.

Thus we can assume that the differences $\alpha - \beta$ $(\alpha, \beta \in S)$ generate the whole Z^n.

Now let X' be the orbit of the point $(1, \ldots, 1) \in V$ under the action $\tilde{\psi}_c$. From the assumption above X is equivalent to X', hence the intersection number of X and a generic $P(W)$ concides with the intersection number of X' and a generic subspace $W \subset V$ of codimension n, that is to say

$$\deg(X) = \deg(X').$$

Using an argument similar to the one above, we can assume that S generates the lattice $\mathbf{Z}^n$. This means that X' is a faithful orbit, i.e. the group T_c^n acts freely on it. From this we conclude that the intersection number of X' and a generic W is $N(S)$:

$$\deg(X') = N(S)$$

if W is defined by the equations

$$\sum_{\alpha \in S} c_\alpha^j v_\alpha = 0 \quad j = 1, \ldots, n.$$

This means that

$$n! V(C(S)) = N(S),$$

as required.

Remark. Atiyah in [2] shows how his convexity results can be used for proving Bernstein's generalization of Theorem 1.1. We will prove this by using toric varieties in subsection 11.3.

8 Toric varieties

8.1 Introduction

In the second part of this dissertation we establish the theory of toric varieties.

In the literature one can find different ways of introducing toric varieties and indeed the name of the main object could be altered. For example the expression toric embedding can be used for toric variety or toric compactification for a special kind of toric variety (see Chapter 27 of Khovanski in [5]). Both names indicate a description of toric variety as an algebraic variety containing a dense open complex torus, where the variety is equipped with an algebraic action of this complex torus extending the canonical action of the torus on itself.

We will give a more theoretical way of introducing toric varieties, and prove the above properties as consequences of the definitions.

Toric varieties are very special kind of algebraic varieties but their structure is rich enough to use them for testing algebro-geometrical ideas.

Our final aim is to establish a fruitful link with convex geometry and give applications of it.

In section 8 we establish the basic notions. In section 9 we discuss the theory of divisors. In section 10 we collect the notions which are needed to state the Hirzebruch-Riemann-Roch theorem. In the final section we create a useful connection between toric and convex geometry and show a few applications of it, included the Alexander-Fenchel inequality and the theorem of Bernstein which generalizes our central Theorem 1.1.

The second part of this work is based on the book of Fulton [7]. We used Hartshorne [10] and Griffiths-Harris [9] for general results in algebraic geometry and the book of Burago-Zalgaller [5] for convex geometry, especially chapter 27 of it which was written by A.G. Khovanski.

8.2 Cones and fans

In this subsection we collect the elemantary information from the theory of convex polyhedral cones we need. We define the notion of convex polyhedral cones and that of fans and give basic results about them.

Definition 8.1 *A convex polyhedral cone σ in an n-dimensional real vector space V is a set*
$$\sigma = \{r_1 v_1 + \ldots + r_s v_s \in V : r_i \geq 0\}$$
generated by any finite set of vectors $v_1, \ldots, v_s$ in V.

A convex polyhedral cone is called strictly convex *if it contains no nontrivial linear subspace.*

The dimension $\dim \sigma$ is the dimension of the vector subspace generated by σ.

The dual σ^* of any set σ is the set

$$\sigma^* = \{u \in V^* : \langle u, v \rangle \geq 0 \ \forall v \in \sigma\}.$$

A face τ of σ is the intersection of σ with any supporting hyperplane:

$$\tau = \{v \in \sigma : \langle u, v \rangle = 0\},$$

where $u \in \sigma^*$.

The basic fact is the following

Proposition 8.1 (Separation theorem) *If σ is a convex polyhedral cone and $v \notin V$ then there is some $u \in \sigma^*$ with $\langle u, v \rangle < 0$.*

We will make use of the well-known

Theorem 8.1 (Farkas's theorem) *The dual of a convex polyhedral cone is a convex polyhedral cone.*

Now we consider lattices and rational convex polyhedral cones in order to be able to introduce fans.

Definition 8.2 *A* lattice *of dimension n is a finitely generated free abelian group of rank n, i.e. that is isomorphic to $\mathbf{Z}^n$. Then $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$ is an n-dimensional real vector space. The dual of the lattice N is denoted by M.*
A convex polyhedral cone in $N_{\mathbf{R}}$ is rational *if it can be generated by vectors from the lattice N.*
If σ is a rational convex polyhedral cone then S_σ stands for $\sigma^* \cap M$.
A cone in N *is a rational, strictly convex cone in $N_{\mathbf{R}}$.*

In the next chapter we will need the following proposition.

Proposition 8.2 (Gordon's Lemma) *If σ is a rational polyhedral cone then S_σ is a finitely generated semigroup, a subsemigroup of the group M.*

Proof. σ^* is a convex polyhedral cone from Farkas's theorem. Hence we have that $u_1, \ldots, u_s \in M$ generate the cone σ^*. Now

$$K = \{\sum t_i u_i : 0 \le t_i \le 1\}$$

is a compact set so $K \cap M$ is finite which clearly generates S_σ as a semigroup.

Now we are in a position to introduce the notion of fan.

Definition 8.3 *A* fan Δ *in N is meant a set of cones in N such that each face of a cone in Δ is also a cone in Δ and the intersection of two cones in Δ is a face of both.*
The support $\mid \Delta \mid$ *of the fan Δ is the union of the cones in the fan.*

Remark. Note the similarities of fans and simplicial complexes.

Example. Now we give an example of a fan which will later link convex geometrical objects with objects in torical geometry. We need some more definitions.

Definition 8.4 . *A the convex hull of finitely many points in $M_{\mathbf{R}}$ is called a* convex polytope.

A face *of a polytope is an intersection of it with a supporting hyperplane as in Definition 8.1.*

If P is a convex polytope of maximal dimension with the origin in its interior then the polar polytope is defined by

$$P^0 = \{v \in N_{\mathbf{R}} : \langle u, v \rangle \geq -1 \text{ for all } u \in P\}.$$

A convex polytope is rational *if the cones over the faces of the polytope are rational.*

Thus let P be an n dimensional convex rational polytope in $M_{\mathbf{R}}$ containing the origin in its interior. Then we see that the set of cones in N over the faces of P^0 which we denote by Δ_P is a fan in $N_{\mathbf{R}}$.

If P does not contain the origin then we can translate it by a vector $u \in M_{\mathbf{R}}$ such that $P + u$ is rational and contains the origin in its interior. Now let us define Δ_P to be the fan Δ_{P+u}. Clearly this is well-defined.

If P is of less dimension than n, then we can consider the subspace $\tilde{M}_{\mathbf{R}}$ which is spanned by P. Construct Δ_P in $\tilde{N}_{\mathbf{R}}$ where $\tilde{N}$ is the dual of $\tilde{M}$. Since $\tilde{N}$ is a sublattice of N we can consider Δ_P as a fan in N.

8.3   Affine toric varieties

Construction of affine varieties   We will follow the standard commutative algebraic way to construct affine toric varieties from cones in $N_{\mathbf{R}}$.

Thus let us given a cone σ in N. Consider the commutative semigroup $S_\sigma = \sigma^* \cap M$ which is finitely generated by Gordon's lemma. Now we can form the group ring $A_\sigma = \mathbf{C}[S_\sigma]$, which is a commutative $\mathbf{C}$-algebra. As a complex vector space it has a basis x^u where $u \in S_\sigma$ and the multiplication is determined by the addition in S_σ, i.e. $x^u \cdot x^{u'} = x^{u+u'}$. Therefore A_σ is a finitely generated commutative $\mathbf{C}$-algebra.

Now we proceed as usual to get an affine variety from the finitely generated $\mathbf{C}$-algebra A_σ. We find a surjective $\mathbf{C}$-algebra homomorphism

$$j_\sigma : \mathbf{C}[X_1, \ldots, X_k] \to A_\sigma$$

from a finitely generated free $\mathbf{C}$-algebra to A_σ. Now the ideal $I_\sigma = \ker(j_\sigma)$ in $\mathbf{C}[X_1, \ldots, X_k]$ determines the affine variety U_σ in the k-dimensinal complex affine space $\mathbf{A}^k$ as the points of $\mathbf{A}^k$ which vanishes at each element of I_σ. Note that the coordinate ring of U_σ is A_σ.

Examples.   1. If $\sigma = \{O\}$ is the 0-dimensional cone in N then σ^* is the space $M_{\mathbf{R}}$. Now the semigroup $S_{\{0\}}$ is in fact the free abelian group M of rank n. Therefore we see that

$$A_{\{O\}} = \mathbf{C}[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$$

the ring of Laurent polynomials in n variables. Now the map

$$j_{\{O\}} : C[X_1, Z_1, \ldots, X_n, Z_n] \to \mathbf{C}[Y_1, Y_1^{-1}, \ldots, Y_n, Y_n^{-1}]$$

is the obvious one, which sends X_i to Y_i and Z_i to Y_i^{-1}. Hence $I_{\{O\}}$ is generated by the polynomials $X_i Z_i - 1$. Thus, we finally get that $U_{\{O\}}$ is isomorphic to the n-dimensional complex torus $\mathbf{T}^n$ the set of points in $\mathbf{A}^n$ having nonzero coordinates. (This was denoted by T_c^n in Section 7.)

2. Let $v_1, \ldots, v_n$ be a basis of N. Consider the cone σ in N which is generated by these vectors. We show that the corresponding affine toric variety is $\mathbf{A}^n$ itself. If $u_1, \ldots, u_n$ denotes the associated basis of M, then σ^* becomes the cone generated by $u_1, \ldots, u_n$. Therefore $A_\sigma = \mathbf{C}[Y_1, \ldots, Y_n]$ and $U_\sigma = \mathbf{A}^n$ as stated.

3. Combining the preceding two examples we can produce plenty of affine toric varieties. Indeed, if τ is the cone generated by $u_1, \ldots, u_k$ a subset of a basis of N, then U_σ is isomorphic with the affine variety $\mathbf{A}^k \times \mathbf{T}^{n-k}$. Note that if $k = 0$ then we get Example 1 and if $k = n$ we get Example 2 above. Note also that each of the affine varieties of this kind are non-singular. In fact these are the only non-singular affine toric varieties.

Theorem 8.2 *An affine toric variety is non-singular if and only if σ is generated by a part of a basis of N.*

Proof. Observe that it is sufficient to prove that a cone σ in N which spans the whole space $N_\mathbf{R}$ determines a non-singular affine toric variety if and only if σ is generated by a basis of $N_\mathbf{R}$, in other words when U_σ is an affine space.

To see this we recall that an affine variety with coordinate ring A is non-singular[2] at a point z if the cotangent space $\mathcal{M}_z/\mathcal{M}_z^2$ is of dimension n, where $\mathcal{M}_z$ is the maximal ideal of A corresponding to the point z (i.e. the maximal ideal of regular functions vanishing at z) and n is the dimension of the affine variety, which is the transcendency degree of $K(A)$ over $\mathbf{C}$, where $K(A)$ is the field of fractions of A.

If there is given a cone σ in N which is of maximal dimension n, then we see that the transcendency degree of $K(A_\sigma)$ over $\mathbf{C}$ is n since now the dual cone σ^* is a strictly convex polyhedral cone of maximal dimension n. Therefore the dimension of U_σ is n.

Now we can find a distinguished point in U_σ denoted by z_σ whose associated maximal ideal is $\mathcal{M}_\sigma$, where $\mathcal{M}_\sigma$ is generated by $\{x^u : O \neq u \in S_\sigma\}$. Thus $\mathcal{M}_\sigma^2$ is generated by elements of the form x^u where $u \in S_\sigma$ is the sum of two vectors in $S_\sigma \setminus \{O\}$. The cotangent space $\mathcal{M}_\sigma/\mathcal{M}_\sigma^2$ therefore has a basis of images of elements x^u for those $u \in S_\sigma \setminus \{O\}$ that are not the sums of two such vectors. In particular it means that the first elements in M lying along the edges of σ^* are vectors of this kind. Hence if the variety is non-singular at z_σ, then we have that the dual cone σ^* has at most n edges, and since σ^* spans $M_\mathbf{R}$ we see that in fact these minimal vectors along these edges must generate S_σ which means that σ^* is generated by a basis of M i.e. U_σ is isomorphic to $\mathbf{A}^n$ as required.

Definition 8.5 *A cone in N is called* non-singular *if it is generated by a part of a basis. A fan in N is called* non-singular *if it is formed by non-singular cones.*

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[2]see Theorem 5 in Hartshorne [10] p. 32

The next lemma follows from basic facts in commutative algebra, hence we omit the proof[3].

Lemma 8.1 *If τ is a face of σ then U_τ is a principal open subset of U_σ, i.e. is a set of points of U_σ where a particular regular function in A_σ does not vanish.*

Remark. In particular, $\{O\}$ is a face of every cone σ, which means that U_σ contains the complex torus $U_{\{O\}} = \mathbf{T}^n$ as a principal open subset. Thus we can conclude that every affine toric variety contains a dense complex torus.

The torus action. We can now describe the $\mathbf{T}^n$ action on an n-dimensional affine toric variety U_σ, which was mentioned in the introduction. We need a morphism $\mathbf{T}^n \times U_\sigma \to U_\sigma$ which extends the canonical morhipsm $\mathbf{T}^n \times \mathbf{T}^n \to \mathbf{T}^n$ given by the multiplication on $\mathbf{T}^n$. This morphism of affine varieties corresponds to the morphism of $\mathbf{C}$-algebras: $\mathbf{C}[S_\sigma] \to \mathbf{C}[S_\sigma] \otimes C[M]$ which sends x^u to $x^u \otimes x^u$. This is well-defined since $\mathbf{C}[S^\sigma] \subset \mathbf{C}[M]$ as we have just seen.

8.4 Algebraic toric varieties

Construction of algebraic toric varieties An algebraic variety is obtained by 'gluing' together affine varieties (affine pieces) along its principal open subsets, where it is demanded that these 'gluings' are compatible with each other[4].

We are going to construct an algebraic toric variety (or a toric variety for short) from a given fan Δ in N by 'gluing' together the affine toric varieties assiotiated with the cones in the fan Δ. Two such affine pieces are glued together identifying the principal open subsets in both corresponding to the intersection cone. These 'gluings' are compatible with each other which is a consequence of the simplical structure of the fan. The resulting algebraic toric variety is denoted by $X(\Delta)$.

Example. To see this construction in a detailed example, we construct the complex projective space $\mathbf{P}^n$ as a toric variety.

Thus we have to begin with a fan Δ in an n-dimensional lattice N. Let $v_1, \ldots, v_n$ be a basis of N, and consider the cones which are generated by any at most n vectors from the set $v_1, \ldots, v_n, -v_1 - v_2 - \ldots - v_n$. These cones form a fan Δ. (Note that this fan can be constructed -as indicated at the end of the subsection 8.2- from the simplex in M with vertices $-u_1, \ldots, -u_n, u_1 + \ldots + u_n$.) Now applying the gluing procedure we claim that finally we get the algebraic variety $\mathbf{P}^n$. The maximal dimensional cones in the fan Δ are the ones which are generated by exactly n vectors from the above vector set. There are $n + 1$ of them. Each of them is generated by a basis of N and so determines the affine toric variety $\mathbf{A}^n$ (as we saw in the preceding subsection) so we have to check only that

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[3]see Fulton [7] p. 18

[4]for a more abstract but precise definition see Hartshorne [10] p. 74 and for the 'gluing' method see Example 2.3.5 on p. 75

any two of them is glued together in the standard way eventually yielding the algebraic variety $\mathbf{P}^n$ with these affine spaces as canonical affine pieces.

Consider therefore two such affine spaces, for example the one constructed from the cone σ_0 generated by the vectors $v_1, \ldots, v_n$ and the other one associated with the cone σ_n generated by $v_1, \ldots, -v_{n-1}, v_1 - \ldots - v_n$.

The dual cone σ_0^* is generated by the dual basis $u_1, \ldots, u_n$ while the dual cone σ_n^* has generators $u_1 - u_n, u_2 - u_n, \ldots u_{n-1} - u_n, -u_n$. Hence the corresponding coordinate rings are

$$A_{\sigma_0} = \mathbf{C}[X_1, \ldots, X_n]$$

and

$$A_{\sigma_n} = \mathbf{C}[X_1 X_n^{-1}, X_2 X_n^{-1}, \ldots, X_{n-1} X_n^{-1}, X_n^{-1}].$$

The principal open subsets correspondig to the intersection cone $\sigma_0 \cap \sigma_n$ are determined by $X_n \neq 0$ and $X_n^{-1} \neq 0$, respectively. This gives the usual gluing map of two affine pieces of the projective space $\mathbf{P}^n$. Thus the resulting algebraic variety is $\mathbf{P}^n$, indeed.

Remark. The actions of the torus $\mathbf{T}^n$ on the varieties U_σ described in the preceding section are compatible with the patching isomorphisms, giving an action of $\mathbf{T}^n$ on $X(\Delta)$. This extends the product in $\mathbf{T}^n$.

We remark that in fact the converse is also true, i.e. any (separated, normal) variety X containing a torus $\mathbf{T}^n$, with compatible action as above, can be realized as a toric variety $X(\Delta)$ for a unique fan Δ in N.

The proof of the following fundamental theorem is rather algebraical hence we omit the proof[5].

Theorem 8.3 *A toric variety* $X(\Delta)$ *is always Hausdorff.*

It is compact if and only if the fan Δ *is complete i.e. the maximal dimensional cones cover the whole space* $N_{\mathbf{R}}$.

9 Divisors

After introducing the basic notions we have to understand the specific behaviour of divisors on toric varieties since these objects have crucial importance in our point of view.

Therefore we go through the general theory of divisors and find the specific features of divisors on toric varieties.

9.1 Divisors in general.

Definition 9.1 *A Weil divisor* *on a variety* X[6] *is a finite formal sum* $\sum a_i V_i$ *of irreducible closed subvarieties* V_i *of codimension 1.*

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[5]For the first statement see the lemma in Fulton [7] p. 21 and for the second one see p. 39.

[6]which is a ∗-variety (noetherian, integral, separated and regular in codimension one) see Hartshorne [10] p. 130

A Cartier divisor D *is given by the data of a covering of X by affine open sets U_α and nonzero rational functions $f_\alpha \in K(A_\alpha)$ (where $K(A_\alpha)$ is the field of fractions of A_α) called* local equations *such that the ratios f_α/f_β are nowhere zero regular functions on $U_\alpha \cap U_\beta$.*

A nonzero rational function f on a variety X determines a Cartier divisor $\mathrm{div}(f)$ *called a* principal divisor *by the rule $f_\alpha = f\mid_{U_\alpha}$.*

Two Cartier divisors D_1 and D_2 are called linearly equivalent *if $D_1 - D_2$ is principal.*

A line bundle *on a variety is determined by the data of a covering of X by affine open sets U_α and nowhere zero rational functions $g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that the cocyclic condition*

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

holds on $U_\alpha \cap U_\beta \cap U_\gamma$ for any α,β and γ.

Remarks. 1. We remark that every algebraic toric variety is a $*$-variety. Therefore Weil divisors are defined on them.

2. On a nice enough variety[7] (like for example non-singular varieties) Cartier divisors and Weil divisors can be naturally identified.

If D is a Cartier divisor then it determines a Weil divisor denoted $[D]$, by the rule

$$[D] = \sum_{\mathrm{cod}(V,X)=1} \mathrm{ord}_V(D) \cdot V,$$

where $\mathrm{ord}_V(D)$ is the order of vanishing of an equation for D in the local ring along the subvariety V, i.e. if $f_V \in U_\alpha$ determines V (i.e. $V = f_V^{-1}(0)$) on an affine piece U_α then $\mathrm{ord}_V(D)$ is the greatest non-negative integer k such that $f_V = gf_\alpha^k$ holds with a regular function $g \in A_\alpha$. This gives a well-defined finite sum if X is a $*$-variety in particular if it is an algebraic toric variety.

If X is nice enough (see footnote 6) then a Weil divisor $D = \sum a_i V_i$ gives a Cartier divisor in the following way. We can find an affine open cover $\{U_\alpha\}$ of X such that in each U_α, every V_i appearing in D has a local defining function $g_{i\alpha} \in \mathcal{O}(U_\alpha)$. We can then set

$$f_\alpha = \prod_i g_{i\alpha}^{a_i} \in \mathcal{R}^*(U_\alpha)$$

to obtain the data which determines a Cartier divisor.

In particular if X is a non-singular variety then the notion of Weil divisor and Cartier divisor can be identified. In this case we will speak of *divisors*.

3. The data which determines a Cartier divisor is nothing else but a section of the sheaf $\mathcal{R}^*/\mathcal{O}^*$, where $\mathcal{R}^*$ is the multiplicative sheaf of invertible (i.e. nonzero) rational functions on X and $\mathcal{O}^*$ is the multiplicative sheaf of invertible (i.e. nowhere zero) regular functions on X. Thus Cartier divisors form the group $H^0(X, \mathcal{R}^*/\mathcal{O}^*)$.

3. Similarly, we see that the set of isomorphism classes of line bundles is just the first cohomology $H^1(X, \mathcal{O}^*)$ of the sheaf $\mathcal{O}^*$. This forms a group, where the multiplication is the tensor product of line bundles.

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[7]integral, separeted noetherian and locally factorial (see Hartshorne [10] p. 141)

4. The short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^* \overset{i}{\longrightarrow} \mathcal{R}^* \overset{j}{\longrightarrow} \mathcal{R}^*/\mathcal{O}^* \longrightarrow 0$$

gives us, in part, the exact sequence

$$H^0(X, \mathcal{R}^*) \overset{j^*}{\longrightarrow} H^0(X, \mathcal{R}^*/\mathcal{O}^*) \overset{\delta}{\longrightarrow} H^1(X, \mathcal{O}^*)$$

of cohomology groups.

j^* is the operation described in the above definition which corresponds to every nonzero rational function a principal Cartier divisor.

δ is the line bundle operation which corresponds to every Cartier divisor D a line bundle denoted by $\mathcal{O}(D)$.

The exactness means that two Cartier divisors have isomorphic line bundles if and only if they are linearly equivalent.

5. If the variety X is non-singular it can be considered as a complex manifold. Let $\mathcal{O}_a$ denote the structure sheaf of X as a complex manifold, i.e. the sheaf of holomorphic functions and $\mathcal{O}_a^*$ be the multiplicative sheaf of nowhere zero holomorphic functions. Then the obvious exponential map $\exp : \mathcal{O}_a^* \to \mathcal{O}_a$ gives the short exact sequence of sheaves

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_a \overset{\exp}{\longrightarrow} \mathcal{O}_a^* \longrightarrow 0,$$

where $\mathbf{Z}$ is the constant sheaf, the sheaf of locally constant maps from X to $\mathbf{Z}$. This yields a long exact sequence and in particular a map

$$\delta : H^1(X, \mathcal{O}_a^*) \to H^2(X, \mathbf{Z}),$$

which associates a second integer cohomology class with every holomorphic line bundle L on X which is called the first Chern class denoted by $c_1(L) = \delta(L)$.

Moreover, if L is an algebraic line bundle (a line bundle as defined above) on the non-singular variety X, then in particular it is a holomorphic line bundle and therefore it determines its first Chern class $c_1(L)$.

9.2 Divisors on toric varieties.

Our next goul is the characterization of divisors on toric varieties.

We start with remarking that any divisor on a toric variety is linearly equivalent with a divisor which is invariant under the action of $\mathbf{T}^n$.

Definition 9.2 *A Weil-divisor on a toric variety is a* $\mathbf{T}$-*Weil-divisor if it is invariant under the canonical action of* $\mathbf{T}^n$.

A Cartier-divisor on a toric variety is a $\mathbf{T}$-*Cartier-divisor if it is equivariant by* $\mathbf{T}^n$.

Remark. A **T**-Weil divisor is a formal linear combination of irreducible subvarieties of codimension 1 which are invariant under the action of $\mathbf{T}^n$. These are clearly the closures of certain orbits of the torus action. More profound arguments show that a $\mathbf{T}^n$-invariant subvariety is the closure of the orbit of the point z_τ where τ is a 1-dimensional face called an *edge* of the fan Δ. (Recall that z_τ is the point in U_τ which corresponds to that maximal ideal of A_τ which is generated by the standard generators of A_τ except $x^O = 1$.) This codimension 1 subvariety is denoted by V_τ.

Number the edges $\tau_1, \ldots, \tau_d$ in Δ and let v_i be the first lattice point met along the edge τ_i. We will denote by D_i the codimension one 1 subvariety V_{τ_1}.

We now give a charactarization of **T**-Cartier divisors on affine toric varieties.

Theorem 9.1 *On an affine toric variety X a* **T**-*Cartier divisor is a principal divisor of the form* $\mathrm{div}(x^u)$ *for some uniqe $u \in M$.*

Moreover, as a T-Weil divisor it has the form:

$$[\mathrm{div}(x^u)] = \sum_i \langle u, v_i \rangle D_i,$$

Proof. The proof of the first claim uses lots of algebra and hence we omit it[8].

For the second part it is sufficient to prove that if v is the first lattice point along an edge τ then

$$\mathrm{ord}_{V(\tau)}(\mathrm{div}(x^u)) = \langle u, v \rangle.$$

The order can be calculated on the open set $U_\tau \cong \mathbf{A} \times \mathbf{T}^{n-1}$, on which $V(\tau)$ correponds to $\{0\} \times \mathbf{T}^{n-1}$. This reduces the calculation to the one-dimensional case, i.e., to the case where $N = \mathbf{Z}$, τ is generated by $v = 1$, and $u \in M$. Then x^u is the monomial X^u, whose order of vanishing at the origin is u.

Remark. Now we see that a **T**-Cartier divisor on an algebraic toric variety $X(\Delta)$ is given by the data of nonzero rational functions $x^{u(\sigma)}$ on the affine open sets U_σ, where σ is a maximal cone of Δ such that if σ_1 and σ_2 are maximal cones of Δ with intersection cone τ then the ratio $x^{u(\sigma_1)-u(\sigma_2)}$ is regular on U_τ which is equivalent with requiring that $\langle u(\sigma_1) - u(\sigma_2), v \rangle = 0$ for every $v \in \tau$. This means that the linear function ψ_D on $\mid \Delta \mid$ (which denotes the support of Δ) defined by the equation $\psi_D(v) = \langle u(\sigma), v \rangle$ on a maximal cone σ is well defined. Conversely, every such function comes from a unique **T**-Cartier divisor.

If $[D] = \sum a_i D_i$ then the function ψ_D is determined by the property that $\psi_D(v_i) = -a_i$ and equivalently

$$[D] = \sum -\psi_D(v_i) D_i.$$

Moreover, a **T**-Weil divisor determines a convex rational polytope P_D in $M_\mathbf{R}$ by the following definiton.

---

[8]see Fulton [7] p.61

Definition 9.3 *For a* **T***-Weil divisor* $D = \sum a_i D_i$ *on* $X(\Delta)$, *we set*

$$
\begin{aligned}
P_D &= \{u \in M_{\mathbf{R}} : \langle u, v_i \rangle \geq -a_i \text{ for all } i\} \\
&= \{u \in M_{\mathbf{R}} : u \geq \psi_D \text{ on } \mid \Delta \mid\}.
\end{aligned}
$$

The following lemma describes the sections of the line bundle $\mathcal{O}(D)$ in terms of the lattice points in P_D.

Lemma 9.1 *The global sections of the line bundle* $\mathcal{O}(D)$ *are*

$$
H^0(X, \mathcal{O}(D)) = \bigoplus_{u \in P_D \cap M} \mathbf{C} \cdot x^u.
$$

Proof.   First note that it is sufficient to show the theorem for an affine toric variety U_σ. In this case Theorem 9.1 yields that the **T**-Cartier divisor D has the form $\mathrm{div}(x^{u_D})$ for a unique $u_D \in M_{\mathbf{R}}$. The second part of Theorem 9.1 shows that $P_D = \sigma^* + (-u_D)$, where the $+$ is the Minkowski sum[9]. On the other hand $H^0(X, \mathcal{O}(D))$ is the fractional ideal $A_\sigma \cdot x^{-u_D}$. The result follows.

The following lemma collects the very first properties of the rational convex polytope P_D.

Lemma 9.2 *If* D *is a* **T***-Cartier divisor on a toric variety* $X(\Delta)$ *then for a positive integer*

$$
P_{\lambda D} = \lambda P_D.
$$

If $u \in M$, *then*

$$
P_{D+\mathrm{div}(x^u)} = P_D + (-u).
$$

Now we describe **T**-Cartier divisors whose line bundle $\mathcal{O}(D)$ is generated by its sections.

Theorem 9.2 *Let* D *be a* **T***-Cartier divisor on* $X(\Delta)$, *where all maximal cones of* Δ *are* n*-dimensional. Then* $\mathcal{O}(D)$ *is generated by its sections if and only if* ψ_D *is convex.*

The proof of this theorem is straightforward but rather technical hence we omit the proof[10].

The following corollary gives a useful formula for the polytope of the sum of divisors whose line bundle is generated by its sections.

Corollary 9.1 *If* $\mathcal{O}(D)$ *and* $\mathcal{O}(E)$ *are generated by their sections then*

$$
P_{D+E} = P_D + P_E,
$$

where on the right hand side $+$ *means Minkowski sum.*

---

[9] see Definition 11.1

[10] see the proposition in Fulton [7] p. 68

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The divisor constructed from the rational convex polytope P. If P is a rational convex polytope in $M_\mathbf{R}$ then we can construct a divisor D_P on the toric variety Δ_P corresponding to P (see the example agter Definition 8.3) in the following way. As we saw D_P is determined by ψ_{D_P}. Thus define

$$\psi_{D_P} = \min_{u \in P}\langle u, v \rangle.$$

By definition ψ_{D_P} is linear on each cone σ in Δ_P. Moreover, ψ_D is clearly convex hence by Theorem 9.2 the $\mathbf{T}$-Cartier divisor D_P is generated by its sections.

More generally we say that a fan Δ in N is *compatible* with a rational convex polytope in P if $\psi_{D_P} = \min_{u \in P}\langle u, v \rangle$ is linear on each cone σ of Δ. Since ψ_{D_P} is convex by definition, the corresponding $\mathbf{T}$-Cartier divisor D_P is generated by its sections.

Now if we have rational convex polytopes in $M_\mathbf{R}$ each containing the origin, then we can consider the fans $\Delta_{P_1}, \dots, \Delta_{P_k}$. As we saw Δ_{P_i} is compatible with P_i. As it is easy to see there exists a non-singular complete fan $\Delta_{(P_1, \dots, P_k)}$ which is a refinement of every Δ_{P_i}, i.e. every edge in Δ_{P_i} is an edge in $\Delta_{(P_1, \dots, P_k)}$. Since $\Delta_{(P_1, \dots, P_k)}$ is still compatible with every Δ_{P_i} we have the following

Theorem 9.3 *Let* $P_1, \dots, P_k$ *be rational convex polytopes in* $M_\mathbf{R}$ *each containing the origin. There exists a non-singular fan* $\Delta_{(P_1, \dots, P_k)}$ *which is compatible with each of the polytopes.*

Moreover, there are unique divisors $D_{P_1}, \dots D_{P_k}$ *on* $X(\Delta_{(P_1, \dots, P_k)})$ *whose line bundle is generated by its sections.*

The line bundle of our divisors therefore will be generated by its sections. The following theorem contains important cohomology information of such line bundles, however we will not prove it since its proof is rather technical[11].

Theorem 9.4 *If* D *is a* $\mathbf{T}$*-Cartier divisor on a complete non-singular toric variety such that* $\mathcal{O}(D)$ *is generated by its sections then the positive cohomologies of the line bundle* $\mathcal{O}(D)$ *vanish:* $H^p(X, \mathcal{O}(D)) = 0$ *for all* $p > 0$.

9.3 Intersections

We need to develop intersection theory of divisors:

Definition 9.4 *Let* X *be an algebraic variety of dimension* n.

The k*'th Chow group* $A_k(X)$ *is the free abelian group on the* k*-dimensional irreducible closed subvarieties of* X, *modulo the subgroup generated by the cycles of the form* $[div(f)]$, *where* f *is a nonzero rational function on a* $(k+1)$*-dimensional subvariety of* X. *A finite formal sum of irreducible closed subvarieties of* X *is called a* k*-cycle. An element in* $A_k(X)$ *is called a* k*-cycle class.*

If D *is a Cartier divisor on* X, V *is an irreducible subvariety of* X *then we can find a Cartier divisor* E *which is linearly equivalent with* D *and* $\mathrm{ord}_V(E) = 0$. *Restricting the local equations of* E *yields a Cartier divisor on* V. *This Cartier divisor gives a Weil divisor on* V *which can be considered as an element* $D \cdot V$ *in* $A_{k-1}(X)$.

---

[11]see the corollary in Fulton [7] p. 74

Remarks. 1. By definition A_{n-1} is the group of Weil divisors modulo linear equivalence.

2. The name k-cycle class for an element in $A_k(X)$ indicates the strong connection between $A_k(X)$ and $H_k(X, \mathbf{Z})$. For example we remark that if X is a non-singular and complete toric variety then these two groups coincide.

3. The definition of the intersection of a Weil divisor and an irreducible variety extends by linearity to an intersection map:

$$H^0(\mathcal{R}^*/\mathcal{O}^*) \times A_k \to A_{k-1}.$$

4. If X is non-singular then we have that Cartier and Weil divisors coincide so in this case we have an intersection pairing

$$A_{n-1} \times A_k \to A_{k-1}.$$

Note also that we can define the intersection $D_1 \cdot \ldots \cdot D_k$ of divisors $D_1, \ldots, D_k$ as an element in A_{n-k}. This intersection operation by definition is multilinear.

The product of n divisors $D_1 \cdot \ldots \cdot D_n$ sits in $\mathbf{A}_0$. Such a 0-cycle class is represented by a formal sum of finitely many points and even if X is complete then the sum of the coefficients of these points does not depend on the representation of this 0-cycle class (as we will see after Theorem 9.6). This integer number is called the *intersection number* of the divisors $D_1, \ldots, D_n$ and is denoted by $(D_1 \cdot \ldots \cdot D_n)$.

In particular if $n = 2$ the intersection number determines a bilinear form on $A_1(X)$. The Hodge index theorem (see Theorem 10.2) describes this bilinear form.

The standard correspondence between the intersection defined above and the cup product in cohomology is valid. To work this out we need the following:

Definition 9.5 *Let X be a complete non-singular n-dimensional algebraic variety.*

The cohomology class $\eta_V \in H^{2k}(X, \mathbf{R})$ of a codimension k subvariety V is the Poincare dual of the fundamental class $[V] \in H_{2n-2k}(M, \mathbf{R})$, where the fundamental class of V is given by the linear functional

$$\phi \mapsto \int_V \phi$$

for $\phi \in H^{2n-2k}(X, \mathbf{R})$.

In particular the cohomology class η_α of a k-cycle α represented by a formal sum of k-dimensional irreducible subvarieties $\alpha = \sum a_i V_i$ is given by $\eta_\alpha = \sum a_i \eta_{V_i}$.

Remark. We will show that two different representations of a k-cycle class give the same cohomology class. For this it is sufficient to show that any representation of the k-cycle class 0 gives the cohomology class 0. As it is easy to see the $k = n - 1$ case yields the rest. The next theorem proves this and shows more, namely, that the cohomology class of a divisor sits in fact in $H^2(X, \mathbf{Z})$.

Theorem 9.5 *If D is a divisor on an complete non-singular variety X then*

$$\eta_D = c_1(\mathcal{O}(D)) \in H^2(X, \mathbf{R}),$$

where $c_1(\mathcal{O}(D))$ is the first Chern class of the line bundle $\mathcal{O}(D))$[12].

Our last theorem in this section gives the standard correspondence between intersection of divisors and cup product of cocycles. Since the proof of this theorem is topological and rather standard we omit the proof[13].

Theorem 9.6 *If D is a divisor on a non-singular variety X and V is an irreducible subvariety of X then*

$$\eta_D \cdot \eta_V = \eta_{D \cdot V},$$

where $\cdot$ on the left hand side is the cup product.

10 Hirzebruch-Riemann-Roch

The basic tool in the link we are to establish between convex and toric geometry is the Hirzebruch-Riemann-Roch (HRR) theorem for line bundles. HRR is one of the most fundamental theorem in the intersection theory in algebraic geometry. We have to introduce the topological notions first to understand the content of the theorem.

Definition 10.1 *Let L be a line bundle over an n-dimensional algebraic variety X.*

The Chern character $\mathrm{ch}(L)$ of the line bundle L is an element in the cohomology ring $H^*(X, \mathbf{Q})$ defined as follows:

$$\mathrm{ch}(L) = \exp(c_1(L)) = \sum_{i=0}^{\infty} \frac{1}{i!} c_1(L)^i = \sum_{i=0}^{n} \frac{1}{i!} c_1(L)^i,$$

where $c_1(L)$ is the first Chern class of the line bundle L considered as an element of $H^1(X, \mathbf{Q})$ and the first sum is in fact finite, using the vanishing of the q'th cohomology of X when $q > n$.

The Todd class $\mathrm{td}(L)$ of the line bundle L is defined by the formula:

$$\mathrm{td}(L) = \frac{c_1(L)}{1 - \exp(c_1(L))} = \sum_{i=0}^{n} td_i(L),$$

where $td_i(L) \in H^i(X, L)$ is the i'th part of $td(L)$.

For general vector bundles the Todd class is determined by requiring it to be multiplicative, i.e., for any short exact sequence

$$0 \to E_1 \to E_2 \to E_3 \to 0$$

of vector bundles over X the equation $\mathrm{td}(E_1)\mathrm{td}(E_3) = \mathrm{td}(E_2)$ holds.

The Euler characteristic of the vector bundle E is an integer number defined by the formula

$$\chi(X, E) = \sum_{i=0}^{n} (-1)^1 \dim(H^i(X, E)).$$

---

[12]for a proof see Proposition 1 in Griffiths-Harris [9] p. 141

[13]it can be deduced using Griffiths-Harris [9] pp. 58-59

We can now state the theorem[14]

Theorem 10.1 (Hirzebruch-Riemann-Roch) *Let E be a vector bundle on an n-dimensional non-singular, complete variety X. Then*

$$\chi(X, E) = \int_X \mathrm{ch}(E) \cdot \mathrm{td}(T_X).$$

Remarks. (T_X is the tangent bundle of X. The product on the right hand side is the cup product in the cohomology ring $H^*(X, \mathbf{Q})$. The integration is meant to evaluate the n'th part of the integrandum as a cohomology class in $H^n(X, \mathbf{Q})$ at the fundamental class $[X] \in H_n(X, \mathbf{Q})$ of X.)

1. Note that the right hand side concerns only topological properties of the vector vector bundle E, while the left hand side contains analytic (holomorphic) information.

2. When the dimension of the variety is 2 and the bundle is the line bundle of a divisor then HRR gives the well-known Riemann-Roch theorem for divisors on an algebraic surfaces. This Riemann-Roch theorem can be used to prove the following:

Theorem 10.2 (Hodge index theorem.) *Let X be a non-singular projective surface. The intersection number determines hyperbolic form on the $\mathbf{Q}$-vector space $A_1(X) \otimes \mathbf{Q}$, i.e. a non-degenerate form with one positive eigenvalue.*

Remark. We will use this theorem when X is a compact non-singular toric surface. We remark that in this case X is automatically a projective surface.

11 Mixed volumes

Till now we have been working out the algebraic geometry we need. Our attention now is focused on analytic convexity. We introduce the basic notions and, without saying much about the standard methods, we establish the link with toric geometry.

11.1 Analytic convexity

Definition 11.1 *Let $\mathbf{R}^n$ denote the standard Euclidean space, $K \subset \mathbf{R}^n$ be a non-empty convex compact set, called a* convex body. *$V(K)$ stands for the volume of K.*

If L is another convex body then $K + L$ denotes the Minkowski sum *of K and L:*

$$K + L = \{x + y; x \in K, y \in L\}$$

which is a convex compact set.

If $\lambda > 0$ is a positive integer then λK stands for the Minkowski sum of λ copies of K.

Minkowski introduced the notion of mixed volumes by proving the following theorem:

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[14]for a proof see Corollary 15.2.1 Fulton [8] p. 288

Theorem 11.1 *The volume of the linear combination of convex bodies* K_1, $K_2, \ldots, K_s$ *in* $\mathbf{R}^n$ *with non-negative coefficients* $\lambda_1, \ldots, \lambda_s$ *is a homogeneous polynomial of degree* n *with respect to* $\lambda_1, \ldots, \lambda_s$:

$$V(\sum_{i=1}^{s} K_i) = \sum_{i_1=1}^{s} \ldots \sum_{i_n=1}^{s} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \ldots \lambda_{i_n},$$

where it is assumed that for products of the λ_i *which differ only in the order of the factors the coefficients have the same numerical value.*

Definition 11.2 *If* $K_1, \ldots, K_n$ *are convex bodies in* $\mathbf{R}^n$ *then from the previous theorem we have a well-defined real number* $V(K_1, \ldots, K_n)$ *called the* mixed volumes *of the convex bodies* $K_1, \ldots, K_n$.

Remark. 1. What very important for us is the continuity of mixed volumes. This means that the mixed volume map is a continious map considering the topology (called the *Hausdorff topology*) on the space of convex bodies in $\mathbf{R}^n$ where two conves bodies are close to each other if for any point of the first body there is a close point in the second one. For example we will prove the Alexander-Fenchel inequality for rational convex polytopes and claim that the statement follows by continuity for general convex bodies. The point is that rational convex polytopes are dense in the Hausdorff topology on the space of convex bodies.

2. The mixed volume is the central notion of analytic convexity. For an example the well-known isoperimetric inequality can be formulated in terms of mixed volumes. To see this we need one more definition.

Definition 11.3 *Let* B *denote the unit ball in* $\mathbf{R}^n$ *and* K *be a convex body then the* m*'th cross-sectional measure of* K *is defined by the formula:*

$$V_m(K) = V(\underbrace{K, \ldots, K}_{m}, \underbrace{B, \ldots, B}_{n-m}).$$

Remark. Clearly, $V_n(K) = V(K)$ is the volume of K.

For a convex polytope P it is relatively easy to see that

$$V_{n-1}(P) = \frac{1}{n} S(P),$$

where $S(P)$ is the $(n-1)$-dimensional boundary area (which is the sum of the $(n-1)$-dimensional volumes of the hyperfaces of P), and by continuity this holds for general convex bodies

$$V_{n-1}(K) = \frac{1}{n} S(K).$$

The classical isoperimetric inequality takes the form:

Theorem 11.2 (Isoperimetric inequality.) *For a convex body* K *the following holds:*

$$S^n(K) \geq n^n v_n V^{n-1}(K),$$

where v_n *is the volume of the unit ball* B.

Remark. We can reformulate this inequality by means of mixed volumes as follows:

$$V_{n-1}^n(K) \geq V(B)V^{n-1}(K).$$

We will prove this inequality as a corollary of the Alexander-Fenchel inequality.

11.2   The Link

We can now establish the link between mixed volumes of rational convex polytopes and intersections of divisors.

Thus let us given a non-singular toric variety X and a divisor D on it whose line bundle is generated by its sections. Let $P = P_D$ be the corresponding convex rational polytope in $M_{\mathbf{R}}$. Let $\lambda > 0$ be a positive integer. We saw in Lemma 9.2 that $P_{\lambda D} = \lambda P_D = \lambda P$.

We want to use HRR for the line bundle $\mathcal{O}(\lambda D)$. Since λD is generated by its sections hence by Theorem 9.4 the positive cohomologies of $\mathcal{O}(\lambda D)$ vanish: $H^p(X, \mathcal{O}(\lambda D)) = 0$ if $p > 0$. Using Lemma 9.1 this gives that the left hand side of HRR takes the form:

$$\chi(X, \mathcal{O}(\lambda D)) = \dim H^0(X, \mathcal{O}(\lambda D)) = \mathrm{card}(\lambda P \cap M)$$

Using Theorem 9.5 the right hand side has the form:

$$\begin{aligned}
\int_X ch(\mathcal{O}(\lambda D)) \cdot td(T_X) &= \int_X \sum_{i=0}^n \frac{c_1(\mathcal{O}(\lambda D))^i \cdot td_{n-i}(T_X)}{i!} \\
&= \int_X \sum_{i=0}^n \frac{\eta_{\lambda D}^i \cdot td_{n-i}(T_X)}{i!} \\
&= \sum_{i=0}^n \int_X \frac{\lambda^i \eta_D^i \cdot td_{n-i}(T_X)}{i!} \\
&= \sum_{i=0}^n a_i \lambda^i,
\end{aligned}$$

where $a_i = \int_X \frac{\eta_D^i \cdot td_{n-i}(T_X)}{i!}$. Since $td_0(T_X) = 1$ we get by using Theorem 9.6 that

$$\lim_{\lambda \to \infty} \frac{\sum_{i=0}^n a_i \lambda^i}{\lambda^n} = a_n = \int_X \frac{\eta_D^n}{n!} = \frac{(D^n)}{n!}.$$

Hence HRR gives the formula:

$$\frac{(D^n)}{n!} = \lim_{\lambda \to \infty} \frac{\mathrm{card}(\lambda P \cap M)}{\lambda^n}.$$

The following easy lemma gives a more simple form for the right hand side.

Lemma 11.1 *If the lattice M is unimodular (i.e. the unit cube generated by a basis of M has volume 1 in $M_{\mathbf{R}}$) then for a convex polytope P:*

$$\lim_{\lambda \to \infty} \frac{\mathrm{card}(\lambda P \cap M)}{\lambda^n} = V(P).$$

Proof. The statement follows from the fact that the error term in estimating the volume by using unit cubes centered at lattice points of a polytope is bounded by the $(n-1)$-dimensional area of the polytope, and this vanishes in the limit.

Therefore,

Proposition 11.1 *If the line bundle of the divisor D on a non-singular and complete toric variety X is generated by its sections and P is the corresponding polytope in the unimodular lattice M then*

$$V(P) = \frac{(D^n)}{n!}.$$

Remark. If we start with a rational convex polytope P in $M_{\mathbf{R}}$, then we can construct a non-singular and complete fan $\Delta_{(P)}$ in N which is compatible with P. Now we have a divisor $D = D_P$ on the non-singular and complete toric variety $X(\Delta_{(P)})$ which is generated by its sections and whose corresponding polytope is P. Thus the Proposition tells us the connection between the volume of P and the intersection number of n copies of D.

Now starting with n rational convex polytopes $P_1, \ldots, P_n$ in $M_{\mathbf{R}}$ and using Theorem 9.3 we consider the divisors $D_1, \ldots, D_n$ on the toric variety $X(\Delta_{(P_1,\ldots,P_n)})$. Lemma 9.2 together with Corollary 9.1 yields that for positive integers $\lambda_1, \ldots, \lambda_n$ the rational convex polytope $\lambda_1 P_1 + \ldots + \lambda_n P_n$ corresponds to the divisor $\lambda_1 D_1 + \ldots + \lambda_n D_n$. Hence by Proposition 11.1

$$V(\lambda_1 P_1 + \ldots + \ldots + \lambda_n P_n) = \frac{((\lambda_1 D_1 + \ldots + \lambda_n D_n)^n)}{n!}.$$

Since the intersection number is multilinear we get the formula:

$$V(P_1, \ldots, P_n) = \frac{(D_1 \cdot \ldots \cdot D_n)}{n!}.$$

This is the formula we have intended so far.

11.3   Applications

In this subsection we give two applications of the preceding ideas. The first one is the Alexander-Fenchel inequality the heart of analytic convexity. As a consequence we prove the isoperimetric inequality. The second application is an alternative proof of the generalization of Kushnirenko's result (see section 7).

Theorem 11.3 (Alexander-Fenchel) *For convex bodies $K_1, \ldots, K_n$ in $\mathbf{R}^n$ the following holds:*

$$V(K_1, \ldots, K_n)^2 \geq V(P_1, P_1, P_3, \ldots, P_n) \cdot V(P_2, P_2, P_3, \ldots, P_n).$$

Proof. By Remark 1 after Definition 11.2 it is sufficies to prove the inequality for rational convex polytopes $P_1, \ldots, P_n$ in $M_\mathbf{R}$, where M is a unimodular lattice.

Thus let us given rational convex polytopes $P_1, \ldots, P_n$ in $M_\mathbf{R}$. Consider the divisors $D_1, \ldots, D_n$ on the complete non-singular toric variety $X = X(\Delta_{(P_1, \ldots, P_n)})$. By the correspondence of the preceding section it is sufficient to show that if $D_1, \ldots, D_n$ are divisors whose line bundle is generated by its sections the following inequality is valid:

$$(D_1 \cdot \ldots \cdot D_n)^2 \geq (D_1 \cdot D_1 \cdot D_2 \cdot \ldots \cdot D_n)(D_2 \cdot D_2 \cdot D_3 \cdot \ldots \cdot D_n).$$

It is possible to find[15] effective divisors $H_3, \ldots, H_n$ which intersect transversally and whose intersecion is a complete non-singular irreducible surface Y (and so is a projective surface) and which are linearly equivalent with $D_3, \ldots, D_n$, respectively. If we define E_1 and E_2 to be the restrictions of D_1 and D_2 to the surface Y the above inequality becomes:

$$(D_1 \cdot D_2)^2 \geq (D_1 \cdot D_1)(D_2 \cdot D_2).$$

We claim that this inequality is a consequence (and in fact an equivalent version) of the Hodge index theorem.

To see this notice that $(D_1^2) \geq 0$ and the inequality is obvious if $(D_1^2) = 0$ so we can suppose that $(D_1^2) > 0$. Moreover, clearly

$$(D_1 \cdot ((D_1 \cdot D_1)D_2 - (D_1 \cdot D_2)D_1) = 0.$$

According to the Hodge index theorem the intersection form on divisors has 1 positive eigenvalue hence the divisor $(D_1 \cdot D_1)D_2 - (D_1 \cdot D_2)D_1$ has nonpositive selfintersection i.e.

$$((D_1 \cdot D_1)D_2 - (D_1 \cdot D_2)D_1)^2 \leq 0,$$

as required. The result follows.

Proof of the Isoperimetric inequality. Using the Alexander-Fenchel inequality an easy induction argument shows that if $K_1, \ldots, K_n$ are convex bodies in $\mathbf{R}^n$ then

$$V(K_1, \ldots, K_n)^n \geq V(K_1) \cdot \ldots \cdot V(K_n).$$

The isoperimetric inequality is a special case of this one.

Alternative proof of Theorem 1.1. We will prove a stronger result, which is due to Bernstein:

Theorem 11.4 (Bernstein) *Let $S_1, \ldots, S_n$ be finite subsets of the n-dimensional lattice M. Then for the number $N(S_1, \ldots, S_n)$ of solutions of the system of n equations*

$$F_j = \sum_{u \in S_j} a_u^j x^u = 0 \quad j = 1, \ldots, n$$

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[15]By a standard Bertini argument

where the x_i's are nonzero complex numbers and the coefficients a_u^j are assumed to be 'general' complex numbers, one gets the formula

$$N(S_1, \ldots, S_n) = n! V(P_1, \ldots, P_n)$$

where P_i is the convex hull of S_i (called the Newton polytope of F_i).

The Laurent polynomials F_i are defined on the complex torus $\mathbf{T}^n$. These are in fact regular functions on $\mathbf{T}^n$. Thus every Laurent polynomial F_i determines a divisor $\mathrm{div}(F_i)$ on $\mathbf{T}^n$. The genericity of these polynomials means that the intersections of these divisors are all isolated and transversal.

We cannot handle intersections on non-complete algebraic varieties therefore we are going to 'compactificate' the complex torus by constructing a complete non-singular toric variety and consider the intersection of the 'closures' (or rather 'continuations' without intersections outside the torus) of the divisors $\mathrm{div}(F_i)$ and use Theorem ?.

We will make use of the following lemma.

Lemma 11.2 *Let Δ be a complete fan compatible with the Newton polytope P of the Laurent polynomial $F = \sum_{u \in M} a_u x^u$, and let D be the $\mathbf{T}$-Cartier divisor on $X(\Delta)$ corresponding to P. Then $D + \mathrm{div}(F)$ is an effective divisor on $X(\Delta)$.*

Proof. Firs note that the Laurent polynomial F can be considered as a rational function on $X(\Delta)$ since Δ is a complete fan.

The assertion is that the order of $D + \mathrm{div}(F)$ along any codimension one subvariety V of $X(\Delta)$ is nonnegative. This is clear if V meets $\mathbf{T}^n$, since F is regular on $\mathbf{T}$ and V is disjoint from $\mathbf{T}^n$. Otherwise V is the subvariety corresponding to an edge of Δ (see the remark after Definition 9.2). With v the generator of this edge, we have

$$\mathrm{ord}_V(F) \geq \min_{a_u \neq 0} \mathrm{ord}_V(x^u) = \min_{u \in P \cap M} \langle u, v \rangle = \psi_V(v) = -\mathrm{ord}_V(D).$$

The result follows.

Proof of Theorem 11.4. Consider the divisors D_i on the complete non-singular toric variety $X = X(\Delta_{(P_1, \ldots, P_n)})$. As we saw at the end of the preceding section the right hand side of Bernstein's formula is equal to the intersection number $(D_1 \cdot \ldots \cdot D_n)$. Since the effective divisor $E_i = D_i + \mathrm{div}(F)$ is linearly equivalent to D_i, this intersection number is the same as $(E_1 \cdot \ldots \cdot E_n)$. By the genericity of the polynomials F_i we can suppose that these effective divisors have no intersection on the complement of the torus $\mathbf{T}^n$ since this is an $(n-1)$-dimensional subvariety of X. However, if we restrict the divisor D_i to the affine open subset $\mathbf{T}^n$, we get the divisor $\mathrm{div}(F)$ using that E_i is $\mathbf{T}$-Cartier and so is disjoint from the torus. Therefore the intersection number of the divisors $E_1, \ldots, E_n$ coincides with the number of common points of the hypersurfaces $F_i = 0$ in the torus $\mathbf{T}^n$.

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