Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 35 (1994), No. 1, 119-123.

Coating by Cubes

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1. Introduction

Let $P_0, P_1, \ldots, P_n$ be convex d–polytopes in d–dimensional Euclidean space with pairwise disjoint interiors. We say that P_0 is *coated* by $P_1, \ldots, P_n$ if $P_0 \subset \mathrm{int}\big(\bigcup_{i=0}^{n} P_i\big)$, where $\mathrm{int}(\cdot)$ stands for the interior of the corresponding set. Coating occurs very often in a very natural way. For example, in each tiling every tile is coated by its neighbors. Thus, if we take an arbitrary triangulation of $\mathbf{E}^d$, then the number of neighbors of any tile is at least as large as the minimum number of d–simplices that can coat a d–simplex in $\mathbf{E}^d$. In this connection the following problem is a rather very basic question.

Problem 1. Find the minimum number of d–simplices that can coat a d–simplex in $\mathbf{E}^d$.

The answer to the above question is obviously three in $\mathbf{E}^2$. In general, we know only the following.

Proposition. *Every d–simplex can be coated by $(2d-1)$ d–simplices in $\mathbf{E}^d$, where $d \geq 2$.*

Since the number of facets of a d–cube in $\mathbf{E}^d$ is $2d$, the number of d–cubes that can coat a fixed d–cube is at least $2d$. The following theorem formulates a sharper statement under some conditions.

Theorem. *Let P_0 be a d–cube of edgelength λ with edges parallel to the coordinate–axes of $\mathbf{E}^d$. Moreover, let $P_1, \ldots, P_n$ be a collection of unit d–cubes with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that P_0 is coated by $P_1, \ldots, P_n$.*
(1) *If $0 < \lambda < 1$, then $n \geq 2^d$, where equality can be achieved for any $0 < \lambda < 1$ and $d \geq 1$.*
(2) *If $\lambda = k$ is a positive integer, then $n \geq 2(k+1)^d - 2k^d$, where equality can be achieved for any $k \geq 1$ and $d \geq 1$.*

[1] The work was partially supported by the Hung. Nat. Foundation for Sci. Research number 326–0413.

0138-4821/93 $ 2.50

As a result we get the following:

Corollary. *The minimum number of the translates of a d–cube that can coat a given d–cube in* $\mathbf{E}^d$ *is at least* 2^d*, where* $d \geq 1$*. If all d-cubes are translates of each other, then* 2^d *can be replaced by* $2^{d+1}-2$*.*

Problem 2. Prove or disprove that the minimum number of d–cubes that can coat a d–cube in $\mathbf{E}^d$ is $2^{d-1}+2$, where $d \geq 2$.

2. Proof of the Proposition

We prove the statement by induction on the dimension d. As the claim is obviously true for $d=2$ we may assume that it is true for any $d' < d$ with $d \geq 3$. Thus, let S be a d–simplex in $\mathbf{E}^d$ with vertices $v_1, v_2, \dots, v_{d+1}$. Moreover, let H be the hyperplane in $\mathbf{E}^d$ spanned by the vertices $v_1, v_2, \dots, v_d$ and let S_0 be the $(d-1)$–simplex with vertices $v_1, v_2, \dots, v_d$. By induction there are $(d-1)$–simplices $S_1, S_2, \dots, S_{2d-3}$ that coat S_0 in H. Let v be a point in $\mathbf{E}^d$ such that v_{d+1} is the relative interior point of the segment v_1v and let v' be a point in $\mathbf{E}^d$ that is strictly separated from v by H. Then it is easy to see that the d–simplices $\mathrm{conv}(S_1 \cup \{v\})$, $\mathrm{conv}(S_2 \cup \{v\}), \dots, \mathrm{conv}(S_{2d-3} \cup \{v\})$, $\mathrm{conv}\{v_2, v_3, \dots, v_{d+1}, v\}$ and $\mathrm{conv}(S'_0 \cup \{v'\})$ coat the d–simplex S, where S'_0 is a simplex in H containing S_0 in its relative interior. This completes the proof of the Proposition.

3. Proof of the Theorem

Proof of (1). In the following proof we assume only that the edgelengths of the d–cubes $P_1, \dots, P_n$ are larger than λ.

At first, remove the d–cubes of the collection $P_1, \dots, P_n$ that are disjoint from P_0. Let $P_1, \dots, P_n$ denote the system left. Obviously, $P_1, \dots, P_n$ still coat P_0. We are going to show that $n = 2^d$. Recall that an orthant in $\mathbf{E}^d$ is the closure of a connected component of the complement of d pairwise orthogonal hyperplanes of $\mathbf{E}^d$.

Lemma 1. *Each d–cube* P_i, $1 \leq i \leq n$ *can be replaced by an orthant* O_i *with* $P_i \subset O_i$ *such that the edges of the orthants* $O_1, \dots, O_n$ *are parallel to the coordinate–axes of* $\mathbf{E}^d$ *and the interiors of the orthants* $O_1, \dots, O_n$ *are pairwise disjoint.*

Proof. Take a d cube P_i, $1 \leq i \leq n$. Let v_i be the vertex of P_i that lies closest to the d–cube P_0. Then let O_i be the orthant with apex v_i and with edges parallel to the coordinate–axes of $\mathbf{E}^d$ and with $P_i \subset O_i$. We are going to show that each O_i is disjoint from the interiors of the d–cubes $P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_n$ and then we prove that the interiors of the orthants $O_1, \dots, O_n$ are pairwise disjoint indeed. In order to do so we need the following:

Lemma 2. *Let* H *be the hyperplane of any facet of* P_i *that does not contain* v_i*. Then* $H \cap P_0 = \emptyset$*.*

Proof. (Indirect) Assume that $H \cap P_0 \neq \emptyset$. Then take the orthogonal projection of v_i onto H. This is a vertex say v'_i of P_i. Moreover, let w_i be the point of P_0 that is closest to v_i

and let w_i' be the orthogonal projection of w_i onto H. Obviously, as $H \cap P_0 \neq \emptyset$ we have $w_i' \in P_0$. Finally, as the edgelength of P_0 is smaller than the edgelength of P_i we get that $\text{dist}(v_i, w_i) > \text{dist}(v_i', w_i')$. Thus, $\text{dist}(v_i, P_0) > \text{dist}(v_i', P_0)$, a contradiction. □

Now imagine a d–cube P_j, $j \neq i$ with $\text{int}\, P_j \cap \text{int}\, O_i \neq \emptyset$. Recall that $\text{int}\, P_j \cap \text{int}\, P_i = \emptyset$. Then obviously, there exists a facet of P_i the hyperplane H of which separates $\text{int}\, P_j$ from $\text{int}\, P_i$. As $\text{int}\, P_j \cap \text{int}\, O_i \neq \emptyset$ therefore $v_i \notin H$. Hence, Lemma 2 implies that $H \cap P_0 = \emptyset$. Now, recall that $P_i \cap P_0 \neq \emptyset$ and $P_j \cap P_0 \neq \emptyset$. Consequently, H (that separates P_i from P_j) must intersect (the convex set) P_0, a contradiction. Hence, we proved that $\text{int}\, O_i \cap \text{int}\, P_j = \emptyset$ for any $i \neq j \in \{1, \dots, n\}$. In order to finish the proof of Lemma 1 we proceed as follows. Take O_1 and enlarge P_1 from v_1 by a very large factor obtaining the cube P_1' the vertex v_1 of which is still the closest vertex to P_0. As a result of the previous arguments $P_1', P_2, \dots, P_n$ coat P_0. Then enlarge $P_2, P_3, \dots, P_n$ after each other in order to get a coating system of P_0 using rather large d–cubes. Keep doing this to see that the orthants $O_1, \dots, O_n$ have pairwise disjoint interiors. This completes the proof of Lemma 1. □

Apply Lemma 1 to get a system $\{O_1, \dots, O_n\}$ of orthants with $P_i \subset O_i$ and with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that the orthants $O_1, \dots, O_n$ have pairwise disjoint interiors, where $1 \leq i \leq n$. Obviously, no two of the orthants $O_1, \dots, O_n$ are translates of each other and they coat P_0. Thus, $n \leq 2^d$. Take the $2^d - 1$ orthants with edges parallel to the coordinate–axes of $\mathbf{E}^d$ that share the same vertex of P_0 as an apex and are disjoint from the interior of P_0. Then it is easy to see that any O_i intersects the interior of 2^l orthants out of $2^d - 1$. Hence, there must be an O_i that intersects the interior of one orthant out of $2^d - 1$ implying that its apex v_i is a vertex of P_0. Thus, $n \geq 2^d$ and so $n = 2^d$.

We are left with the proof of showing the existence of 2^d orthants $O_1, \dots, O_{2^d}$ that coat P_0. As a result of the previous arguments we look for 2^d orthants with the property that the apex of each orthant is a vertex of P_0 and each vertex of P_0 is an apex of exactly one orthant. We prove the existence of such orthants by induction on the dimension d. They obviously exist in case $d = 2$. So assume that if P_0' is a $(d-1)$–cube of edgelength λ with edges parallel to the coordinate–axes of $\mathbf{E}^{d-1}$, then there are 2^{d-1} orthants in $\mathbf{E}^{d-1}$ say, $O_1', \dots, O_{2^{d-1}}'$ that coat P_0' in $\mathbf{E}^{d-1}$. Also, assume that $\mathbf{E}^{d-1}$ is a hyperplane of $\mathbf{E}^d$. Then for each orthant O_i', $1 \leq i \leq 2^{d-1}$ in $\mathbf{E}^{d-1}$ we assign two orthants of $\mathbf{E}^d$ say, $+O_i$ and $-O_i$ such that the distinct orthants $+O_i$ and $-O_i$ share the $(d-1)$–dimensional orthant O_i' as a facet in common. Let F_1' and F_2' be two opposite (i.e., disjoint) facets of P_0'. Without loss of generality we may assume that the apexes of the orthants $O_1', \dots, O_{2^{d-2}}'$ belong to F_1' and the apexes of the orthants $O_{2^{d-2}+1}', \dots, O_{2^{d-1}}'$ belong to F_2'. Finally, let e_1 be the vector of length λ with $e_1 + F_1' = F_2'$ and let e_d be a vector of length λ orthogonal to $\mathbf{E}^{d-1}$. Without loss of generality we may assume that the d–dimensional orthants $+O_1, \dots, +O_{2^{d-1}}$ lie in that closed half–space of $\mathbf{E}^d$ bounded by the $\mathbf{E}^{d-1}$ into which e_d points.

Then take the following 2^d orthants in $\mathbf{E}^d$:

$$e_d + (-O_1), e_d + (-O_2), \dots, e_d + (-O_{2^{d-2}});$$
$$e_1 + e_d + (+O_1), e_1 + e_d + (+O_2), \dots, e_1 + e_d + (+O_{2^{d-2}});$$
$$-e_1 + (-O_{2^{d-2}+1}), -e_1 + (-O_{2^{d-2}+2}), \dots, -e_1 + (-O_{2^{d-1}});$$

$$+O_{2^{d-2}+1}, +O_{2^{d-2}+2}, \ldots, +O_{2^{d-1}}.$$

If P_0 is the d–cube $\mathrm{conv}(P'_0 \cup (e_d + P'_0))$, then using the induction hypothesis that P'_0 is coated by the $(d-1)$–dimensional orthants $O'_1, \ldots, 0'_{2^{d-1}}$ in $\mathbf{E}^{d-1}$ it is easy to see that the above 2^d d–dimensional orthants coat P_0 in $\mathbf{E}^d$. This completes the proof of (1). □

Proof of (2). At first, we show that if P_0 has an integer edgelength say $k \geq 1$, then P_0 can be coated by $2(k+1)^d - 2k^d$ unit d–cubes with edges parallel to the coordinate axes of $\mathbf{E}^d$, i.e., parallel to the edges of P_0. We prove this by induction on the dimension d. The claim is obviously true for the case $d = 1$. So assume that it is true for every $d' < d$ and take a d–cube P_0 of $\mathbf{E}^d$ with integer edgelength $k \geq 1$. Let H_0 be a supporting hyperplane of P_0 that intersects P_0 in a facet F_0. Then let H_l be the translate of H_0 by the vector of length l orthogonal to H_0 that intersects P_0 in a $(d-1)$–cube $F_l = H_l \cap P_0$ of edgelength k, where $l = 1, \ldots, k$. By induction each F_l can be coated by $2(k+1)^{d-1} - 2k^{d-1}$ unit $(d-1)$–cubes in H_l, where $l = 0, 1, \ldots, k$. Thus, if we place $k(2(k+1)^{d-1} - 2k^{d-1})$ unit d–cubes between the consecutive hyperplanes H_i, H_{i+1}, $0 \leq i \leq k-1$ properly, then we are left with the problem to coat P_0 along the facets F_0 and F_k only. This can be done easily by $2(k+1)^{d-1}$ unit d–cubes. Thus, P_0 is coated by $k(2(k+1)^{d-1} - 2k^{d-1}) + 2(k+1)^{d-1} = 2(k+1)^d - 2k^d$ unit d–cubes in $\mathbf{E}^d$ finishing the construction.

At second, notice that each unit d–cube of the above construction has a $(d-1)$–dimensional intersection with P_0. The following easy lemma is the key to prove the claim (2) completely.

Lemma 3. *Let P_0 be a d–cube of $\mathbf{E}^d$ with integer vertices and with edges parallel to the coordinate–axes of $\mathbf{E}^d$. We assign to P_0 each orthant of $\mathbf{E}^d$ that has an integer apex belonging to P_0 and the edges of which are parallel to the coordinate–axes of $\mathbf{E}^d$ such that the interior of the orthant is disjoint from P_0. If P is a unit d–cube of $\mathbf{E}^d$ with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that $P \cap P_0$ is $(d-1)$–dimensional, then the number of the orthants assigned to P_0, each of whose interior intersects P and each of whose apex belongs to P, is always 2^{d-1}.*

Proof. We leave the rather easy proof to the reader. □

To complete the proof of (2) assume that P_0 is coated by the unit d–cubes $P_1, \ldots, P_n$ in $\mathbf{E}^d$. Without loss of generality we may assume that P_0 is a d–cube of $\mathbf{E}^d$ with integer vertices and with edges parallel to the coordinate–axes of $\mathbf{E}^d$. Assign to P_0 all orthants described in Lemma 3. As $P_i \cap P_0$ is at most $(d-1)$–dimensional $(1 \leq i \leq n)$ it is easy to see (using Lemma 3) that the number of the orthants assigned to P_0 the interior of each of which intersects P_i and the apex of each of which belongs to P_i is at most 2^{k-1}. Thus, a very simple counting argument implies that n is at least as large as the number of unit d–cubes in the above construction, i.e., $2(k+1)^d - 2k^d$. This completes the proof of (2). □

4. Proof of the Corollary

Without loss of generality we may assume that P_0 is a d–cube of edge length λ with edges parallel to the coordinate–axes of $\mathbf{E}^d$ such that it is coated by the unit d–cubes $P_1, \ldots, P_n$

of $\mathbf{E}^d$ the edges of which are par allel to the coordinate–axes of $\mathbf{E}^d$. If $\lambda \leq 1$, then the claim follows from the Theorem in a straight way. So, we are left with the case, when $\lambda > 1$. Then take two opposite facets say, F and F' of P_0. Obviously, F (F', resp.) is a $(d-1)$–cube of edge length λ that is covered by some d–cubes of the collection $P_1, \ldots, P_n$ each of which has edge length smaller than λ. Thus, the number of d–cubes of the collection $P_1, \ldots, P_n$ that cover F (F', resp.) is obviously at least 2^{d-1}. Hence, $n \geq 2^{d-1} + 2^{d-1} = 2^d$. □

Acknowledgement. We are indebted to the referee for the valuable remarks which made the presentation more clear.

Received 14.10.93