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Arithmetic harmonic analysis on character and quiver varieties II☆

Tamás Hausel[a], Emmanuel Letellier[b,*], Fernando Rodriguez-Villegas[c,d]

[a] EPF Lausanne, Switzerland
[b] Université de Caen Basse-Normandie and CNRS UMR 6139, France
[c] University of Texas at Austin, United States
[d] ICTP, Italy

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Abstract

We study connections between the topology of generic character varieties of fundamental groups of punctured Riemann surfaces, Macdonald polynomials, quiver representations, Hilbert schemes on $\mathbb{C}^\times \times \mathbb{C}^\times$, modular forms and multiplicities in tensor products of irreducible characters of finite general linear groups.

Keywords: Character varieties; Quiver representations; Hilbert schemes; Representations of finite general linear groups

Contents

☆ With an appendix by Gergely Harcos.
* Corresponding author.
E-mail addresses: tamas.hausel@epfl.ch (T. Hausel), emmanuel.letellier@unicaen.fr, letellier.emmanuel@math.unicaen.fr (E. Letellier), villegas@math.utexas.edu, villegas@ictp.it (F. Rodriguez-Villegas).

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1. Introduction

1.1. Character varieties

Given a non-negative integer g and a k-tuple $\boldsymbol{\mu} = (\mu^1, \mu^2, \ldots, \mu^k)$ of partitions of n, we define a generic character variety $\mathcal{M}_{\boldsymbol{\mu}}$ of type $\boldsymbol{\mu}$ as follows (see [9] for more details). Choose a *generic* tuple $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ such that for each $i = 1, 2, \ldots, k$ the multiplicities of the eigenvalues of $\mathcal{C}_i$ are given by the parts of μ^i.

Define $\mathcal{Z}_{\boldsymbol{\mu}}$ as

$$\mathcal{Z}_{\boldsymbol{\mu}} := \left\{ (a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_k) \in (\mathrm{GL}_n)^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \;\middle|\; \prod_{j=1}^{g} (a_i, b_i) \prod_{i=1}^{k} x_i = 1 \right\},$$

where $(a, b) := aba^{-1}b^{-1}$. The group GL_n acts diagonally by conjugation on $\mathcal{Z}_{\boldsymbol{\mu}}$ and we define $\mathcal{M}_{\boldsymbol{\mu}}$ as the affine GIT quotient

$$\mathcal{M}_{\boldsymbol{\mu}} := \mathcal{Z}_{\boldsymbol{\mu}} /\!\!/ \mathrm{GL}_n := \mathrm{Spec}\left(\mathbb{C}[\mathcal{Z}_{\boldsymbol{\mu}}]^{\mathrm{GL}_n} \right).$$

Though the variety $\mathcal{M}_{\boldsymbol{\mu}}$ depends on the actual choice of eigenvalues of $\mathcal{C}_i$, we do not make this explicit in the notation as the properties we will consider are insensitive to this choice.

We prove in [9] that, if non-empty, $\mathcal{M}_{\boldsymbol{\mu}}$ is non-singular of pure dimension

$$d_{\boldsymbol{\mu}} := n^2(2g-2+k) - \sum_{i,j}(\mu^i_j)^2 + 2.$$

We also defined an *a priori* rational function $\mathbb{H}_{\boldsymbol{\mu}}(z,w) \in \mathbb{Q}(z,w)$ in terms of Macdonald symmetric functions (see Section 2.1.4 for a precise definition) and we conjecture that the compactly supported mixed Hodge numbers $\{h_c^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}})\}_{i,j,k}$ satisfy $h_c^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}}) = 0$ unless $i = j$ and

$$H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t) \stackrel{?}{=} (t\sqrt{q})^{d_\mu}\mathbb{H}_{\boldsymbol{\mu}}\left(-t\sqrt{q}, \frac{1}{\sqrt{q}}\right), \tag{1.1.1}$$

where $H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t) := \sum_{i,j} h_c^{i,i;j}(\mathcal{M}_{\boldsymbol{\mu}})q^i t^j$ is the compactly supported mixed Hodge polynomial. In particular, $\mathbb{H}_{\boldsymbol{\mu}}(-z,w)$ should actually be a polynomial with non-negative integer coefficients of degree $d_{\boldsymbol{\mu}}$ in each variable.

In [9] we prove that (1.1.1) is true under the specialization $(q,t) \mapsto (q,-1)$, namely,

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) := H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, -1) = q^{\frac{1}{2}d_\mu}\mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right). \tag{1.1.2}$$

This formula is obtained by counting points of $\mathcal{M}_{\boldsymbol{\mu}}$ over finite fields. We compute $\sharp\mathcal{M}_{\boldsymbol{\mu}}(\mathbb{F}_q)$ using a formula involving the values of the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ (a formula that goes back to Frobenius [4]). The calculation shows that $\mathcal{M}_{\boldsymbol{\mu}}$ has *polynomial count*; i.e., there exists a polynomial $P \in \mathbb{C}[T]$ such that for any finite field $\mathbb{F}_q$ of sufficiently large characteristic, $\#\mathcal{M}_{\boldsymbol{\mu}}(\mathbb{F}_q) = P(q)$. Then by a theorem of Katz [9, Appendix] $E(\mathcal{M}_{\boldsymbol{\mu}}; q) = P(q)$. Moreover, $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$ satisfies the following identity

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{d_\mu}E(\mathcal{M}_{\boldsymbol{\mu}}; q^{-1}). \tag{1.1.3}$$

In this paper we use Formula (1.1.2) to prove the following theorem.

Theorem 1.1.1. *If non-empty, the character variety $\mathcal{M}_{\boldsymbol{\mu}}$ is connected.*

The proof of the theorem reduces to proving that the coefficient of the lowest power of q in $\mathbb{H}_{\boldsymbol{\mu}}(\sqrt{q}, 1/\sqrt{q})$, namely $q^{-d_\mu/2}$, equals 1. This turns out to require a rather delicate argument, by far the most technical of the paper, that uses the inequality of Section 6 in a crucial way.

1.2. Relations to Hilbert schemes on $\mathbb{C}^\times \times \mathbb{C}^\times$ and modular forms

Here we assume that $g = k = 1$. Put $X = \mathbb{C}^\times \times \mathbb{C}^\times$ and denote by $X^{[n]}$ the Hilbert scheme of n points in X. Define $\mathbb{H}^{[n]}(z,w) \in \mathbb{Q}(z,w)$ by

$$\sum_{n\geq 0}\mathbb{H}^{[n]}(z,w)T^n := \prod_{n\geq 1}\frac{(1-zwT^n)^2}{(1-z^2T^n)(1-w^2T^n)}, \tag{1.2.1}$$

with the convention that $\mathbb{H}^{[0]}(z,w) := 1$. It is known by work of Göttsche and Soergel [8] that the mixed Hodge polynomial $H_c\left(X^{[n]}; q, t\right)$ is given by

$$H_c\left(X^{[n]};q,t\right)=(qt^2)^n\mathbb{H}^{[n]}\left(-t\sqrt{q},\frac{1}{\sqrt{q}}\right).$$

Conjecture 1.2.1. *We have*

$$\mathbb{H}^{[n]}(z,w)=\mathbb{H}_{(n-1,1)}(z,w).$$

This together with the conjectural Formula (1.1.1) implies that the Hilbert scheme $X^{[n]}$ and the character variety $\mathcal{M}_{(n-1,1)}$ should have the same mixed Hodge polynomial. Although this is believed to be true (in the analogous additive case this is well-known; see Theorem 4.1.1) there is no complete proof in the literature. (The result follows from known facts modulo some missing arguments in the non-Abelian Hodge theory for punctured Riemann surfaces; see the comment after Conjecture 4.2.1.) We prove the following results which give evidence for Conjecture 1.2.1.

Theorem 1.2.2. *We have*

$$\mathbb{H}^{[n]}(0,w)=\mathbb{H}_{(n-1,1)}(0,w),$$
$$\mathbb{H}^{[n]}(w^{-1},w)=\mathbb{H}_{(n-1,1)}(w^{-1},w).$$

The second identity means that the E-polynomials of $X^{[n]}$ and $\mathcal{M}_{(n-1,1)}$ agree. As a consequence of Theorem 1.2.2 we have the following relation between character varieties and quasi-modular forms.

Corollary 1.2.3. *We have*

$$1+\sum_{n\geq 1}\mathbb{H}_{(n-1,1)}\left(e^{u/2},e^{-u/2}\right)T^n=\frac{1}{u}\left(e^{u/2}-e^{-u/2}\right)\exp\left(2\sum_{k\geq 2}G_k(T)\frac{u^k}{k!}\right),$$

where

$$G_k(T):=\frac{-B_k}{2k}+\sum_{n\geq 1}\sum_{d|n}d^{k-1}T^n$$

(with B_k is the k-th Bernoulli number) is the classical Eisenstein series for $SL_2(\mathbb{Z})$.

In particular, the coefficient of any power of u in the left hand side is in the ring of quasi-modular *forms, generated by the G_k, $k\geq 2$, over $\mathbb{Q}$.*

Relations between Hilbert schemes and modular forms were first investigated by Göttsche [7].

1.3. Quiver representations

For a partition $\mu=\mu_1\geq\cdots\geq\mu_r>0$ of n we denote by $l(\mu)=r$ its length. Given a non-negative integer g and a k-tuple $\boldsymbol{\mu}=(\mu^1,\mu^2,\ldots,\mu^k)$ of partitions of n we define a *comet-shaped* quiver $\Gamma_{\boldsymbol{\mu}}$ with k legs of length $s_1,s_2,\ldots,s_k$ (where $s_i=l(\mu^i)-1$) and g loops at the central vertex (see picture in Section 3.2). The multi-partition $\boldsymbol{\mu}$ defines also a dimension vector $\mathbf{v}_{\boldsymbol{\mu}}$ of $\Gamma_{\boldsymbol{\mu}}$ whose coordinates on the i-th leg are $(n,n-\mu^i_1,n-\mu^i_1-\mu^i_2,\ldots,n-\sum_{r=1}^{s_i}\mu^i_r)$.

By a theorem of Kac [13] there exists a monic polynomial $A_{\boldsymbol{\mu}}(T)\in\mathbb{Z}[T]$ of degree $d_{\boldsymbol{\mu}}/2$ such that the number of absolutely indecomposable representations over $\mathbb{F}_q$ (up to isomorphism) of $\Gamma_{\boldsymbol{\mu}}$ of dimension $\mathbf{v}_{\boldsymbol{\mu}}$ equals $A_{\boldsymbol{\mu}}(q)$.

Let us state the main result of this section.

Theorem 1.3.1. *We have*

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \tag{1.3.1}$$

If we assume that $\mathbf{v}_{\boldsymbol{\mu}}$ is indivisible, i.e., that the gcd of all the parts of the partitions $\mu^1, \ldots, \mu^k$ equals 1, then, as mentioned in [9, Remark 1.4.3], the formula can be proved using the results of Crawley-Boevey and van den Bergh [2] together with the results in [9]. More precisely, the results of Crawley-Boevey and van den Bergh say that $A_{\boldsymbol{\mu}}(q)$ equals (up to some power of q) the compactly supported Pincasé polynomial of certain quiver variety $\mathcal{Q}_{\boldsymbol{\mu}}$ (which exists only if $\mathbf{v}_{\boldsymbol{\mu}}$ is indivisible). In [9] we show that the Poincaré polynomial of $\mathcal{Q}_{\boldsymbol{\mu}}$ agrees with $\mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q})$ up to the same power of q, hence the Formula (1.3.1).

The proof of Formula (1.3.1) we give in this paper is completely combinatorial (and works also in the divisible case). It is based on Hua's formula [11] for the number of absolutely indecomposable representations of quivers over finite fields.

The conjectural Formula (1.1.1) together with Formula (1.3.1) implies the following conjecture.

Conjecture 1.3.2. *We have*

$$A_{\boldsymbol{\mu}}(q) = q^{-\frac{d_{\boldsymbol{\mu}}}{2}} PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q),$$

where $PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q) := \sum_i h_c^{i,i;2i}(\mathcal{M}_{\boldsymbol{\mu}}) q^i$ *is the* pure part *of* $H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t)$.

1.4. Characters of general linear groups over finite fields

Given two irreducible complex characters $\mathcal{X}_1, \mathcal{X}_2$ of $\mathrm{GL}_n(\mathbb{F}_q)$ it is a natural and difficult question to understand the decomposition of the tensor product $\mathcal{X}_1 \otimes \mathcal{X}_2$ as a sum of irreducible characters. Note that the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is known (Green, 1955) and so we can compute in theory the multiplicity $\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X} \rangle$ of any irreducible character $\mathcal{X}$ of $\mathrm{GL}_n(\mathbb{F}_q)$ in $\mathcal{X}_1 \otimes \mathcal{X}_2$ using the scalar product formula

$$\langle \mathcal{X}_1 \otimes \mathcal{X}_2, \mathcal{X} \rangle = \frac{1}{|\mathrm{GL}_n(\mathbb{F}_q)|} \sum_{g \in \mathrm{GL}_n(\mathbb{F}_q)} \mathcal{X}_1(g)\mathcal{X}_2(g)\overline{\mathcal{X}(g)}. \tag{1.4.1}$$

However it is very difficult to extract any interesting information from this formula.

In [9] we define the notion of *generic* tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$. We also consider the character $\Lambda : \mathrm{GL}_n(\mathbb{F}_q) \to \mathbb{C}, x \mapsto q^{g \cdot \dim C_{\mathrm{GL}_n}(x)}$ where $C_{\mathrm{GL}_n}(x)$ denotes the centralizer of x in $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and where g is a non-negative integer. For $g = 1$ this is the character of the conjugation action of $\mathrm{GL}_n(\mathbb{F}_q)$ on the group algebra $\mathbb{C}[\mathfrak{gl}_n(\mathbb{F}_q)]$.

If $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is a partition of n, an irreducible character of $\mathrm{GL}_n(\mathbb{F}_q)$ is said to be of type μ if it is of the form $R_{L_\mu}^{\mathrm{GL}_n}(\alpha)$ where $L_\mu = \mathrm{GL}_{\mu_1} \times \mathrm{GL}_{\mu_2} \times \cdots \times \mathrm{GL}_{\mu_r}$ and where α is a *regular* linear character of $L_\mu(\mathbb{F}_q)$; see Section 3.4 for definitions. Characters of this form are called *split semisimple*.

In [9] we prove that for a generic tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ of split semisimple irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of type $\boldsymbol{\mu}$, we have

$$\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1 \rangle = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \tag{1.4.2}$$

Note that in particular this implies that the left hand side only depends on the combinatorial type $\boldsymbol{\mu}$ not on the specific choice of characters.

Together with Formula (1.3.1) we deduce the following formula.

Theorem 1.4.1. *We have*

$$\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle = A_{\boldsymbol{\mu}}(q).$$

Using Kac's results on quiver representations (see Section 3.1) the above theorem has the following consequence.

Corollary 1.4.2. *Let $\Phi(\Gamma_{\boldsymbol{\mu}})$ denote the root system associated with $\Gamma_{\boldsymbol{\mu}}$ and let $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ be a generic k-tuple of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ of type $\boldsymbol{\mu}$.*

We have $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle \neq 0$ if and only if $\mathbf{v}_{\boldsymbol{\mu}} \in \Phi(\Gamma_{\boldsymbol{\mu}})$. Moreover $\langle \Lambda \otimes \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k, 1\rangle = 1$ if and only if $\mathbf{v}_{\boldsymbol{\mu}}$ is a real root.

In [17] the second author discusses the statement of Corollary 1.4.2 for generic tuples of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ which are not necessarily split semisimple.

2. Preliminaries

We denote by $\mathbb{F}$ an algebraic closure of a finite field $\mathbb{F}_q$.

2.1. Symmetric functions

2.1.1. Partitions, Macdonald polynomials, Green polynomials

We denote by $\mathcal{P}$ the set of all partitions including the unique partition 0 of 0, by $\mathcal{P}^*$ the set of non-zero partitions and by $\mathcal{P}_n$ the set of partitions of n. Partitions λ are denoted by $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. We will also sometimes write a partition as $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_i denotes the multiplicity of i in λ. The *size* of λ is $|\lambda| := \sum_i \lambda_i$; the *length* $l(\lambda)$ of λ is the maximum i with $\lambda_i > 0$. For two partitions λ and μ, we define $\langle\lambda, \mu\rangle$ as $\sum_i \lambda'_i \mu'_i$ where λ' denotes the dual partition of λ. We put $n(\lambda) = \sum_{i>0}(i-1)\lambda_i$. Then $\langle\lambda, \lambda\rangle = 2n(\lambda) + |\lambda|$. For two partitions $\lambda = (1^{n_1}, 2^{n_2}, \ldots)$ and $\mu = (1^{m_1}, 2^{m_2}, \ldots)$, we denote by $\lambda \cup \mu$ the partition $(1^{n_1+m_1}, 2^{n_2+m_2}, \ldots)$. For a non-negative integer d and a partition λ, we denote by $d \cdot \lambda$ the partition $(d\lambda_1, d\lambda_2, \ldots)$. The *dominance ordering* for partitions is defined as follows: $\mu \trianglelefteq \lambda$ if and only if $\mu_1 + \cdots + \mu_j \leq \lambda_1 + \cdots + \lambda_j$ for all $j \geq 1$.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be an infinite set of variables and $\Lambda(\mathbf{x})$ the corresponding ring of symmetric functions. As usual we will denote by $s_\lambda(\mathbf{x}), h_\lambda(\mathbf{x}), p_\lambda(\mathbf{x})$, and $m_\lambda(\mathbf{x})$, the Schur symmetric functions, the complete symmetric functions, the power symmetric functions and the monomial symmetric functions.

We will deal with elements of the ring $\Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(z, w)$ and their images under two specializations: their *pure part*, $z = 0, w = \sqrt{q}$ and their *Euler specialization*, $z = \sqrt{q}, w = 1/\sqrt{q}$.

For a partition λ, let $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(q, t)$ be the *Macdonald symmetric function* defined in Garsia and Haiman [6, I.11]. We collect in this section some basic properties of these functions that we will need.

We have the duality

$$\tilde{H}_\lambda(\mathbf{x}; q, t) = \tilde{H}_{\lambda'}(\mathbf{x}; t, q); \tag{2.1.1}$$

see [6, Corollary 3.2]. We define the (transformed) *Hall–Littlewood symmetric function* as

$$\tilde{H}_\lambda(\mathbf{x}; q) := \tilde{H}_\lambda(\mathbf{x}; 0, q). \tag{2.1.2}$$

In the notation just introduced $\tilde{H}_\lambda(\mathbf{x}; q)$ is the pure part of $\tilde{H}_\lambda(\mathbf{x}; z^2, w^2)$.

Under the Euler specialization of $\tilde{H}_\lambda(\mathbf{x}; z^2, w^2)$ we have [9, Lemma 2.3.4]

$$\tilde{H}_\lambda(\mathbf{x}; q, q^{-1}) = q^{-n(\lambda)} H_\lambda(q) s_\lambda(\mathbf{xy}), \tag{2.1.3}$$

where $y_i = q^{i-1}$ and $H_\lambda(q) := \prod_{s\in\lambda}(1 - q^{h(s)})$ is the *hook polynomial* [19, I,3, Example 2].

Define the (q, t)*-Kostka polynomials* $\tilde{K}_{\nu\lambda}(q, t)$ by

$$\tilde{H}_\lambda(\mathbf{x}; q, t) = \sum_\nu \tilde{K}_{\nu\lambda}(q, t) s_\nu(\mathbf{x}). \tag{2.1.4}$$

These are (q, t) generalizations of the $\tilde{K}_{\nu\lambda}(q)$ Kostka–Foulkes polynomial in Macdonald [19, III, (7.11)], which are obtained as $q^{n(\lambda)} K_{\nu\lambda}(q^{-1}) = \tilde{K}_{\nu\lambda}(q) = \tilde{K}_{\nu\lambda}(0, q)$, i.e., by taking their pure part. In particular,

$$\tilde{H}_\lambda(\mathbf{x}; q) = \sum_\nu \tilde{K}_{\nu\lambda}(q) s_\nu(\mathbf{x}). \tag{2.1.5}$$

For a partition λ, we denote by χ^λ the corresponding irreducible character of $S_{|\lambda|}$ as in Macdonald [19]. Under this parameterization, the character $\chi^{(1^n)}$ is the sign character of $S_{|\lambda|}$ and $\chi^{(n^1)}$ is the trivial character. Recall also that the decomposition into disjoint cycles provides a natural parameterization of the conjugacy classes of S_n by the partitions of n. We then denote by χ^λ_μ the value of χ^λ at the conjugacy class of $S_{|\lambda|}$ corresponding to μ (we use the convention that $\chi^\lambda_\mu = 0$ if $|\lambda| \neq |\mu|$). The *Green polynomials* $\{Q^\tau_\lambda(q)\}_{\lambda,\tau\in\mathcal{P}}$ are defined as

$$Q^\tau_\lambda(q) = \sum_\nu \chi^\nu_\lambda \tilde{K}_{\nu\tau}(q) \tag{2.1.6}$$

if $|\lambda| = |\tau|$ and $Q^\tau_\lambda = 0$ otherwise.

2.1.2. *Exp and Log*

Let $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) := \Lambda(\mathbf{x}_1) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \Lambda(\mathbf{x}_k)$ be the ring of functions separately symmetric in each set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of infinitely many variables. To ease the notation we will simply write Λ_k for the ring $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_\mathbb{Z} \mathbb{Q}(q, t)$.

The power series ring $\Lambda_k[[T]]$ is endowed with a natural λ-ring structure in which the Adams operations are

$$\psi_d(f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, q, t; T)) := f(\mathbf{x}_1^d, \mathbf{x}_2^d, \ldots, \mathbf{x}_k^d, q^d, t^d; T^d).$$

Let $\Lambda_k[[T]]^+$ be the ideal $T\Lambda_k[[T]]$ of $\Lambda_k[[T]]$. Define $\Psi : \Lambda_k[[T]]^+ \to \Lambda_k[[T]]^+$ by

$$\Psi(f) := \sum_{n\geq 1} \frac{\psi_n(f)}{n},$$

and $\mathrm{Exp} : \Lambda_k[[T]]^+ \to 1 + \Lambda_k[[T]]^+$ by

$$\mathrm{Exp}(f) = \exp(\Psi(f)).$$

The inverse $\Psi^{-1} : \Lambda_k[\![T]\!]^+ \to \Lambda_k[\![T]\!]^+$ of Ψ is given by

$$\Psi^{-1}(f) = \sum_{n\geq 1} \mu(n) \frac{\psi_n(f)}{n}$$

where μ is the ordinary Möbius function.

The inverse $\mathrm{Log} : 1 + \Lambda_k[\![T]\!]^+ \to \Lambda_k[\![T]\!]^+$ of Exp is given by

$$\mathrm{Log}(f) = \Psi^{-1}(\log(f)).$$

Remark 2.1.1. Let $f = 1 + \sum_{n\geq 1} f_n T^n \in 1 + \Lambda_k[\![T]\!]^+$. If we write

$$\log(f) = \sum_{n\geq 1} \frac{1}{n} U_n T^n, \qquad \mathrm{Log}(f) = \sum_{n\geq 1} V_n T^n,$$

then

$$V_r = \frac{1}{r} \sum_{d|r} \mu(d) \psi_d(U_{r/d}).$$

We will need the following properties (details may be found for instance in Mozgovoy [20]). For $g \in \Lambda_k$ and $n \geq 1$ we put

$$g_n := \frac{1}{n} \sum_{d|n} \mu(d) \psi_{\frac{n}{d}}(g).$$

This is the Möbius inversion formula of $\psi_n(g) = \sum_{d|n} d \cdot g_d$.

Lemma 2.1.2. *Let $g \in \Lambda_k$ and $f_1, f_2 \in 1 + \Lambda_k[\![T]\!]^+$ such that*

$$\log(f_1) = \sum_{d=1}^{\infty} g_d \cdot \log(\psi_d(f_2)).$$

Then

$$\mathrm{Log}(f_1) = g \cdot \mathrm{Log}(f_2).$$

Lemma 2.1.3. *Assume that $f \in \Lambda_k[\![T]\!]^+$. If it has coefficients in $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_{\mathbb{Z}} \mathbb{Z}[q,t] \subset \Lambda_k$, then* $\mathrm{Exp}(f)$ *has also coefficients in $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_{\mathbb{Z}} \mathbb{Z}[q,t]$.*

2.1.3. Types

We choose once and for all a total ordering $\geq$ on $\mathcal{P}$ (e.g. the lexicographic ordering) and we continue to denote by $\geq$ the total ordering defined on the set of pairs $\mathbb{Z}^*_{\geq 0} \times \mathcal{P}^*$ as follows: if $\lambda \neq \mu$ and $\lambda \geq \mu$, then $(d, \lambda) \geq (d', \mu)$, and $(d, \lambda) \geq (d', \lambda)$ if $d \geq d'$. We denote by $\mathbf{T}$ the set of non-increasing sequences $\omega = (d_1, \omega^1) \geq (d_2, \omega^2) \geq \cdots \geq (d_r, \omega^r)$, which we will call a *type*. To alleviate the notation we will then omit the symbol $\geq$ and write simply $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r)$. The *size* of a type ω is $|\omega| := \sum_i d_i |\lambda^i|$. We denote by $\mathbf{T}_n$ the set of types of size n. We denote by $m_{d,\lambda}(\omega)$ the multiplicity of (d, λ) in ω. As with partitions it is sometimes convenient to consider a type as a collection of integers $m_{d,\lambda} \geq 0$ indexed by pairs

$(d, \lambda) \in \mathbb{Z}_{>0} \times \mathcal{P}^*$. For a type $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r)$, we put $n(\omega) = \sum_i d_i n(\omega^i)$ and $[\omega] := \cup_i d_i \cdot \omega^i$.

When considering elements $a_{\boldsymbol{\mu}} \in \Lambda_k$ indexed by multi-partitions $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, we will always assume that they are homogeneous of degree $(|\mu^1|, \ldots, |\mu^k|)$ in the set of variables $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Let $\{a_{\boldsymbol{\mu}}\}_{\boldsymbol{\mu} \in \mathcal{P}^k}$ be a family of symmetric functions in Λ_k indexed by multi-partitions. We extend its definition to a *multi-type* $\boldsymbol{\omega} = (d_1, \boldsymbol{\omega}^1) \cdots (d_s, \boldsymbol{\omega}^s)$ with $\boldsymbol{\omega}^p \in (\mathcal{P}_{n_p})^k$, by

$$a_{\boldsymbol{\omega}} := \prod_p \psi_{d_p}(A_{\boldsymbol{\omega}^p}).$$

For a multi-type $\boldsymbol{\omega}$ as above, we put

$$C^o_{\boldsymbol{\omega}} := \begin{cases} \dfrac{\mu(d)}{d}(-1)^{r-1} \dfrac{(r-1)!}{\prod_{\boldsymbol{\mu}} m_{d,\boldsymbol{\mu}}(\boldsymbol{\omega})!} & \text{if } d_1 = \cdots = d_r = d \\ 0 \quad \text{otherwise} \end{cases}$$

where $m_{d,\boldsymbol{\mu}}(\boldsymbol{\omega})$ with $\boldsymbol{\mu} \in \mathcal{P}^k$ denotes the multiplicity of $(d, \boldsymbol{\mu})$ in $\boldsymbol{\omega}$.

We have the following lemma (see [9, Section 2.3.3] for a proof).

Lemma 2.1.4. *Let $\{A_{\boldsymbol{\mu}}\}_{\boldsymbol{\mu} \in \mathcal{P}^k}$ be a family of symmetric functions in Λ_k with $A_0 = 1$. Then*

$$\operatorname{Log}\left(\sum_{\boldsymbol{\mu} \in \mathcal{P}^k} A_{\boldsymbol{\mu}} T^{|\boldsymbol{\mu}|}\right) = \sum_{\boldsymbol{\omega}} C^o_{\boldsymbol{\omega}} A_{\boldsymbol{\omega}} T^{|\boldsymbol{\omega}|} \tag{2.1.7}$$

where $\boldsymbol{\omega}$ runs over multi-types $(d_1, \boldsymbol{\omega}^1) \cdots (d_s, \boldsymbol{\omega}^s)$.

The formal power series $\sum_{n \geq 0} a_n T^n$ with $a_n \in \Lambda_k$ that we will consider in what follows will all have a_n homogeneous of degree n. Hence we will typically scale the variables of Λ_k by $1/T$ and eliminate T altogether.

Given any family $\{a_\mu\}$ of symmetric functions indexed by partitions $\mu \in \mathcal{P}$ and a multi-partition $\boldsymbol{\mu} \in \mathcal{P}^k$ as above define

$$a_{\boldsymbol{\mu}} := a_{\mu^1}(\mathbf{x}_1) \cdots a_{\mu^k}(\mathbf{x}_k).$$

Let $\langle \cdot, \cdot \rangle$ be the Hall pairing on $\Lambda(\mathbf{x})$, extend its definition to $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ by setting

$$\langle a_1(\mathbf{x}_1) \cdots a_k(\mathbf{x}_k), b_1(\mathbf{x}_1) \cdots b_k(\mathbf{x}_k) \rangle = \langle a_1, b_1 \rangle \cdots \langle a_k, b_k \rangle, \tag{2.1.8}$$

for any $a_1, \ldots, a_k; b_1, \ldots, b_k \in \Lambda(\mathbf{x})$ and to formal series by linearity.

2.1.4. Cauchy identity

Given a partition $\lambda \in \mathcal{P}_n$ we define the genus g *hook function* $\mathcal{H}_\lambda(z, w)$ by

$$\mathcal{H}_\lambda(z, w) := \prod_{s \in \lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^{2g}}{(z^{2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})},$$

where the product is over all cells s of λ with $a(s)$ and $l(s)$ its arm and leg length, respectively. For details on the hook function we refer the reader to [10].

Recall the specialization (cf. [9, Section 2.3.5])

$$\mathcal{H}_\lambda(0, \sqrt{q}) = \frac{q^{g\langle\lambda,\lambda\rangle}}{a_\lambda(q)} \tag{2.1.9}$$

where $a_\lambda(q)$ is the cardinality of the centralizer of a unipotent element of $\mathrm{GL}_n(\mathbb{F}_q)$ with Jordan form of type λ.

It is also not difficult to verify that the Euler specialization of $\mathcal{H}_\lambda$ is

$$\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) = \left(q^{-\frac{1}{2}\langle\lambda,\lambda\rangle} H_\lambda(q)\right)^{2g-2}. \tag{2.1.10}$$

We have

$$\mathcal{H}_\lambda(z, w) = \mathcal{H}_{\lambda'}(w, z) \quad \text{and} \quad \mathcal{H}_\lambda(-z, -w) = \mathcal{H}_\lambda(z, w). \tag{2.1.11}$$

Let

$$\Omega(z, w) = \Omega(\mathbf{x}_1, \ldots, \mathbf{x}_k; z, w) := \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(z, w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2).$$

By (2.1.1) and (2.1.11) we have

$$\Omega(z, w) = \Omega(w, z) \quad \text{and} \quad \Omega(-z, -w) = \Omega(z, w). \tag{2.1.12}$$

For $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, we let

$$\mathbb{H}_{\boldsymbol{\mu}}(z, w) := (z^2 - 1)(1 - w^2)\left\langle \mathrm{Log}\,\Omega(z, w), h_{\boldsymbol{\mu}}\right\rangle. \tag{2.1.13}$$

By (2.1.12) we have the symmetries

$$\mathbb{H}_{\boldsymbol{\mu}}(z, w) = \mathbb{H}_{\boldsymbol{\mu}}(w, z) \quad \text{and} \quad \mathbb{H}_{\boldsymbol{\mu}}(-z, -w) = \mathbb{H}_{\boldsymbol{\mu}}(z, w). \tag{2.1.14}$$

We may recover $\Omega(z, w)$ from the $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$'s by the formula:

$$\Omega(z, w) = \mathrm{Exp}\left(\sum_{\boldsymbol{\mu}\in\mathcal{P}^k} \frac{\mathbb{H}_{\boldsymbol{\mu}}(z, w)}{(z^2 - 1)(1 - w^2)} m_{\boldsymbol{\mu}}\right). \tag{2.1.15}$$

From Formulas (2.1.3) and (2.1.10) we have the following.

Lemma 2.1.5. *With the specialization $y_i = q^{i-1}$,*

$$\Omega\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right) = \sum_{\lambda\in\mathcal{P}} q^{(1-g)|\lambda|} \left(q^{-n(\lambda)} H_\lambda(q)\right)^{2g+k-2} \prod_{i=1}^{k} s_\lambda(\mathbf{x}_i \mathbf{y}).$$

Conjecture 2.1.6. *The rational function $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ is a polynomial with integer coefficients. It has degree*

$$d_{\boldsymbol{\mu}} := n^2(2g - 2 + k) - \sum_{i,j} (\mu^i_j)^2 + 2$$

in each variable and the coefficients of $\mathbb{H}_{\boldsymbol{\mu}}(-z, w)$ are non-negative.

The function $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ is computed in many cases in [9, Section 1.5].

2.2. Characters and Fourier transforms

2.2.1. Characters of finite general linear groups

For a finite group H let us denote by Mod_H the category of finite dimensional $\mathbb{C}[H]$ left modules. Let K be an other finite group. By an *H-module-K* we mean a finite dimensional $\mathbb{C}$-vector space M endowed with a left action of H and with a right action of K which commute together. Such a module M defines a functor $R_K^H : \mathrm{Mod}_K \to \mathrm{Mod}_H$ by $V \mapsto M \otimes_{\mathbb{C}[K]} V$. Let $\mathbb{C}(H)$ denotes the $\mathbb{C}$-vector space of all functions $H \to \mathbb{C}$ which are constant on conjugacy classes. We continue to denote by R_K^H the $\mathbb{C}$-linear map $\mathbb{C}(K) \to \mathbb{C}(H)$ induced by the functor R_K^H (we first define it on irreducible characters and then extend it by linearity to the whole $\mathbb{C}(K)$). Then for any $f \in \mathbb{C}(K)$, we have

$$R_K^H(f)(g) = |K|^{-1} \sum_{k \in K} \mathrm{Trace}\Big((g, k^{-1}) \mid M\Big) f(k). \tag{2.2.1}$$

Let $G = \mathrm{GL}_n(\mathbb{F}_q)$ with $\mathbb{F}_q$ a finite field. Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n and let $\mathcal{F}_\lambda = \mathcal{F}_\lambda(\mathbb{F}_q)$ be the variety of partial flags of $\mathbb{F}_q$-vector spaces

$$\{0\} = E^r \subset E^{r-1} \subset \cdots \subset E^1 \subset E^0 = (\mathbb{F}_q)^n$$

such that $\dim(E^{i-1}/E^i) = \lambda_i$.

Let G acts on $\mathcal{F}_\lambda$ in the natural way. Fix an element

$$X_o = \Big(\{0\} = E^r \subset E^{r-1} \subset \cdots \subset E^1 \subset E^0 = (\mathbb{F}_q)^n\Big) \in \mathcal{F}_\lambda$$

and denote by P_λ the stabilizer of X_o in G and by U_λ the subgroup of elements $g \in P_\lambda$ which induces the identity on E^i/E^{i+1} for all $i = 0, 1, \ldots, r-1$.

Put $L_\lambda := \mathrm{GL}_{\lambda_r}(\mathbb{F}_q) \times \cdots \times \mathrm{GL}_{\lambda_1}(\mathbb{F}_q)$. Recall that U_λ is a normal subgroup of P_λ and that $P_\lambda = L_\lambda \ltimes U_\lambda$.

Denote by $\mathbb{C}[G/U_\lambda]$ the $\mathbb{C}$-vector space generated by the finite set $G/U_\lambda = \{gU_\lambda \mid g \in G\}$. The group L_λ (resp. G) acts on $\mathbb{C}[G/U_\lambda]$ as $(gU_\lambda)\cdot l = glU_\lambda$ (resp. as $g \cdot (hU_\lambda) = ghU_\lambda$). These two actions make $\mathbb{C}[G/U_\lambda]$ into a G-module-L_λ. The associated functor $R_{L_\lambda}^G : \mathrm{Mod}_{L_\lambda} \to \mathrm{Mod}_G$ is the so-called *Harish-Chandra functor*.

We have the following well-known lemma.

Lemma 2.2.1. *We denote by* 1 *the identity character of L_λ. Then for all $g \in G$, we have*

$$R_{L_\lambda}^G(1)(g) = \#\{X \in \mathcal{F}_\lambda \mid g \cdot X = X\}.$$

Proof. By Formula (2.2.1) we have

$$\begin{aligned} R_{L_\lambda}^G(1)(g) &= |L_\lambda|^{-1} \sum_{k \in L_\lambda} \#\{hU_\lambda \mid ghU_\lambda = hkU_\lambda\} \\ &= |L_\lambda|^{-1} \sum_{k \in L_\lambda} \#\{hU_\lambda \mid gh \in hkU_\lambda\} \\ &= |L_\lambda|^{-1} \#\{hU_\lambda \mid gh \in hP_\lambda\} \\ &= \#\{hP_\lambda \mid ghP_\lambda = hP_\lambda\}. \end{aligned}$$

We deduce the lemma from last equality by noticing that the map $G \to \mathcal{F}_\lambda$, $g \mapsto g \cdot X_o$ induces a bijection $G/P_\lambda \to \mathcal{F}_\lambda$. $\square$

We now recall the definition of the type of a conjugacy class C of G (cf. [9, 4.1]). The Frobenius $f : \mathbb{F} \to \mathbb{F}, x \mapsto x^q$ acts on the set of eigenvalues of C. Let us write the set of eigenvalues of C as a disjoint union

$$\{\gamma_1, \gamma_1^q, \ldots\} \coprod \{\gamma_2, \gamma_2^q, \ldots\} \coprod \cdots \coprod \{\gamma_r, \gamma_r^q, \ldots\}$$

of $\langle f\rangle$-orbits, and let m_i be the multiplicity of γ_i. The unipotent part of an element of C defines a unique partition ω^i of m_i. Re-ordering if necessary we may assume that $(d_1, \omega^1) \geq (d_2, \omega^2) \geq \cdots \geq (d_r, \omega^r)$. We then call $\omega = (d_1, \omega^1)\cdots(d_r, \omega^r) \in \mathbf{T}_n$ the *type* of C.

Put $T := L_{(1,1,\ldots,1)}$. It is the subgroup of diagonal matrices of G. The decomposition of $R_T^{L_\lambda}(1)$ as a sum of irreducible characters reads

$$R_T^{L_\lambda}(1) = \sum_{\chi\in \mathrm{Irr}(W_{L_\lambda})} \chi(1)\cdot \mathcal{U}_\chi,$$

where $W_{L_\lambda} := N_{L_\lambda}(T)/T$ is the Weyl group of L_λ. We call the irreducible characters $\{\mathcal{U}_\chi\}_\chi$ the *unipotent* characters of L_λ. The character $\mathcal{U}_1$ is the trivial character of L_λ. Since $W_{L_\lambda} \simeq S_{\lambda_1} \times \cdots \times S_{\lambda_r}$, the irreducible characters of W_{L_λ} are $\chi^\tau := \chi^{\tau^1}\cdots\chi^{\tau^r}$ where τ runs over the set of types $\tau = \{(1, \tau^i)\}_{i=1,\ldots,r}$ with τ^i a partition of λ_i. We denote by $\mathcal{U}_\tau$ the unipotent character of L_λ corresponding to such a type τ.

Theorem 2.2.2. *Let $\mathcal{U}_\tau$ be a unipotent character of L_λ and let C be a conjugacy class of type ω. Then*

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \left\langle \tilde{H}_\omega(\mathbf{x}, q), s_\tau(\mathbf{x})\right\rangle.$$

Proof. The proof is contained in [9] although the formula is not explicitly written there. For the convenience of the reader we now explain how to extract the proof from [9]. For $w \in W_\lambda$, we denote by $R_{T_w}^G(1)$ the corresponding *Deligne–Lusztig character* of G. Its construction is outlined in [9, 2.6.4]. The character $\mathcal{U}_\tau$ of L_λ decomposes as,

$$\mathcal{U}_\tau = |W_\lambda|^{-1} \sum_{w\in W_\lambda} \chi_w^\tau \cdot R_{T_w}^{L_\lambda}$$

where χ_w^τ denotes the value of χ^τ at w. Applying the Harish-Chandra induction $R_{L_\lambda}^G$ to both side and using the transitivity of induction we find that

$$R_{L_\lambda}^G(\mathcal{U}_\tau) = |W_\lambda|^{-1} \sum_{w\in W_\lambda} \chi_w^\tau \cdot R_{T_w}^G(1).$$

We are now in a position to use the calculation in [9]. Notice that the right hand side of the above formula is the right hand side of the first formula displayed in the proof of [9, Theorem 4.3.1] with $(M, \theta^{T_w}, \tilde{\varphi}) = (L_\lambda, 1, \chi^\tau)$ and so the same calculation to get [9, (4.3.2)] together with [9, (4.3.3)] gives in our case

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \sum_\alpha z_\alpha^{-1}\chi_\alpha^\tau \sum_{\{\beta|[\beta]=[\alpha]\}} Q_\beta^\omega(q) z_{[\alpha]} z_\beta^{-1}$$

where the notation are those of [9, 4.3]. We now apply [9, Lemma 2.3.5] to get

$$R_{L_\lambda}^G(\mathcal{U}_\tau)(C) = \left\langle \tilde{H}_\omega(\mathbf{x}; q), s_\tau(\mathbf{x})\right\rangle. \quad \square$$

If α is the type $(1,(\lambda_1))\cdots(1,(\lambda_r))$, then $s_\alpha(\mathbf{x})=h_\lambda(\mathbf{x})$. Hence we have the following.

Corollary 2.2.3. *If C is a conjugacy class of G type ω, then*

$$R_{L_\lambda}^G(1)(C)=\left\langle \tilde{H}_\omega(\mathbf{x},q),h_\lambda(\mathbf{x})\right\rangle.$$

Corollary 2.2.4. *Put $\mathcal{F}^{\#}_{\lambda,\omega}(q):=\#\{X\in\mathcal{F}_\lambda\mid g\cdot X=X\}$ where $g\in G$ is an element in a conjugacy class of type ω. Then*

$$\tilde{H}_\omega(\mathbf{x},q)=\sum_\lambda \mathcal{F}^{\#}_{\lambda,\omega}(q)m_\lambda(\mathbf{x}).$$

Proof. It follows from Lemma 2.2.1 and Corollary 2.2.3. □

We now recall how to construct from a partition $\lambda=(\lambda_1,\ldots,\lambda_r)$ of n a certain family of irreducible characters of G. Choose r distinct linear characters $\alpha_1,\ldots,\alpha_r$ of $\mathbb{F}_q^\times$. This defines for each i a linear character $\tilde{\alpha}_i:\mathrm{GL}_{\lambda_i}(\mathbb{F}_q)\to\mathbb{C}^\times$, $g\mapsto\alpha_i(\det(g))$, and hence a linear character $\tilde{\boldsymbol{\alpha}}:L_\lambda\to\mathbb{C}^\times$, $(g_i)\mapsto\tilde{\alpha}_r(g_r)\cdots\tilde{\alpha}_1(g_1)$. This linear character has the following property: for an element $g\in N_G(L_\lambda)$, we have $\tilde{\alpha}(g^{-1}lg)=\tilde{\alpha}(l)$ for all $l\in L_\lambda$ if and only if $g\in L_\lambda$. A linear character of L_λ which satisfies this property is called a *regular* character of L_λ.

It is a well-known fact that $R_{L_\lambda}^G(\tilde{\boldsymbol{\alpha}})$ is an irreducible character of G. Note that the irreducible characters of G are not all obtained in this way (see [18] for the complete description of the irreducible characters of G in terms of Deligne–Lusztig induction).

We now recall the definition of generic tuples of irreducible characters (cf. [9, Definition 4.2.2]). Since in this paper we are only considering irreducible characters of the form $R_{L_\lambda}^G(\tilde{\boldsymbol{\alpha}})$, the definition given in [9, Definition 4.2.2] simplifies.

Definition 2.2.5. Consider irreducible characters $R_{L_{\lambda^1}}^G(\tilde{\boldsymbol{\alpha}}_1),\ldots,R_{L_{\lambda^k}}^G(\tilde{\boldsymbol{\alpha}}_k)$ of G as above for a multi-partition $\boldsymbol{\lambda}=(\lambda^1,\ldots,\lambda^k)\in(\mathcal{P}_n)^k$. Let T be the subgroup of G of diagonal matrices. Note that $T\subset L_\lambda$ for all partition λ, and so T contains the center Z_λ of any L_λ. Consider the linear character $\boldsymbol{\alpha}=(\tilde{\boldsymbol{\alpha}}_1|_T)\cdots(\tilde{\boldsymbol{\alpha}}_k|_T)$ of T. Then we say that the tuple $\left(R_{L_{\lambda^1}}^G(\tilde{\boldsymbol{\alpha}}_1),\ldots,R_{L_{\lambda^k}}^G(\tilde{\boldsymbol{\alpha}}_k)\right)$ is *generic* if the restriction $\boldsymbol{\alpha}|_{Z_\lambda}$ of $\boldsymbol{\alpha}$ to any subtori Z_λ, with $\lambda\in\mathcal{P}_n-\{(n)\}$, is non-trivial and if $\boldsymbol{\alpha}|_{Z_{(n)}}$ is trivial (the center $Z_{(n)}\simeq\mathbb{F}_q^\times$ consists of scalar matrices $a.I_n$).

We can show as for conjugacy classes [9, Lemma 2.1.2] that if the characteristic p of $\mathbb{F}_q$ and q are sufficiently large, generic tuples of irreducible characters of a given type $\boldsymbol{\lambda}$ always exist.

Put $\mathfrak{g}:=\mathfrak{gl}_n(\mathbb{F}_q)$. For $X\in\mathfrak{g}$, put

$$\Lambda^1(X):=\#\{Y\in\mathfrak{g}\mid[X,Y]=0\}.$$

The restriction $\Lambda^1:G\to\mathbb{C}$ of Λ^1 to $G\subset\mathfrak{g}$ is the character of the representation $G\to\mathrm{GL}(\mathbb{C}[\mathfrak{g}])$ induced by the conjugation action of G on $\mathfrak{g}$. Fix a non-negative integer g and put $\Lambda:=(\Lambda^1)^{\otimes g}$.

For a multi-partition $\boldsymbol{\mu}=(\mu^1,\ldots,\mu^k)\in(\mathcal{P}_n)^k$ and a generic tuple $\left(R_{L_{\mu^1}}^G(\tilde{\boldsymbol{\alpha}}_1),\ldots,R_{L_{\mu^k}}^G(\tilde{\boldsymbol{\alpha}}_k)\right)$ of irreducible characters we put

$$R_{\boldsymbol{\mu}}:=R_{L_{\mu^1}}^G(\tilde{\boldsymbol{\alpha}}_1)\otimes\cdots\otimes R_{L_{\mu^k}}^G(\tilde{\boldsymbol{\alpha}}_k).$$

For two class functions $f, g \in \mathbb{C}(G)$, we define

$$\langle f, g\rangle := |G|^{-1}\sum_{h\in G} f(h)\overline{g(h)}.$$

We have the following theorem [9, Theorem 1.4.1].

Theorem 2.2.6. *We have*

$$\left\langle \Lambda\otimes R_{\boldsymbol{\mu}}, 1\right\rangle = \mathbb{H}_{\boldsymbol{\mu}}\left(0, \sqrt{q}\right)$$

where $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ is the function defined in Section 2.1.4.

Corollary 2.2.7. *The multiplicity $\left\langle \Lambda\otimes R_{\boldsymbol{\mu}}, 1\right\rangle$ depends only on $\boldsymbol{\mu}$ and not on the choice of linear characters $(\tilde{\boldsymbol{\alpha}}_1, \ldots, \tilde{\boldsymbol{\alpha}}_k)$.*

2.2.2. Fourier transforms

Let $\mathrm{Fun}(\mathfrak{g})$ be the $\mathbb{C}$-vector space of all functions $\mathfrak{g}\to\mathbb{C}$ and $\mathbb{C}(\mathfrak{g})$ the subspace of functions $\mathfrak{g}\to\mathbb{C}$ which are constant on G-orbits of $\mathfrak{g}$ for the conjugation action of G on $\mathfrak{g}$.

Let $\Psi : \mathbb{F}_q \to \mathbb{C}^\times$ be a non-trivial additive character and consider the trace pairing $\mathrm{Tr} : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}^\times$. Define the Fourier transform $\mathcal{F}^{\mathfrak{g}} : \mathrm{Fun}(\mathfrak{g})\to\mathrm{Fun}(\mathfrak{g})$ by the formula

$$\mathcal{F}^{\mathfrak{g}}(f)(x) = \sum_{y\in\mathfrak{g}} \Psi\left(\mathrm{Tr}(xy)\right) f(y)$$

for all $f\in\mathrm{Fun}(\mathfrak{g})$ and $x\in\mathfrak{g}$.

The Fourier transform satisfies the following easy property.

Proposition 2.2.8. *For any $f\in\mathrm{Fun}(\mathfrak{g})$ we have:*

$$|\mathfrak{g}|\cdot f(0) = \sum_{x\in\mathfrak{g}}\mathcal{F}^{\mathfrak{g}}(f)(x).$$

Let $*$ be the convolution product on $\mathrm{Fun}(\mathfrak{g})$ defined by

$$(f*g)(a) = \sum_{x+y=a} f(x)g(y)$$

for any two functions $f, g\in\mathrm{Fun}(\mathfrak{g})$.

Recall that

$$\mathcal{F}^{\mathfrak{g}}(f*g) = \mathcal{F}^{\mathfrak{g}}(f)\cdot\mathcal{F}^{\mathfrak{g}}(g). \tag{2.2.2}$$

For a partition λ of n, let $\mathfrak{p}_\lambda, \mathfrak{l}_\lambda, \mathfrak{u}_\lambda$ be the Lie sub-algebras of $\mathfrak{g}$ corresponding respectively to the subgroups $P_\lambda, L_\lambda, U_\lambda$ defined in Section 2.2, namely $\mathfrak{l}_\lambda = \bigoplus_i \mathfrak{gl}_{\lambda_i}(\mathbb{F}_q)$, $\mathfrak{p}_\lambda$ is the parabolic sub-algebra of $\mathfrak{g}$ having $\mathfrak{l}_\lambda$ as a Levi sub-algebra and containing the upper triangular matrices. We have $\mathfrak{p}_\lambda = \mathfrak{l}_\lambda\oplus\mathfrak{u}_\lambda$.

Define the two functions $R^{\mathfrak{g}}_{\mathfrak{l}_\lambda}(1), Q^{\mathfrak{g}}_{\mathfrak{l}_\lambda}\in\mathbb{C}(\mathfrak{g})$ by

$$R^{\mathfrak{g}}_{\mathfrak{l}_\lambda}(1)(x) = |P_\lambda|^{-1}\#\{g\in G \mid g^{-1}xg\in\mathfrak{p}_\lambda\},$$
$$Q^{\mathfrak{g}}_{\mathfrak{l}_\lambda}(x) = |P_\lambda|^{-1}\#\{g\in G \mid g^{-1}xg\in\mathfrak{u}_\lambda\}.$$

We define the type of a G-orbit of $\mathfrak{g}$ similarly as in the group setting (see Corollary 2.2.3). The types of the G-orbits of $\mathfrak{g}$ are then also parameterized by $\mathbf{T}_n$.

Remark 2.2.9. From Lemma 2.2.1, we see that $R_{L_\lambda}^G(1)(x) = |P_\lambda|^{-1}\#\{g \in G \mid g^{-1}xg \in P_\lambda\}$; hence $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)$ is the Lie algebra analogue of $R_{L_\lambda}^G(1)$ and the two functions take the same values on elements of same type.

Proposition 2.2.10. *We have*

$$\mathcal{F}^{\mathfrak{g}}\left(Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}}\right) = q^{\frac{1}{2}(n^2-\sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1).$$

Proof. Consider the $\mathbb{C}$-linear map $R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} : \mathbb{C}(\mathfrak{l}_\lambda) \to \mathbb{C}(\mathfrak{g})$ defined by

$$R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(f)(x) = |P_\lambda|^{-1} \sum_{\{g\in G | g^{-1}xg\in\mathfrak{p}_\lambda\}} f(\pi(g^{-1}xg))$$

where $\pi : \mathfrak{p}_\lambda \to \mathfrak{l}_\lambda$ is the canonical projection. Then it is easy to see that $Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}} = R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1_0)$ where $1_0 \in \mathbb{C}(\mathfrak{l}_\lambda)$ is the characteristic function of $0 \in \mathfrak{l}_\lambda$, i.e., $1_0(x) = 1$ if $x = 0$ and $1_0(x) = 0$ otherwise. The result follows from the easy fact that $\mathcal{F}^{\mathfrak{l}_\lambda}(1_0)$ is the identity function 1 on $\mathfrak{l}_\lambda$ and the fact (see Lehrer [16]) that

$$\mathcal{F}^{\mathfrak{g}} \circ R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} = q^{\frac{1}{2}(n^2-\sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}} \circ \mathcal{F}^{\mathfrak{l}_\lambda}. \quad \square$$

Remark 2.2.11. For $x \in \mathfrak{g}$, denote by $1_x \in \text{Fun}(\mathfrak{g})$ the characteristic function of x that takes the value 1 at x and the value 0 elsewhere. Note that $\mathcal{F}^{\mathfrak{g}}(1_x)$ is the linear character $\mathfrak{g} \to \mathbb{C}, t \mapsto \Psi(\text{Tr}\,(xt))$ of the abelian group $(\mathfrak{g}, +)$. Hence if $f : \mathfrak{g} \to \mathbb{C}$ is a function which takes integer values, then $\mathcal{F}^{\mathfrak{g}}(f)$ is a character (not necessarily irreducible) of $(\mathfrak{g}, +)$. Since the Green functions $Q_{\mathfrak{l}_\lambda}^{\mathfrak{g}}$ take integer values, by Proposition 2.2.10 the function $q^{\frac{1}{2}(n^2-\sum_i \lambda_i^2)} R_{\mathfrak{l}_\lambda}^{\mathfrak{g}}(1)$ is a character of $(\mathfrak{g}, +)$.

3. Absolutely indecomposable representations

3.1. Generalities on quiver representations

Let Γ be a finite quiver, I be the set of its vertices and let Ω be the set of its arrows. For $\gamma \in \Omega$, we denote by $h(\gamma), t(\gamma) \in I$ the head and the tail of γ. A *dimension vector* of Γ is a collection of non-negative integers $\mathbf{v} = \{v_i\}_{i\in I}$ and a *representation* φ of Γ of dimension $\mathbf{v}$ over a field $\mathbb{K}$ is a collection of $\mathbb{K}$-linear maps $\boldsymbol{\varphi} = \{\varphi_\gamma : V_{t(\gamma)} \to V_{h(\gamma)}\}_{\gamma\in\Omega}$ with $\dim V_i = v_i$. Let $\text{Rep}_{\Gamma,\mathbf{v}}(\mathbb{K})$ be the $\mathbb{K}$-vector space of all representations of Γ of dimension $\mathbf{v}$ over $\mathbb{K}$. If $\boldsymbol{\varphi} \in \text{Rep}_{\Gamma,\mathbf{v}}(\mathbb{K})$, $\boldsymbol{\varphi}' \in \text{Rep}_{\Gamma,\mathbf{v}'}(\mathbb{K})$, then a morphism $f : \boldsymbol{\varphi} \to \boldsymbol{\varphi}'$ is a collection of $\mathbb{K}$-linear maps $f_i : V_i \to V_i'$, $i \in I$ such that for all $\gamma \in \Omega$, we have $f_{h(\gamma)} \circ \varphi_\gamma = \varphi'_\gamma \circ f_{t(\gamma)}$.

We define in the obvious way direct sums $\boldsymbol{\varphi} \oplus \boldsymbol{\varphi}' \in \text{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v} + \mathbf{v}')$ of representations. A representation of Γ is said to be *indecomposable* over $\mathbb{K}$ if it is not isomorphic to a direct sum of two non-zero representations of Γ. If an indecomposable representation of Γ remains indecomposable over any finite extension of $\mathbb{K}$, we say that it is *absolutely indecomposable*. Denote by $\text{M}_{\Gamma,\mathbf{v}}(\mathbb{K})$ the set of isomorphism classes of $\text{Rep}_{\Gamma,\mathbf{v}}(\mathbb{K})$ and by $\text{A}_{\Gamma,\mathbf{v}}(\mathbb{K})$ the subset of absolutely indecomposable representations of $\text{Rep}_{\Gamma,\mathbf{v}}(\mathbb{K})$.

By a theorem of Kac there exists a polynomial $A_{\Gamma,\mathbf{v}}(T) \in \mathbb{Z}[T]$ such that for any finite field with q elements $A_{\Gamma,\mathbf{v}}(q) = \#\text{A}_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$. We call $A_{\Gamma,\mathbf{v}}$ the *Kac polynomial* of $(\Gamma, \mathbf{v})$.

Let $\Phi(\Gamma) \subset \mathbb{Z}^I$ be the root system associated with the quiver Γ following Kac [13] and let $\Phi(\Gamma)^+ \subset \left(\mathbb{Z}_{\geq 0}\right)^I$ be the subset of positive roots. Let $\mathbf{C} = (c_{ij})_{i,j}$ be the Cartan matrix of Γ, namely

$$c_{ij} = \begin{cases} 2 - 2(\text{the number of edges joining } i \text{ to itself}) & \text{if } i = j \\ -(\text{the number of edges joining } i \text{ to } j) & \text{otherwise.} \end{cases}$$

Then we have the following well-known theorem (see Kac [13]).

Theorem 3.1.1. *$A_{\Gamma,\mathbf{v}}(q) \neq 0$ if and only if $\mathbf{v} \in \Phi(\Gamma)^+$; $A_{\Gamma,\mathbf{v}}(q) = 1$ if and only if $\mathbf{v}$ is a real root. The polynomial $A_{\Gamma,\mathbf{v}}$, if non-zero, is monic of degree $1 - \frac{1}{2}{}^t\mathbf{v}\mathbf{C}\mathbf{v}$.*

By Kac [13], there exists a polynomial $M_{\Gamma,\mathbf{v}}(q) \in \mathbb{Q}[T]$ such that $M_{\Gamma,\mathbf{v}}(q) := \#\mathrm{M}_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$. The following formula is a reformation of Hua's formula [11].

Theorem 3.1.2. . *We have*

$$\operatorname{Log}\left(\sum_{\mathbf{v}\in(\mathbb{Z}_{\geq 0})^I} M_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}}\right) = \sum_{\mathbf{v}\in(\mathbb{Z}_{\geq 0})^I - \{0\}} A_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}},$$

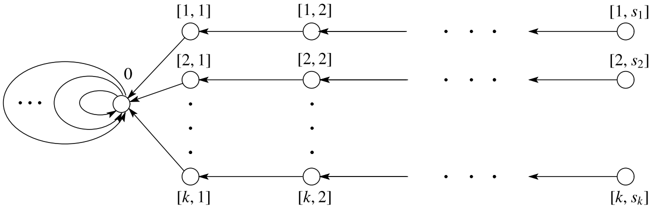
where $X^{\mathbf{v}}$ is the monomial $\prod_{i\in I} X_i^{v_i}$ for some independent commuting variables $\{X_i\}_{i\in I}$.

Since $A_{\Gamma,\mathbf{v}}(q) \in \mathbb{Z}[q]$, we see by Theorem 3.1.2 and Lemma 2.1.3, that $M_{\Gamma,\mathbf{v}}(q)$ also has integer coefficients.

3.2. Comet-shaped quivers

Fix strictly positive integers $g, k, s_1, \ldots, s_k$ and consider the following (comet-shaped) quiver Γ with g loops on the central vertex and with set of vertices

$$I = \{0\} \cup \left\{[i, j] \mid i = 1, \ldots, k;\ j = 1, \ldots, s_i\right\}.$$



Let Ω^0 denote the set of arrows $\gamma \in \Omega$ such that $h(\gamma) \neq t(\gamma)$.

Lemma 3.2.1. *Let $\mathbb{K}$ be any field. Let $\boldsymbol{\varphi} \in \mathrm{Rep}_{\Gamma,\mathbf{v}}(\mathbb{K})$ and assume that $v_0 > 0$. If $\boldsymbol{\varphi}$ is indecomposable, then the linear maps φ_γ, with $\gamma \in \Omega^0$, are all injective.*

Proof. If γ is the arrow $[i, j] \to [i, j-1]$, with $j = 1, \ldots, s_i$ and with the convention that $[i, 0] = 0$, we use the notation $\varphi_{ij} : V_{[i,j]} \to V_{[i,j-1]}$ rather than $\varphi_\gamma : V_{t(\gamma)} \to V_{h(\gamma)}$. Assume that φ_{ij} is not injective. We define a graded vector subspace $\boldsymbol{V}' = \bigoplus_{i\in I} V_i'$ of $\boldsymbol{V} = \bigoplus_{i\in I} V_i$ as follows.

If the vertex i is not one of the vertices $[i, j], [i, j+1], \ldots, [i, s_i]$, we put $V_i' := \{0\}$. We put $V'_{[i,j]} := \operatorname{Ker}\varphi_{ij}$, $V'_{[i,j+1]} := \varphi^{-1}_{i(j+1)}(V'_{[i,j]}), \ldots, V'_{[i,s_i]} := \varphi^{-1}_{is_i}(V'_{i(s_i-1)})$. Let $\mathbf{v}'$ be the dimension of the graded space $\boldsymbol{V}' = \bigoplus_{i\in I} V'_i$ which we consider as a dimension vector of Γ. Define $\boldsymbol{\varphi}' \in \mathrm{Rep}_{\Gamma,\mathbf{v}'}(\mathbb{K})$ as the restriction of $\boldsymbol{\varphi}$ to $\boldsymbol{V}'$. It is a non-zero subrepresentation of $\boldsymbol{\varphi}$. It is now possible to define a graded vector subspace $\boldsymbol{V}'' = \bigoplus_{i\in I} V''_i$ of $\boldsymbol{V}$ such that the restriction $\boldsymbol{\varphi}''$ of $\boldsymbol{\varphi}$ to $\boldsymbol{V}''$ satisfies $\boldsymbol{\varphi} = \boldsymbol{\varphi}'' \oplus \boldsymbol{\varphi}'$: we start by taking any subspace $V''_{[i,j]}$ such that $V_{[i,j]} = V'_{[i,j]} \oplus V''_{[i,j]}$, then define $V''_{[i,j+r]}$ from $V''_{[i,j]}$ as $V'_{[i,j+r]}$ was defined from $V_{[i,j]}$, and finally put $V''_i := V_i$ if the vertex i is not one of the vertices $[i,j], [i,j+1], \ldots, [i,s_i]$. As $v_0 > 0$, the subrepresentation $\boldsymbol{\varphi}''$ is non-zero, and so $\boldsymbol{\varphi}$ is not indecomposable. □

We denote by $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ the subspace of representation $\boldsymbol{\varphi} \in \mathrm{Rep}_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ such that φ_γ is injective for all $\gamma \in \Omega^0$, and by $\mathrm{M}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ the set of isomorphism classes of $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$. Put $M^*_{\Gamma,\mathbf{v}}(q) = \#\left\{\mathrm{M}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)\right\}$. Following [1] we say that a dimension vector $\mathbf{v}$ of Γ is *strict* if for each $i = 1, \ldots, k$ we have $n_0 \geq v_{[i,1]} \geq v_{[i,2]} \geq \cdots \geq v_{[i,s_i]}$. Let us denote by $\mathcal{S}$ the set of strict dimension vector of Γ.

Proposition 3.2.2.

$$\mathrm{Log}\left(\sum_{\mathbf{v}\in\mathcal{S}} M^*_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}}\right) = \sum_{\mathbf{v}\in\mathcal{S}-\{0\}} A_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}}.$$

Proof. Let us denote by $I_{\Gamma,\mathbf{v}}(q)$ the number of isomorphism classes of indecomposable representations in $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$. By the Krull–Schmidt theorem, a representation of Γ decomposes as a direct sum of indecomposable representation in a unique way up to permutation of the summands. Notice that, for $\mathbf{v} \in \mathcal{S}$, each summand of an element of $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ lives in some $\mathrm{Rep}^*_{\Gamma,\mathbf{w}}(\mathbb{F}_q)$ for some $\mathbf{w} \in \mathcal{S}$. On the other hand, by Lemma 3.2.1, $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ contains all the indecomposable representations in $\mathrm{Rep}_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$. This implies the following identity

$$\sum_{\mathbf{v}\in\mathcal{S}} M^*_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}} = \prod_{\mathbf{v}\in\mathcal{S}-\{0\}} (1 - X^{\mathbf{v}})^{-I_{\Gamma,\mathbf{v}}(q)},$$

where $X^{\mathbf{v}}$ denotes the monomial $\prod_{i\in I} X_i^{v_i}$ for some fixed independent commuting variables $\{X_i\}_{i\in I}$. Exactly as Hua [11, Proof of Lemma 4.5] does we show from this formal identity that

$$\mathrm{Log}\left(\sum_{\mathbf{v}\in\mathcal{S}} M^*_{\Gamma,\mathbf{v}}(q) X^{\mathbf{v}}\right) = \sum_{\mathbf{v}\in\mathcal{S}-\{0\}} A_{\Gamma,\mathbf{v}}(q)\, X^{\mathbf{v}}. \quad \square$$

It follows from Proposition 3.2.2 that since $A_{\Gamma,\mathbf{v}}(T) \in \mathbb{Z}[T]$ the quantity $M^*_{\Gamma,\mathbf{v}}(q)$ is also the evaluation of a polynomial with integer coefficients at $T = q$.

Given a non-increasing sequence $u = (n_0 \geq n_1 \geq \cdots)$ of non-negative integers we let Δu be the sequence of successive differences $n_0 - n_1, n_1 - n_2 \ldots$. We extend the notation of Section 2.2.1 and denote by $\mathcal{F}_{\Delta u}$ the set of partial flags of $\mathbb{F}_q$-vector spaces

$$\{0\} \subseteq E^r \subseteq \cdots \subseteq E^1 \subseteq E^0 = (\mathbb{F}_q)^{n_0}$$

such that $\dim(E^i) = n_i$.

Assume that $\mathbf{v} \in \mathcal{S}$ and let $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$, where μ^i is the partition obtained from $\Delta\mathbf{v}_i$ by reordering, where $\mathbf{v}_i := (v_0 \geq v_{[i,1]} \geq \cdots \geq v_{[i,s_i]})$. Consider the set of orbits

$$\mathfrak{S}_{\boldsymbol{\mu}}(\mathbb{F}_q) := \left(\mathrm{Mat}_{n_0}(\mathbb{F}_q)^g \times \prod_{i=1}^k \mathcal{F}_{\mu^i}(\mathbb{F}_q)\right) \Big/ \mathrm{GL}_{v_0}(\mathbb{F}_q),$$

where $\mathrm{GL}_{v_0}(\mathbb{F}_q)$ acts by conjugation on the first g coordinates and in the obvious way on each $\mathcal{F}_{\mu^i}(\mathbb{F}_q)$.

Let $\boldsymbol{\varphi} \in \mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ with underlying graded vector space $\boldsymbol{V} = V_0 \oplus \bigoplus_{i,j} V_{[i,j]}$. We choose a basis of V_0 and we identify V_0 with $(\mathbb{F}_q)^{v_0}$. In the chosen basis, the g maps φ_γ, with $\gamma \in \Omega - \Omega^0$, give an element in $\mathrm{Mat}_{v_0}(\mathbb{F}_q)^g$. For each $i = 1, \ldots, k$, we obtain a partial flag by taking the images in $(\mathbb{F}_q)^{v_0}$ of the $V_{[i,j]}$'s via the compositions of the φ_γ's where γ runs over the arrows of the i-th leg of Γ. We thus have defined a map

$$\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q) \longrightarrow \left(\mathrm{Mat}_{v_0}(\mathbb{F}_q)^g \times \prod_{i=1}^k \mathcal{F}_{\Delta\mathbf{v}_i}(\mathbb{F}_q)\right) \Big/ \mathrm{GL}_{v_0}(\mathbb{F}_q). \tag{3.2.1}$$

The target set is clearly in bijection with $\mathfrak{G}_{\boldsymbol{\mu}}(\mathbb{F}_q)$ as $\mathcal{F}_{\Delta\mathbf{v}_i}(\mathbb{F}_q)$ is in bijection with $\mathcal{F}_{\mu^i}(\mathbb{F}_q)$. On the other hand two elements of $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q)$ have the same image if and only if they are isomorphic. Indeed, if $\mathbf{v}_i^> = (v_0 > v^>_{[i,1]} > \cdots > v^>_{[i,r_i]})$ is the longest strictly decreasing subsequence of $\mathbf{v}_i$, then $\mathbf{v}^>$ is a dimension vector of the comet-shaped quiver $\Gamma^>$ obtained from $(\Gamma, \mathbf{v})$ by gluing together the vertices on each leg on which $\mathbf{v}$ has the same coordinate. Then the natural projection $\mathrm{Rep}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q) \to \mathrm{Rep}^*_{\Gamma^>,\mathbf{v}^>}(\mathbb{F}_q)$ induces a bijection $\mathrm{M}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q) \simeq \mathrm{M}^*_{\Gamma^>,\mathbf{v}^>}(\mathbb{F}_q)$ on isomorphism classes whose target is clearly in bijection with the target of the map (3.2.1). The map (3.2.1) induces thus a bijection $\mathrm{M}^*_{\Gamma,\mathbf{v}}(\mathbb{F}_q) \simeq \mathfrak{G}_{\boldsymbol{\mu}}(\mathbb{F}_q)$.

For a multi-partition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ define a new comet-shaped quiver $\Gamma_{\boldsymbol{\mu}}$ consisting of g loops on a central vertex and k legs of length $l(\mu^i) - 1$ and let $\mathbf{v}_{\boldsymbol{\mu}}$ be the dimension vector as in Section 1.3 (for $\mathbf{v}$ and $\boldsymbol{\mu}$ as above, $\Gamma_{\boldsymbol{\mu}} = \Gamma^>$). Applying the above construction to the pair $(\Gamma_{\boldsymbol{\mu}}, \mathbf{v}_{\boldsymbol{\mu}})$ we obtain a bijection $\mathrm{M}^*_{\Gamma_{\boldsymbol{\mu}},\mathbf{v}_{\boldsymbol{\mu}}}(\mathbb{F}_q) \simeq \mathfrak{G}_{\boldsymbol{\mu}}(\mathbb{F}_q)$. Put $G_{\boldsymbol{\mu}}(q) := \#\mathfrak{G}_{\boldsymbol{\mu}}(\mathbb{F}_q)$ and let $A_{\boldsymbol{\mu}}(q)$ be the Kac polynomial of the quiver $\Gamma_{\boldsymbol{\mu}}$ for the dimension vector $\mathbf{v}_{\boldsymbol{\mu}}$.

Theorem 3.2.3. *We have*

$$\mathrm{Log}\left(\sum_{\boldsymbol{\mu}\in\mathcal{P}^k} G_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}}\right) = \sum_{\boldsymbol{\mu}\in\mathcal{P}^k-\{0\}} A_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}}.$$

Proof. In Proposition 3.2.2 make the change of variables

$$X_0 := x_{1,1}\cdots x_{k,1}, \qquad X_{[i,j]} := x_{i,j}^{-1} x_{i,j+1}, \quad i = 1, 2, \ldots, k, \ j = 1, 2, \ldots.$$

Since the terms on both sides are invariant under permutation of the entries $v_{[i,1]}, v_{[i,2]}, \ldots$ of $\mathbf{v}$ we can collect all terms that yield the same multipartition $\boldsymbol{\mu}$. The resulting sum of $X^{\mathbf{v}}$ gives the monomial symmetric function $m_{\boldsymbol{\mu}}(x)$. $\square$

Remark 3.2.4. Since $A_{\boldsymbol{\mu}}(q) \in \mathbb{Z}[q]$, it follows from Theorem 3.2.3 that $G(q) \in \mathbb{Z}[q]$.

Recall that $\mathbb{F}$ denotes an algebraic closure of $\mathbb{F}_q$ and $f : \mathbb{F} \to \mathbb{F}, x \mapsto x^q$ is the Frobenius endomorphism.

Proposition 3.2.5. *We have*

$$\log\left(\sum_{\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}}\right) = \sum_{d=1}^{\infty} \phi_d(q) \cdot \log\left(\Omega\left(\mathbf{x}_1^d, \ldots, \mathbf{x}_k^d; 0, q^{d/2}\right)\right)$$

where $\phi_n(q) = \frac{1}{n}\sum_{d|n} \mu(d)(q^{n/d} - 1)$ *is the number of* $\langle f\rangle$*-orbits of* $\mathbb{F}^\times := \mathbb{F} - \{0\}$ *of size* n.

Proof. If X is a finite set on which a finite group H acts, recall Burnside's formula which says that

$$\#X/H = \frac{1}{|H|}\sum_{h\in H}\#\{x\in X \mid h\cdot x = x\}.$$

Denote by $\boldsymbol{C}_n$ the set of conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$. Applying Burnside's formula to $\mathfrak{G}_{\boldsymbol{\mu}}(\mathbb{F}_q)$, with $\boldsymbol{\mu}\in(\mathcal{P}_n)^k$, we find that

$$\begin{aligned} G_{\boldsymbol{\mu}}(q) &= |\mathrm{GL}_n(\mathbb{F}_q)|^{-1}\sum_{g\in\mathrm{GL}_n(\mathbb{F}_q)}\Lambda(g)\prod_{i=1}^k\#\{X\in\mathcal{F}_{\mu^i}\mid g\cdot X = X\}\\ &= |\mathrm{GL}_n(\mathbb{F}_q)|^{-1}\sum_{g\in\mathrm{GL}_n(\mathbb{F}_q)}\Lambda(g)\prod_{i=1}^k R_{L_{\mu^i}}^G(1)(g)\\ &= \sum_{\mathcal{O}\in\boldsymbol{C}_n}\frac{\Lambda(\mathcal{O})}{|Z_{\mathcal{O}}|}\prod_{i=1}^k R_{L_{\mu^i}}^G(1)(\mathcal{O}). \end{aligned}$$

For a conjugacy class $\mathcal{O}$ of $\mathrm{GL}_n(\mathbb{F}_q)$, let $\omega(\mathcal{O})$ denote its type. By Formula (2.1.9), we have

$$\frac{\Lambda(\mathcal{O})}{|Z_{\mathcal{O}}|} = \mathcal{H}_{\omega(\mathcal{O})}(0,\sqrt{q}).$$

By Corollary 2.2.3, we deduce that

$$\sum_{\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(q)m_{\boldsymbol{\mu}} = \sum_{\mathcal{O}\in\boldsymbol{C}}\mathcal{H}_{\omega(\mathcal{O})}(0,\sqrt{q})\prod_{i=1}^k\tilde{H}_{\omega(\mathcal{O})}(\mathbf{x}_i,q)$$

where $\boldsymbol{C} := \bigcup_{n\geq 1}\boldsymbol{C}_n$.

We denote by $\mathbf{F}^\times$ the set of $\langle f\rangle$-orbits of $\mathbb{F}^\times$. There is a natural bijection from the set $\boldsymbol{C}$ to the set of all maps $\mathbf{F}^\times\to\mathcal{P}$ with finite support [19, IV,2]. If $C\in\boldsymbol{C}$ corresponds to $\alpha:\mathbf{F}^\times\to\mathcal{P}$, then we may enumerate the elements of $\{s\in\mathbf{F}^\times\mid\alpha(s)\neq 0\}$ as $c_1,\ldots,c_r$ such that $\omega(\alpha) := (d(c_1),\alpha(c_1))\cdots(d(c_r),\alpha(c_r))$, where $d(c)$ denotes the size of c, is the type $\omega(C)$.

We have

$$\begin{aligned} \sum_{\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(q)m_{\boldsymbol{\mu}} &= \sum_{\alpha\in\mathcal{P}^{\mathbf{F}^\times}}\mathcal{H}_{\omega(\alpha)}(0,\sqrt{q})\prod_{i=1}^k\tilde{H}_{\omega(\alpha)}(\mathbf{x}_i,q)\\ &= \prod_{c\in\mathbf{F}^\times}\Omega\left(\mathbf{x}_1^{d(c)},\ldots,\mathbf{x}_k^{d(c)};0,q^{d(c)/2}\right)\\ &= \prod_{d=1}^\infty\Omega\left(\mathbf{x}_1^{d},\ldots,\mathbf{x}_k^{d};0,q^{d/2}\right)^{\phi_d(q)}. \quad\square \end{aligned}$$

Remark 3.2.6. The second formula displayed in the proof of Proposition 3.2.5 shows that

$$G_{\boldsymbol{\mu}}(q) = \left\langle\Lambda\otimes R_{\boldsymbol{\mu}}(1),1\right\rangle$$

where $R_{\boldsymbol{\mu}}(1) := R_{L_{\mu^1}}^G(1)\otimes\cdots\otimes R_{L_{\mu^k}}^G(1)$.

Theorem 3.2.7. *We have*

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}).$$

Proof. From Formula (2.1.15) we have

$$\sum_{\boldsymbol{\mu}} \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q})\, m_{\boldsymbol{\mu}} = (q-1)\mathrm{Log}\left(\Omega(0, \sqrt{q})\right).$$

We thus need to see that

$$\sum_{\boldsymbol{\mu}} A_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}} = (q-1)\mathrm{Log}\left(\Omega(0, \sqrt{q})\right). \tag{3.2.2}$$

From Theorem 3.2.3 we are reduced to prove that

$$\mathrm{Log}\left(\sum_{\boldsymbol{\mu}} G_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}}\right) = (q-1)\mathrm{Log}\left(\Omega(0, \sqrt{q})\right).$$

But this follows from Lemma 2.1.2 and Proposition 3.2.5. □

3.3. Another formula for Kac polynomials

When the dimension vector $\mathbf{v}_{\boldsymbol{\mu}}$ is indivisible, it is known by Crawley-Boevey and van den Bergh [2] that the polynomial $A_{\boldsymbol{\mu}}(q)$ equals (up to some power of q) the polynomial which counts the number of points of some quiver variety over $\mathbb{F}_q$.

Here we prove some relation between $A_{\boldsymbol{\mu}}(q)$ and some variety which is closely related to quiver varieties. This relation holds for any $\boldsymbol{\mu}$ (in particular $\mathbf{v}_{\boldsymbol{\mu}}$ can be divisible). We continue to use the notation $G, P_\lambda, L_\lambda, U_\lambda, \mathcal{F}_\lambda$ of Section 2.2 and the notation $\mathfrak{g}, \mathfrak{p}_\lambda, \mathfrak{l}_\lambda, \mathfrak{u}_\lambda$ of Section 2.2.2.

For a partition λ of n, define

$$\mathbb{X}_\lambda := \left\{(X, gP_\lambda) \in \mathfrak{g} \times (G/P_\lambda) \mid g^{-1}Xg \in \mathfrak{u}_\lambda\right\}.$$

It is well-known that the image of the projection $p : \mathbb{X}_\lambda(\mathbb{F}) \to \mathfrak{g}(\mathbb{F}), (X, gP_\lambda) \mapsto X$ is the Zariski closure $\overline{\mathcal{O}}_{\lambda'}$ of the nilpotent adjoint orbit $\mathcal{O}_{\lambda'}$ of $\mathfrak{gl}_n(\mathbb{F})$ whose Jordan form is given by λ', and that p is a desingularization.

Put

$$\mathbb{V}_{\boldsymbol{\mu}} := \left\{\left(a_1, b_1, \ldots, a_g, b_g, (X_1, g_1 P_{\mu^1}), \ldots, (X_k, g_k P_{\mu^k})\right) \in \mathfrak{g}^{2g} \times \mathbb{X}_{\mu^1} \times \cdots \times \mathbb{X}_{\mu^k} \,\middle|\, \sum_i [a_i, b_i] + \sum_j X_j = 0\right\}$$

where $[a, b] = ab - ba$.

Define $\Lambda^{\sim} : \mathfrak{g} \to \mathbb{C}, z \mapsto q^{gn^2}\Lambda(z)$. By [9, Proposition 3.2.2] we know that

$$\Lambda^{\sim} = \mathcal{F}^{\mathfrak{g}}(F)$$

where for $z \in \mathfrak{g}$,

$$F(z) := \#\left\{(a_1, b_1, \ldots, a_g, b_g) \in \mathfrak{g}^{2g} \,\middle|\, \sum_i [a_i, b_i] = z\right\}.$$

By Remark 2.2.11, the functions $\Lambda^{\sim}$ and $\mathfrak{R}^{\mathfrak{g}}_{\mathfrak{l}_\lambda} := q^{\frac{1}{2}(n^2-\sum_i \lambda_i^2)} R^{\mathfrak{g}}_{\mathfrak{l}_\lambda}$ are characters of $\mathfrak{g}$. Put

$$\mathfrak{R}_{\boldsymbol{\mu}}(1) := \mathfrak{R}^{\mathfrak{g}}_{\mathfrak{l}_{\mu^1}}(1) \otimes \cdots \otimes \mathfrak{R}^{\mathfrak{g}}_{\mathfrak{l}_{\mu^k}}(1).$$

For two functions $f, g : \mathfrak{g} \to \mathbb{C}$, define their inner product as

$$\langle f, g \rangle = |\mathfrak{g}|^{-1} \sum_{X \in \mathfrak{g}} f(X)\overline{g(X)}.$$

Proposition 3.3.1. *We have*

$$|\mathbb{V}_{\boldsymbol{\mu}}| = \left\langle \Lambda^{\sim} \otimes \mathfrak{R}_{\boldsymbol{\mu}}(1), 1 \right\rangle.$$

Proof. Notice that

$$|\mathbb{V}_{\boldsymbol{\mu}}| = \left(F * Q^{\mathfrak{g}}_{\mathfrak{l}_{\mu^1}} * \cdots * Q^{\mathfrak{g}}_{\mathfrak{l}_{\mu^k}} \right)(0).$$

Hence the result follows from Propositions 2.2.8 and 2.2.10. □

The proposition shows that $|\mathbb{V}_{\boldsymbol{\mu}}|$ is a rational function in q which is an integer for infinitely many values of q. Hence $|\mathbb{V}_{\boldsymbol{\mu}}|$ is a polynomial in q with integer coefficients.

Consider

$$V_{\boldsymbol{\mu}}(q) := \frac{|\mathbb{V}_{\boldsymbol{\mu}}|}{|G|}.$$

Recall that $d_{\boldsymbol{\mu}} = n^2(2g-2+k) - \sum_{i,j}(\mu^i_j)^2 + 2$.

Theorem 3.3.2. *We have*

$$\mathrm{Log}\left(\sum_{\boldsymbol{\mu}} q^{-\frac{1}{2}(d_{\boldsymbol{\mu}}-2)} V_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}} \right) = \frac{q}{q-1} \sum_{\boldsymbol{\mu}} A_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}}.$$

By Lemma 2.1.2 and Formula (3.2.2) we are reduced to prove the following.

Proposition 3.3.3. *We have*

$$\log\left(\sum_{\boldsymbol{\mu}} q^{-\frac{1}{2}(d_{\boldsymbol{\mu}}-2)} V_{\boldsymbol{\mu}}(q) m_{\boldsymbol{\mu}} \right) = \sum_{d=1}^{\infty} \varphi_d(q) \cdot \log\left(\Omega\left(\mathbf{x}_1^d, \ldots, \mathbf{x}_k^d; 0, q^{d/2} \right) \right)$$

where $\varphi_n(q) = \frac{1}{n}\sum_{d|n} \mu(d) q^{n/d}$ *is the number of* $\langle f \rangle$*-orbits of* $\mathbb{F}$ *of size* n.

Proof. By Proposition 3.3.1, we have

$$V_{\boldsymbol{\mu}}(q) = \frac{q^{-n^2+\frac{1}{2}\left(kn^2 - \sum_{i,j}(\mu^i_j)^2\right)}}{|G|} \sum_{x \in \mathfrak{g}} \Lambda^{\sim}(x) R^{\mathfrak{g}}_{\mathfrak{l}_{\mu^1}}(1)(x) \cdots R^{\mathfrak{g}}_{\mathfrak{l}_{\mu^1}}(1)(x).$$

By Remark 2.2.9 and Corollary 2.2.3, we see that $R^{\mathfrak{g}}_{\mathfrak{l}_\lambda}(1)(x) = \left\langle \tilde{H}_\omega(\mathbf{x}; q), h_\lambda(\mathbf{x}) \right\rangle$ when the G-orbit of x is of type ω.

We now proceed exactly as in the proof of Proposition 3.2.5 to prove our formula. □

3.4. Applications to the character theory of finite general linear groups

The following theorem (which is a consequence of Theorems 3.2.7 and 2.2.6) expresses certain fusion rules in the character ring of $\mathrm{GL}_n(\mathbb{F}_q)$ in terms of absolutely indecomposable representations of comet shaped quivers.

Theorem 3.4.1. *We have*

$$\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle = A_{\boldsymbol{\mu}}(q).$$

From Theorems 3.4.1 and 3.1.1 we have the following result.

Corollary 3.4.2. $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle \neq 0$ *if and only if* $\mathbf{v}_{\boldsymbol{\mu}} \in \Phi(\Gamma_{\boldsymbol{\mu}})^+$. *Moreover* $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle = 1$ *if and only if* $\mathbf{v}_{\boldsymbol{\mu}}$ *is a real root.*

Remark 3.4.3. We will see in Section 5.2 that $\mathbf{v}_{\boldsymbol{\mu}}$ is always an imaginary root when $g \geq 1$, hence the second assertion concerns only the case $g = 0$ (i.e. $\Lambda = 1$).

A proof of Theorem 3.4.1 for $\mathbf{v}_{\boldsymbol{\mu}}$ indivisible is given in [9] by expressing $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle$ as the Poincaré polynomial of a comet-shaped quiver variety. This quiver variety exists only when $\mathbf{v}_{\boldsymbol{\mu}}$ is indivisible.

4. Example: Hilbert scheme of n points on $\mathbb{C}^\times \times \mathbb{C}^\times$

Throughout this section we will have $g = k = 1$ and $\boldsymbol{\mu}$ will be either the partition (n) or $(n-1, 1)$.

In this section we illustrate our conjectures and formulas in these cases.

4.1. Hilbert schemes: review

For a non-singular complex surface S we denote by $S^{[n]}$ the Hilbert scheme of n points in S. Recall that $S^{[n]}$ is non-singular and has dimension $2n$. We denote by $Y^{[n]}$ the Hilbert scheme of n points in $\mathbb{C}^2$.

Recall (see for instance [21, Section 5.2]) that $h_c^i(Y^{[n]}) = 0$ unless i is even and that the compactly supported Poincaré polynomial $P_c(Y^{[n]}; q) := \sum_i h_c^{2i}(Y^{[n]})q^i$ is given by the following explicit formula

$$\sum_{n\geq 0} P_c(Y^{[n]}; q)T^n = \prod_{m\geq 1} \frac{1}{1-q^{m+1}T^m} \tag{4.1.1}$$

which is equivalent to

$$\mathrm{Log}\left(\sum_{n\geq 0} q^{-n} \cdot P_c(Y^{[n]}; q)T^n\right) = q\sum_{n\geq 1} T^n. \tag{4.1.2}$$

For $n \geq 2$, consider the partition $\mu = (n-1, 1)$ of n and let C be a semisimple adjoint orbit of $\mathfrak{gl}_n(\mathbb{C})$ with characteristic polynomial of the form $(-1)^n(x-\alpha)^{n-1}(x-\beta)$ with $\beta = -(n-1)\alpha$ and $\alpha \neq 0$. Consider the variety

$$\mathcal{V}_{(n-1,1)} = \{(a, b, X) \in (\mathfrak{gl}_n)^2 \times C \mid [a, b] + X = 0\}.$$

The group GL_n acts on $\mathcal{V}_{(n-1,1)}$ diagonally by conjugating the coordinates. This action induces a free action of PGL_n on $\mathcal{V}_{(n-1,1)}$ and we put

$$\mathcal{Q}_{(n-1,1)} := \mathcal{V}_{(n-1,1)}/\!\!/\mathrm{PGL}_n = \mathrm{Spec}\left(\mathbb{C}[\mathcal{V}_{(n-1,1)}]^{\mathrm{PGL}_n}\right).$$

The variety $\mathcal{Q}_{(n-1,1)}$ is known to be non-singular of dimension $2n$ (see for instance [9, Section 2.2] and the references therein).

We have the following well-known theorem.

Theorem 4.1.1. *The two varieties $\mathcal{Q}_{(n-1,1)}$ and $Y^{[n]}$ have isomorphic cohomology supporting pure mixed Hodge structures.*

Proof. By [9, Appendix B] it is enough to prove that there is a smooth morphism $f : \mathfrak{M} \to \mathbb{C}$ which satisfies the two following properties.

(1) There exists an action of $\mathbb{C}^\times$ on $\mathfrak{M}$ such that the fixed point set $\mathfrak{M}^{\mathbb{C}^\times}$ is complete and for all $x \in X$ the limit $\lim_{\lambda\mapsto 0} \lambda x$ exists.
(2) $\mathcal{Q}_{(n-1,1)} = f^{-1}(\lambda)$ and $Y^{[n]} = f^{-1}(0)$.

Denote by $\mathbf{v}$ the dimension vector of $\Gamma_{(n-1,1)}$ which has coordinate n on the central vertex (i.e., the vertex supporting the loop) and 1 on the other vertex. It is well-known (see Nakajima [21]) that $Y^{[n]}$ can be identified with the quiver variety $\mathfrak{M}_{0,\theta}(\mathbf{v})$ where θ is the stability parameter with coordinate -1 on the central vertex and n on the other vertex. If we let ξ be the parameter with coordinate $-\alpha$ at the central vertex and $\alpha - \beta$ at the other vertex, then the variety $\mathcal{Q}_{(n-1,1)}$ is isomorphic to the quiver variety $\mathfrak{M}_{\xi,\theta}(\mathbf{v})$ (see for instance [9] and the references therein). Now we can define as in [9, Section 2.2] a map $f : \mathfrak{M} \to \mathbb{C}$ such that $f^{-1}(0) = \mathfrak{M}_{0,\theta}(\mathbf{v})$ and $f^{-1}(\lambda) = \mathfrak{M}_{\xi,\theta}(\mathbf{v})$ and which satisfies the required properties. $\square$

Corollary 4.1.2. *We have*

$$P_c(Y^{[n]}; q) = q^n \cdot A_{(n-1,1)}(q).$$

Proof. We have $P_c(\mathcal{Q}_{(n-1,1)}; q) = q^n \cdot \mathbb{H}_{(n-1,1)}(0, \sqrt{q})$ by [9, Theorem 1.3.1] and so by Theorem 3.2.7 we see that $P_c(\mathcal{Q}_{(n-1,1)}; q) = q^n \cdot A_{(n-1,1)}(q)$. Hence the result follows from Theorem 4.1.1. $\square$

Now put $X := \mathbb{C}^* \times \mathbb{C}^*$. Unlike $Y^{[n]}$, the mixed Hodge structure on $X^{[n]}$ is not pure. By Göttsche and Soergel [8, Theorem 2] we have the following result.

Theorem 4.1.3. *We have $h_c^{i,j;k}(X^{[n]}) = 0$ unless $i = j$ and*

$$1 + \sum_{n\geq 1} H_c(X^{[n]}; q, t)T^n = \prod_{n\geq 1} \frac{(1 + t^{2n+1}q^nT^n)^2}{(1 - q^{n-1}t^{2n}T^n)(1 - t^{2n+2}q^{n+1}T^n)} \tag{4.1.3}$$

with $H_c\left(X^{[n]}; q, t\right) := \sum_{i,k} h_c^{i,i;k}(X^{[n]})q^it^k$.

Define $\mathbb{H}^{[n]}(z, w)$ such that

$$H_c\left(X^{[n]}; q, t\right) = (t\sqrt{q})^{2n}\mathbb{H}^{[n]}\left(-t\sqrt{q}, \frac{1}{\sqrt{q}}\right).$$

Then Formula (4.1.3) reads

$$\sum_{n\geq 0}\mathbb{H}^{[n]}(z,w)T^n=\prod_{n\geq 1}\frac{(1-zwT^n)^2}{(1-z^2T^n)(1-w^2T^n)}, \tag{4.1.4}$$

with the convention that $\mathbb{H}^{[0]}(z,w)=1$. Hence we may re-write Formula (4.1.3) as

$$\mathrm{Log}\left(\sum_{n\geq 0}\mathbb{H}^{[n]}(z,w)T^n\right)=(z-w)^2\sum_{n\geq 1}T^n. \tag{4.1.5}$$

Specializing Formula (4.1.5) with $(z,w)\mapsto(0,\sqrt{q})$ we see from Formula (4.1.2) that

$$P_c(Y^{[n]};q)=q^n\cdot\mathbb{H}^{[n]}(0,\sqrt{q}). \tag{4.1.6}$$

We thus have the following result.

Proposition 4.1.4. *We have*

$$PH_c(X^{[n]};T)=P_c(Y^{[n]};T)$$

where $PH_c(X^{[n]};T):=\sum_i h_c^{i,i;2i}(X^{[n]})T^i$ *is the Poincaré polynomial of the pure part of the cohomology of* $X^{[n]}$.

4.2. A conjecture

The aim of this section is to discuss the following conjecture.

Conjecture 4.2.1. *We have*

$$\mathbb{H}_{(n-1,1)}(z,w)=\mathbb{H}^{[n]}(z,w). \tag{4.2.1}$$

Modulo the conjectural Formula (1.1.1), Formula (4.2.1) says that the two mixed Hodge polynomials $H_c(X^{[n]};q,t)$ and $H_c(\mathcal{M}_{(n-1,1)};q,t)$ agree. This would be a multiplicative analogue of Theorem 4.1.1. Unfortunately the proof of Theorem 4.1.1 does not work in the multiplicative case. This is because the natural family $g:\mathfrak{X}\to\mathbb{C}$ with $X^{[n]}=g^{-1}(0)$ and $\mathcal{M}_{(n-1,1)}=g^{-1}(\lambda)$ for $0\neq\lambda\in\mathbb{C}$ does not support a $\mathbb{C}^\times$-action with a projective fixed point set and so [9, Appendix B] does not apply.

One can still attempt to prove that the restriction map $H^*(\mathfrak{X};\mathbb{Q})\to H^*(g^{-1}(\lambda);\mathbb{Q})$ is an isomorphism for every fiber over $\lambda\in\mathbb{C}$ by using a family version of the non-Abelian Hodge theory as developed in the tamely ramified case in [22]. In other words one would construct a family $g_{\mathrm{Dol}}:\mathfrak{X}_{\mathrm{Dol}}\to\mathbb{C}$ such that $g_{\mathrm{Dol}}^{-1}(0)$ would be isomorphic with the moduli space of parabolic Higgs bundles on an elliptic curve C with one puncture and flag type $(n-1,1)$ and meromorphic Higgs field with a nilpotent residue at the puncture, and $g_{\mathrm{Dol}}^{-1}(\lambda)$ for $\lambda\neq 0$ would be isomorphic with parabolic Higgs bundles on C with one puncture and semisimple residue at the puncture of type $(n-1,1)$. In this family one should have a $\mathbb{C}^\times$ action satisfying the assumptions of [9, Appendix B] and so could conclude that $H^*(\mathfrak{X}_{\mathrm{Dol}};\mathbb{Q})\to H^*(g_{\mathrm{Dol}}^{-1}(\lambda);\mathbb{Q})$ is an isomorphism for every fiber over $\lambda\in\mathbb{C}$. Then a family version of non-Abelian Hodge theory in the tamely ramified case would yield that the two families $\mathfrak{X}_{\mathrm{Dol}}$ and $\mathfrak{X}$ are diffeomorphic, and so one could conclude the desired isomorphism $H^*(X^{[n]};\mathbb{Q})\cong H^*(\mathcal{M}_{(n-1,1)})$ preserving mixed Hodge structures. However a family version of the non-Abelian Hodge theory in the tamely ramified case (which was initiated in [22]) is not available in the literature.

Proposition 4.2.2. *Conjecture 4.2.1 is true under the specialization $z = 0$, $w = \sqrt{q}$.*

Proof. The left hand side specializes to $A_{(n-1,1)}(q)$ by Theorem 3.2.7, which by (4.1.5) and Corollary 4.1.2 agrees with the right hand side. □

The *Young diagram* of a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is defined as the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \le j \le \lambda_i$. We adopt the convention that the coordinate i of (i, j) increases as one goes down and the second coordinate j increases as one goes to the right.

For $\lambda \neq 0$, we define $\phi_\lambda(z, w) := \sum_{(i,j)\in\lambda} z^{j-1}w^{i-1}$, and for $\lambda = 0$, we put $\phi_\lambda(z, w) = 0$. Define

$$A_1(z, w; T) := \sum_\lambda \mathcal{H}_\lambda(z, w)\phi_\lambda(z^2, w^2)T^{|\lambda|},$$
$$A_0(z, w; T) := \sum_\lambda \mathcal{H}_\lambda(z, w)T^{|\lambda|}.$$

Proposition 4.2.3. *We have*

$$\sum_{n\ge 1} \mathbb{H}_{(n-1,1)}(z, w)T^n = (z^2 - 1)(1 - w^2)\frac{A_1(z, w; T)}{A_0(z, w; T)}.$$

Proof. The coefficient of the monomial symmetric function $m_{(n-1,1)}(\mathbf{x})$ in a symmetric function in $\Lambda(\mathbf{x})$ of homogeneous degree n is the coefficient of u when specializing the variables $\mathbf{x} = \{x_1, x_2, \ldots\}$ to $\{1, u, 0, 0, \ldots\}$. Hence, the generating series $\sum_{n\ge 1} \mathbb{H}_{(n-1,1)}(z, w)T^n$ is the coefficient of u in

$$(z^2 - 1)(1 - w^2)\mathrm{Log}\left(\sum_\lambda \mathcal{H}_\lambda(z, w)\tilde{H}_\lambda(1, u, 0, 0, \ldots; z^2, w^2)\, T^{|\lambda|}\right).$$

We know that

$$\tilde{H}_\lambda(\mathbf{x}; z, w) = \sum_\rho \tilde{K}_{\rho\lambda}(z, w)s_\rho(\mathbf{x}),$$

and $s_\rho(\mathbf{x}) = \sum_{\mu\trianglelefteq\rho} K_{\rho\mu}m_\mu(\mathbf{x})$ where $K_{\rho\mu}$ are the Kostka numbers. We have

$$s_{(n)}(1, u, 0, 0, \ldots) = 1 + u + O(u^2)$$
$$s_{(n-1,1)}(1, u, 0, 0, \ldots) = u + O(u^2)$$

and

$$s_\rho(1, u, 0, 0, \ldots) = O(u^2)$$

for any other partition ρ. Hence,

$$\tilde{H}_\lambda(1, u, 0, 0, \ldots; z, w) = \tilde{K}_{(n)\lambda}(z, w)(1 + u) + \tilde{K}_{(n-1,1)\lambda}(z, w)u + O(u^2).$$

From Macdonald [19, p. 362] we obtain $\tilde{K}_{(n)\lambda}(a, b) = 1$ and $\tilde{K}_{(n-1,1)\lambda}(a, b) = \phi_\lambda(a, b) - 1$. Hence, finally,

$$\tilde{H}_\lambda(1, u, 0, 0, \ldots; z, w) = 1 + \phi_\lambda(z, w)u + O(u^2). \tag{4.2.2}$$

It follows that $(z^2-1)^{-1}(1-w^2)^{-1}\sum_{n\geq 1}\mathbb{H}_{(n-1,1)}(z,w)T^n$ equals the coefficient of u in

$$\begin{aligned}&\mathrm{Log}\left(\sum_{\lambda}\mathcal{H}_{\lambda}(z,w)\left(1+\phi_{\lambda}(z^2,w^2)u+O(u^2)\right)T^{|\lambda|}\right)\\&\quad=\mathrm{Log}\left(A_0(T)+A_1(T)u+O(u^2)\right).\end{aligned}$$

The claim follows from the general fact

$$\mathrm{Log}\left(A_0(T)+A_1(T)u+O(u^2)\right)=\mathrm{Log}\,A_0(T)+\frac{A_1(T)}{A_0(T)}u+O(u^2).\quad\square$$

Combining Proposition 4.2.3 with (4.1.4) we obtain the following.

Corollary 4.2.4. *Conjecture 4.2.1 is equivalent to the following combinatorial identity*

$$1+(z^2-1)(1-w^2)\frac{A_1(z,w;T)}{A_0(z,w;T)}=\prod_{n\geq 1}\frac{(1-zwT^n)^2}{(1-z^2T^n)(1-w^2T^n)}. \tag{4.2.3}$$

The main result of this section is the following theorem.

Theorem 4.2.5. *Formula (4.2.3) is true under the Euler specialization* $(z,w)\mapsto\left(\sqrt{q},1/\sqrt{q}\right)$*; namely, we have*

$$\mathbb{H}_{(n-1,1)}(z,z^{-1})=\mathbb{H}^{[n]}(z,z^{-1}). \tag{4.2.4}$$

Equivalently, the two varieties $\mathcal{M}_{(n-1,1)}$ *and* $X^{[n]}$ *have the same* E*-polynomial.*

Proof. Consider the generating function

$$F:=(1-z)(1-w)\sum_{\lambda}\phi_{\lambda}(z,w)T^{|\lambda|}.$$

It is straightforward to see that for $\lambda\neq 0$ we have

$$\begin{aligned}(1-z)(1-w)\phi_{\lambda}(z,w)&=1+\sum_{i=1}^{l(\lambda)}(w^i-w^{i-1})z^{\lambda_i}-w^{l(\lambda)}\\&=1+\sum_{i\geq 1}(w^i-w^{i-1})z^{\lambda_i}.\end{aligned}$$

Interchanging summations we find

$$F=\sum_{i\geq 1}(w^i-w^{i-1})\sum_{\lambda\neq 0}z^{\lambda_i}T^{|\lambda|}+\sum_{\lambda\neq 0}T^{|\lambda|}.$$

To compute the sum over λ for a fixed i we break the partitions as follows:

$$\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_{i-1}\geq\underbrace{\lambda_i\geq\lambda_{i+1}\geq\cdots}_{\rho}$$

and we put

$$\begin{aligned}&\rho:=(\lambda_i,\lambda_{i+1},\ldots)\\&\mu:=(\lambda_1-\lambda_i,\lambda_2-\lambda_i,\ldots,\lambda_{i-1}-\lambda_i).\end{aligned}$$

Notice that $\mu'_1 = l(\mu) < i$, $\rho_1 = l(\rho') = \lambda_i$ and $|\lambda| = |\mu| + |\rho| + l(\rho')(i-1)$.

We then have

$$\sum_{\lambda} z^{\lambda_i} T^{|\lambda|} = \sum_{\mu_1 < i} T^{|\mu|} \sum_{\rho} z^{l(\rho)} T^{|\rho|+(i-1)l(\rho)}$$

(changing ρ to ρ' and μ to μ'). Each sum can be written as an infinite product, namely

$$\sum_{\lambda} z^{\lambda_i} T^{|\lambda|} = \prod_{k=1}^{i-1} (1-T^k)^{-1} \prod_{n\geq 1} (1-zT^{n+i-1})^{-1}.$$

So

$$F = \sum_{\lambda \neq 0} T^{|\lambda|} + \sum_{i\geq 1} (w^i - w^{i-1}) \left(\prod_{k=1}^{i-1} (1-T^k)^{-1} \prod_{n\geq 1} (1-zT^{n+i-1})^{-1} - 1 \right)$$

$$= \sum_{\lambda \neq 0} T^{|\lambda|} + \prod_{n\geq 1} (1-zT^n)^{-1} \sum_{i\geq 1} (w^i - w^{i-1}) \prod_{k=1}^{i-1} \frac{(1-zT^k)}{(1-T^k)} - \sum_{i\geq 1} (w^i - w^{i-1}).$$

The last sum telescopes to 1 and we find

$$F = \sum_{\lambda} T^{|\lambda|} + \prod_{n\geq 1} (1-zT^n)^{-1} (w-1) \sum_{i\geq 1} w^{i-1} \prod_{k=1}^{i-1} \frac{(1-zT^k)}{(1-T^k)}. \tag{4.2.5}$$

By the Cauchy q-binomial theorem the sum equals

$$\frac{1}{(1-w)} \prod_{n\geq 1} \frac{(1-wzT^n)}{(1-wT^n)}.$$

Also

$$\sum_{\lambda} T^{|\lambda|} = \prod_{n\geq 1} (1-T^n)^{-1}.$$

If we divide Formula (4.2.5) by this we finally get

$$1 - (1-z)(1-w) \prod_{n\geq 1} (1-T^n) \sum_{\lambda} \phi_\lambda(z,w) T^{|\lambda|} = \prod_{n\geq 1} \frac{(1-wzT^n)(1-T^n)}{(1-zT^n)(1-wT^n)}.$$

Putting now $(z,w) = (q, 1/q)$ we find that

$$1 - (1-q)(1-1/q) \prod_{n\geq 1} (1-T^n) \sum_{\lambda} \phi_\lambda(q, 1/q) T^{|\lambda|}$$

$$= \prod_{n\geq 1} \frac{(1-T^n)^2}{(1-qT^n)(1-q^{-1}T^n)}. \tag{4.2.6}$$

From Formula (2.1.10) we have $\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) = 1$ (since $g = 1$) and so

$$A_1\left(\sqrt{q}, \frac{1}{\sqrt{q}}; T\right) = \sum_{\lambda} \phi_\lambda\left(q, \frac{1}{q}\right) T^{|\lambda|}$$

$$A_0\left(\sqrt{q}, \frac{1}{\sqrt{q}}; T\right) = \sum_{\lambda} T^{|\lambda|} = \prod_{n\geq 1} (1-T^n)^{-1}.$$

Hence, under the specialization $(z, w) \mapsto (\sqrt{q}, 1/\sqrt{q})$, the left hand side of Formula (4.2.3) agrees with the left hand side of Formula (4.2.6).

Finally, it is straightforward to see that if we put $(z, w) = (\sqrt{q}, 1/\sqrt{q})$, then the right hand side of Formula (4.2.3) agrees with the right hand side of Formula (4.2.6); hence the theorem. □

4.3. Connection with modular forms

For a positive, even integer k let G_k be the standard Eisenstein series for $SL_2(\mathbb{Z})$

$$G_k(T) = \frac{-B_k}{2k} + \sum_{n\geq 1}\sum_{d|n} d^{k-1}T^n, \tag{4.3.1}$$

where B_k is the k-th Bernoulli number.

For $k > 2$ the G_k's are modular forms of weight k; i.e., they are holomorphic (including at infinity) and satisfy

$$\begin{aligned} &G_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k G_k(\tau) \\ &\text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \qquad T = e^{2\pi i\tau}, \quad \Im\tau > 0. \end{aligned} \tag{4.3.2}$$

For $k = 2$ we have a similar transformation up to an additive term.

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c}{4\pi i}(c\tau+d). \tag{4.3.3}$$

The ring $\mathbb{Q}[G_2,\ G_4,\ G_6]$ is called the ring of *quasi-modular* forms (see [14]).

Theorem 4.3.1. *We have*

$$1 + \sum_{n\geq 1} \mathbb{H}_{(n-1,1)}\left(e^{u/2}, e^{-u/2}\right) T^n = \frac{1}{u}\left(e^{u/2} - e^{-u/2}\right) \exp\left(2\sum_{k\geq 2} G_k(T)\frac{u^k}{k!}\right).$$

In particular, the coefficient of any power of u on the left hand side is in the ring of quasi-modular forms.

Remark 4.3.2. The relation between the E-polynomial of the Hilbert scheme of points on a surface and theta functions goes back to Göttsche [7].

Proof. Consider the classical theta function

$$\theta(w) = (1-w)\prod_{n\geq 1}\frac{(1-q^n w)(1-q^n w^{-1})}{(1-q^n)^2}, \tag{4.3.4}$$

with simple zeros at $q^n,\ n \in \mathbb{Z}$ and functional equations

$$\begin{aligned} &\text{(i) } \theta(qw) = -w^{-1}\theta(w) \\ &\text{(ii) } \theta(w^{-1}) = -w^{-1}\theta(w). \end{aligned} \tag{4.3.5}$$

We have the following expansion

$$\frac{1}{\theta(w)} = \frac{1}{1-w} + \sum_{\substack{n,m>0 \\ n \not\equiv m \bmod 2}} (-1)^n \, q^{\frac{nm}{2}} \, w^{\frac{m-n-1}{2}}. \tag{4.3.6}$$

This is classical but not that well known. For a proof see, for example, [12, Chapter VI, p. 453], where it is deduced from a more general expansion due to Kronecker. Namely,

$$\frac{\theta(uv)}{\theta(u)\theta(v)} = \sum_{m,n\geq 0} q^{mn} u^m v^n - \sum_{m,n\geq 1} q^{mn} u^{-m} v^{-n}.$$

(To see this set $v = u^{-\frac{1}{2}}$ and use the functional equation (4.3.5) to get

$$\frac{1}{\theta(w)} = \frac{1}{1-w} + \sum_{m,n\geq 1} q^{mn} \left(w^{m-\frac{1}{2}(n+1)} - w^{m+\frac{1}{2}(n-1)} \right),$$

which is equivalent to (4.3.6).) It is not hard, as was shown to us by J. Tate, to give a direct proof using (4.3.5).

From (4.3.6) we deduce, switching q to T and w to q, that

$$\prod_{n\geq 1} \frac{(1-T^n)^2}{(1-qT^n)(1-q^{-1}T^n)} = 1 + \sum_{\substack{r,s>0 \\ r \not\equiv s \bmod 2}} (-1)^r T^{\frac{rs}{2}} \left(q^{\frac{s-r-1}{2}} - q^{\frac{2-r+1}{2}} \right) \tag{4.3.7}$$

which combined with Theorem 4.2.5 gives

$$\mathbb{H}_{(n-1,1)}\left(\sqrt{q}, \frac{1}{\sqrt{q}}\right) = \sum_{\substack{rs=2n \\ r \not\equiv s \bmod 2}} (-1)^r \left(q^{\frac{s-r-1}{2}} - q^{\frac{2-r+1}{2}} \right). \tag{4.3.8}$$

We compute the logarithm of the left hand side of (4.3.7) and get

$$\sum_{m,n\geq 1} (q^m + q^{-m} - 2) \frac{T^{mn}}{m}.$$

Applying $(q\frac{d}{dq})^k$ and then setting $q = 1$ we obtain

$$\sum_{m,n\geq 1} (m^k + (-m)^k) \frac{T^{mn}}{m},$$

which vanishes identically if k is odd. For k even, it equals

$$2 \sum_{n\geq 1} \sum_{d|n} d^{k-1} \, T^n.$$

Comparing with (4.3.1) we see that this series equals $2G_k$, up to the constant term.

Note that if $q = e^u$ then

$$q\frac{d}{dq} = \frac{d}{du}, \quad q = 1 \leftrightarrow u = 0.$$

Hence,

$$\log\left(1 + \sum_{n\geq 1} \mathbb{H}_{(n-1,1)}(e^{u/2}, e^{-u/2}) T^n \right) = \sum_{\substack{k\geq 2 \\ \text{even}}} \left(2G_k + \frac{B_k}{k} \right) \frac{u^k}{k!}.$$

On the other hand, it is easy to check that

$$u \exp\left(\sum_{k\geq 2} \frac{B_k}{k} \frac{u^k}{k!}\right) = e^{u/2} - e^{-u/2}$$

($B_k = 0$ if $k > 1$ is odd.) This proves the claim. □

5. Connectedness of character varieties

5.1. The main result

Let $\boldsymbol{\mu}$ be a multi-partition $(\mu^1, \ldots, \mu^k)$ of n and let $\mathcal{M}_{\boldsymbol{\mu}}$ be a genus g generic character variety of type $\boldsymbol{\mu}$ as in Section 1.1.

Theorem 5.1.1. *The character variety $\mathcal{M}_{\boldsymbol{\mu}}$ is connected (if not empty).*

Let us now explain the strategy of the proof. We first need the following lemma.

Lemma 5.1.2. *If $\mathcal{M}_{\boldsymbol{\mu}}$ is not empty, its number of connected components equals the constant term in $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$.*

Proof. The number of connected components of $\mathcal{M}_{\boldsymbol{\mu}}$ is $\dim H^0(\mathcal{M}_{\boldsymbol{\mu}}, \mathbb{C})$ which is also equal to the mixed Hodge number $h^{0,0;0}(\mathcal{M}_{\boldsymbol{\mu}})$.

Poincaré duality implies that

$$h^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}}) = h_c^{d_{\boldsymbol{\mu}}-i, d_{\boldsymbol{\mu}}-j; 2d_{\boldsymbol{\mu}}-k}(\mathcal{M}_{\boldsymbol{\mu}}).$$

From Formula (1.1.3) we thus have

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = \sum_i \left(\sum_k (-1)^k h^{i,i;k}(\mathcal{M}_{\boldsymbol{\mu}})\right) q^i.$$

On the other hand the mixed Hodge numbers $h^{i,j;k}(X)$ of any complex non-singular variety X are zero if $(i, j, k) \notin \{(i, j, k) | i \leq k, j \leq k, k \leq i + j\}$; see [3]. Hence $h^{0,0;k}(\mathcal{M}_{\boldsymbol{\mu}}) = 0$ if $k > 0$.

We thus deduce that the constant term of $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$ is $h^{0,0;0}(\mathcal{M}_{\boldsymbol{\mu}})$. □

From the above lemma and Formula (1.1.2) we are reduced to prove that the coefficient of the lowest power $q^{-\frac{d_{\boldsymbol{\mu}}}{2}}$ of q in $\mathbb{H}_{\boldsymbol{\mu}}(\sqrt{q}, 1/\sqrt{q})$ is equal to 1.

The strategy to prove this is in two steps. First, in Section 5.3.1 we analyze the lowest power of q in $\mathcal{A}_{\lambda\boldsymbol{\mu}}(q)$, where

$$\Omega\left(\sqrt{q}, 1/\sqrt{q}\right) = \sum_{\lambda, \boldsymbol{\mu}} \mathcal{A}_{\lambda\boldsymbol{\mu}}(q)\, m_{\boldsymbol{\mu}}.$$

Then in Section 5.3.2 we see how these combine in $\mathrm{Log}\left(\Omega\left(\sqrt{q}, 1/\sqrt{q}\right)\right)$. In both cases, Lemmas 5.2.8 and 5.3.6, we will use in an essential way the inequality of Section 6. Though very similar, the relation between the partitions ν^p in these lemmas and the matrix of numbers $x_{i,j}$ in Section 6 is dual to each other (the ν^p appear as rows in one and columns in the other).

5.2. Preliminaries

For a multi-partition $\boldsymbol{\mu} \in (\mathcal{P}_n)^k$ we define

$$\Delta(\boldsymbol{\mu}) := \frac{1}{2}d_{\boldsymbol{\mu}} - 1 = \frac{1}{2}(2g-2+k)n^2 - \frac{1}{2}\sum_{i,j}\left(\mu^i_j\right)^2. \tag{5.2.1}$$

Remark 5.2.1. Note that when $g = 0$ the quantity $-2\Delta(\boldsymbol{\mu})$ is Katz's *index of rigidity* of a solution to $X_1 \cdots X_k = I$ with $X_i \in \mathcal{C}_i$ (see for example [15, p. 91]).

From $\boldsymbol{\mu}$ we define as in Theorem 3.2.3 a comet-shaped quiver $\Gamma = \Gamma_{\boldsymbol{\mu}}$ as well as a dimension vector $\mathbf{v} = \mathbf{v}_{\boldsymbol{\mu}}$ of Γ. We denote by I the set of vertices of Γ and by Ω the set of arrows. Recall that $\boldsymbol{\mu}$ and $\mathbf{v}$ are linearly related ($v_0 = n$ and $v_{[i,j]} = n - \sum_{r=1}^{j}\mu^i_r$ for $j > 1$ and conversely, $\mu^i_1 = n - v_{[i,1]}$ and $\mu^i_j = v_{[i,j-1]} - v_{[i,j]}$ for $j > 1$). Hence Δ yields an integral-valued quadratic from on $\mathbb{Z}^I$. Let $(\cdot,\cdot)$ be the associated bilinear form on $\mathbb{Z}^I$ so that

$$(\mathbf{v},\mathbf{v}) = 2\Delta(\boldsymbol{\mu}). \tag{5.2.2}$$

Let $\mathbf{e}_0$ and $\mathbf{e}_{[i,j]}$ be the fundamental roots of Γ (vectors in $\mathbb{Z}^I$ with all zero coordinates except for a 1 at the indicated vertex). We find that

$$(\mathbf{e}_0,\mathbf{e}_0) = 2g-2, \qquad (\mathbf{e}_{[i,j]},\mathbf{e}_{[i,j]}) = -2,$$
$$(\mathbf{e}_0,\mathbf{e}_{[i,1]}) = 1 \qquad (\mathbf{e}_{[i,j]},\mathbf{e}_{[i,j+1]}) = 1,$$

for $i = 1,2,\ldots,k$, $j = 1,2,\ldots,s_i - 1$ and all other pairings are zero. In other words, Δ is the negative of the Tits quadratic form of Γ (with the natural orientation of all edges pointing away from the central vertex).

With this notation we define

$$\delta = \delta(\boldsymbol{\mu}) := (\mathbf{e}_0,\mathbf{v}) = (2g-2+k)n - \sum_{i=1}^{k}\mu^i_1. \tag{5.2.3}$$

Remark 5.2.2. In the case of $g = 0$ the quantity δ is called the *defect* by Simpson (see [23, p. 12]).

Note that $\delta \geq (2g-2)n$ is non-negative unless $g = 0$. On the other hand,

$$(\mathbf{e}_{[i,j]},\mathbf{v}) = \mu^i_j - \mu^i_{j+1} \geq 0. \tag{5.2.4}$$

We now follow the terminology of [13].

Lemma 5.2.3. *The dimension vector* $\mathbf{v}$ *is in the fundamental set of imaginary roots of* Γ *if and only if* $\delta(\boldsymbol{\mu}) \geq 0$.

Proof. Note that $v_{[i,j]} > 0$ if $j < l(\mu^i)$ and $v_{[i,j]} = 0$ for $j \geq l(\mu^i)$; since $n > 0$ the support of $\mathbf{v}$ is then connected. We already have $(\mathbf{e}_{[i,j]},\mathbf{v}) \geq 0$ by (5.2.4); hence $\mathbf{v}$ is in the fundamental set of imaginary roots of Γ if and only if $\delta \geq 0$ (see [13]). □

For a partition $\mu \in \mathcal{P}_n$ we define

$$\sigma(\mu) := n\mu_1 - \sum_j \mu_j^2$$

and extend to a multipartition $\boldsymbol{\mu} \in (\mathcal{P}_n)^k$ by

$$\sigma(\boldsymbol{\mu}) := \sum_{i=1}^{k} \sigma(\mu^i).$$

Remark 5.2.4. Again for $g = 0$ this is called the *superdefect* by Simpson.

We say that $\mu \in \mathcal{P}_n$ is *rectangular* if and only if all of its (non-zero) parts are equal, i.e., $\mu = (t^{n/t})$ for some $t \mid n$. We extend this to multi-partitions: $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in (\mathcal{P}_n)^k$ is rectangular if each μ^i is (the μ^i's are not required to be of the same length). Note that $\boldsymbol{\mu}$ is rectangular if and only if the associated dimension vector $\mathbf{v}$ satisfies $(\mathbf{e}_{[i,j]}, \mathbf{v}) = 0$ for all $[i, j]$ by (5.2.4).

Lemma 5.2.5. *For $\boldsymbol{\mu} \in (\mathcal{P}_n)^k$ we have*

$$\sigma(\boldsymbol{\mu}) \geq 0$$

with equality if and only if $\boldsymbol{\mu}$ is rectangular.

Proof. For any $\mu \in \mathcal{P}_n$ we have $n\mu_1 = \mu_1 \sum_j \mu_j \geq \sum_j \mu_j^2$ and equality holds if and only if $\mu_1 = \mu_j$. □

Since

$$2\Delta(\boldsymbol{\mu}) = n\,\delta(\boldsymbol{\mu}) + \sigma(\boldsymbol{\mu}) \tag{5.2.5}$$

we find that

$$d_{\boldsymbol{\mu}} \geq n\,\delta(\boldsymbol{\mu}) + 2 \tag{5.2.6}$$

and in particular $d_{\boldsymbol{\mu}} \geq 2$ if $\delta(\boldsymbol{\mu}) \geq 0$.

If Γ is affine it is known that the positive imaginary roots are of the form $t\mathbf{v}^*$ for an integer $t \geq 1$ and some $\mathbf{v}^*$. We will call $\mathbf{v}^*$ the *basic positive imaginary root* of Γ. The affine star-shaped quivers are given in the table below; their basic positive imaginary root is the dimension vector associated to the indicated multi-partition $\boldsymbol{\mu}^*$. These $\boldsymbol{\mu}^*$, and hence also any scaled version $t\boldsymbol{\mu}^*$ for $t \geq 1$, are rectangular. Moreover, $\Delta(\boldsymbol{\mu}^*) = 0$ and in fact, $\boldsymbol{\mu}^*$ generates the one-dimensional radical of the quadratic form Δ so that $\Delta(\boldsymbol{\mu}^*, \boldsymbol{\nu}) = 0$ for all $\boldsymbol{\nu}$.

Proposition 5.2.6. *Suppose that $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in (\mathcal{P}_n)^k$ has $\delta(\boldsymbol{\mu}) \geq 0$. Then $d_{\boldsymbol{\mu}} = 2$ if and only if Γ is of affine type, i.e., Γ is either the Jordan quiver J (one loop on one vertex), $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, and $\boldsymbol{\mu} = t\boldsymbol{\mu}^*$ (all parts scaled by t) for some $t \geq 1$, where $\boldsymbol{\mu}^*$, given in the table below, corresponds to the basic imaginary root of Γ.*

Proof. By (5.2.5) and Lemma 5.2.5 $d_{\boldsymbol{\mu}} = 2$ when $\delta(\boldsymbol{\mu}) \geq 0$ if and only if $\delta(\boldsymbol{\mu}) = 0$ and $\boldsymbol{\mu}$ is rectangular. As we observed above $\delta(\boldsymbol{\mu}) \geq (2g-2)n$. Hence if $\delta(\boldsymbol{\mu}) = 0$ then $g = 1$ or $g = 0$. If $g = 1$ then necessarily $\mu^i = (n)$ and Γ is the Jordan quiver J.

If $g = 0$ then $\delta = 0$ is equivalent to the equation

$$\sum_{i=1}^{k} \frac{1}{l_i} = k - 2, \tag{5.2.7}$$

where $l_i := n/t_i$ is the length of $\mu^i = (t_i^{n/t_i})$. In solving this equation, any term with $l_i = 1$ can be ignored. It is elementary to find all of its solutions; they correspond to the cases $\Gamma = \tilde{D}_4, \tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$.

We summarize the results in the following table

Γ	l_i	n	$\boldsymbol{\mu}^*$
J	(1)	1	(1)
$\tilde{D}_4$	(2, 2, 2, 2)	2	(1, 1), (1, 1), (1, 1), (1, 1)
$\tilde{E}_6$	(3, 3, 3)	3	(1, 1, 1), (1, 1, 1), (1, 1, 1)
$\tilde{E}_7$	(2, 4, 4)	4	(2, 2), (1, 1, 1, 1), (1, 1, 1, 1)
$\tilde{E}_8$	(2, 3, 6)	6	(3, 3), (2, 2, 2), (1, 1, 1, 1, 1, 1)

(5.2.8)

where we listed the cases with smallest possible positive values of n and k and the corresponding multi-partition $\boldsymbol{\mu}^*$. □

Proposition 5.2.6 is due to Kostov; see for example [23, p. 14].

We will need the following result about Δ.

Proposition 5.2.7. *Let $\boldsymbol{\mu} \in (\mathcal{P}_n)^k$ and $\boldsymbol{\nu}^p = (\nu^{1,p}, \dots, \nu^{k,p}) \in \left(\mathcal{P}_{n_p}\right)^k$ for $p = 1, \dots, s$ be non-zero multi-partitions such that up to permutations of the parts of $\nu^{i,p}$ we have*

$$\mu^i = \sum_{p=1}^{s} \nu^{i,p}, \quad i = 1, \dots, k.$$

Assume that $\delta(\boldsymbol{\mu}) \geq 0$. Then

$$\sum_{p=1}^{s} \Delta(\boldsymbol{\nu}^p) \leq \Delta(\boldsymbol{\mu}).$$

Equality holds if and only if

(i) *$s = 1$ and $\boldsymbol{\mu} = \boldsymbol{\nu}^1$.*
or
(ii) *Γ is affine and $\boldsymbol{\mu}, \boldsymbol{\nu}^i, \dots, \boldsymbol{\nu}^s$ correspond to positive imaginary roots.*

We start with the following. For partitions μ, ν define

$$\sigma_\mu(\nu) := \mu_1 |\nu|^2 - |\mu| \sum_i \nu_i^2.$$

Note that $\sigma_\mu(\mu) = |\mu|\, \sigma(\mu)$.

Lemma 5.2.8. *Let $\nu^1, \dots, \nu^s$ and μ be non-zero partitions such that up to permutation of the parts of each ν^p we have $\sum_{p=1}^{s} \nu^p = \mu$. Then*

$$\sum_{p=1}^{s} \sigma_\mu(\nu^p) \leq \sigma_\mu(\mu).$$

Equality holds if and only if:

(i) $s = 1$ *and* $\mu = \nu^1$*;*
or
(ii) $\nu^1, \dots, \nu^s$ *and* μ *all are rectangular of the same length.*

Proof. This is just a restatement of the inequality of Section 6 with $x_{i,k} = \nu^k_{\sigma_k(i)}$, for the appropriate permutations σ_k, where $1 \le i \le l(\mu),\ 1 \le k \le s$. □

Lemma 5.2.9. *If the partitions* μ, ν *are rectangular of the same length then*

$$\sigma_\mu(\nu) = 0.$$

Proof. Direct calculation. □

Proof of Proposition 5.2.7. From the definition (5.2.1) we get

$$2n\Delta(\boldsymbol{\mu}) = \delta(\boldsymbol{\mu})n^2 + \sum_{i=1}^{k} \sigma_{\mu^i}(\mu^i)$$

and similarly

$$2n\Delta(\boldsymbol{\nu}^p) = \delta(\boldsymbol{\mu})n_p^2 + \sum_{i=1}^{k} \sigma_{\mu^i}(\nu^{i,p}), \qquad p = 1, \dots, s;$$

hence

$$2n \sum_{p=1}^{s} \Delta(\boldsymbol{\nu}^p) = \delta(\boldsymbol{\mu}) \sum_{p=1}^{s} n_p^2 + \sum_{i=1}^{k} \sum_{p=1}^{s} \sigma_{\mu^i}(\nu^{i,p}).$$

Since $n = \sum_{p=1}^{s} n_p$ and $\delta(\boldsymbol{\mu}) \ge 0$ we get from Lemma 5.2.8 that

$$\sum_{p=1}^{s} \Delta(\boldsymbol{\nu}^p) \le \Delta(\boldsymbol{\mu})$$

as claimed.

Clearly, equality cannot occur if $\delta(\boldsymbol{\mu}) > 0$ and $s > 1$. If $\delta(\boldsymbol{\mu}) = 0$ and $s > 1$ it follows from Lemmas 5.2.8, 5.2.9 and (5.2.5) that $\Delta(\boldsymbol{\mu}) = \Delta(\boldsymbol{\nu}^p) = 0$ for $p = 1, 2, \dots, s$. Now (ii) is a consequence of Proposition 5.2.6. □

5.3. Proof of Theorem 5.1.1

5.3.1. Step I

Let

$$\mathcal{A}_{\lambda\boldsymbol{\mu}}(q) := q^{(1-g)|\lambda|} \left(q^{-n(\lambda)} H_\lambda(q)\right)^{2g+k-2} \prod_{i=1}^{k} \left\langle h_{\mu^i}(\mathbf{x}_i), s_\lambda(\mathbf{x}_i\mathbf{y})\right\rangle, \tag{5.3.1}$$

so that by Lemma 2.1.5

$$\Omega\left(\sqrt{q}, 1/\sqrt{q}\right) = \sum_{\lambda,\boldsymbol{\mu}} \mathcal{A}_{\lambda\boldsymbol{\mu}}(q)\, m_{\boldsymbol{\mu}}.$$

It is easy to verify that $\mathcal{A}_{\lambda\boldsymbol{\mu}}$ is in $\mathbb{Q}(q)$.

For a non-zero rational function $\mathcal{A} \in \mathbb{Q}(q)$ we let $v_q(\mathcal{A}) \in \mathbb{Z}$ be its valuation at q. We will see shortly that $\mathcal{A}_{\lambda\boldsymbol{\mu}}$ is nonzero for all $\lambda, \boldsymbol{\mu}$; let $v(\lambda) := v_q\left(\mathcal{A}_{\lambda\boldsymbol{\mu}}(q)\right)$. The first main step toward the proof of the connectedness is the following theorem.

Theorem 5.3.1. *Let $\boldsymbol{\mu} = (\mu^1, \mu^2, \ldots, \mu^k) \in \mathcal{P}_n{}^k$ with $\delta(\boldsymbol{\mu}) \geq 0$. Then we have the following.*

(i) *The minimum value of $v(\lambda)$ as λ runs over the set of partitions of size n, is*

$$v((1^n)) = -\Delta(\boldsymbol{\mu}).$$

(ii) *There are two cases as to where this minimum occurs.*

Case I: *The quiver Γ is affine and the dimension vector associated to $\boldsymbol{\mu}$ is a positive imaginary root $t\mathbf{v}^*$ for some $t \mid n$. In this case, the minimum is reached at all partitions λ which are the union of n/t copies of any $\lambda_0 \in \mathcal{P}_t$.*

Case II: *Otherwise, the minimum occurs only at $\lambda = (1^n)$.*

Before proving the theorem we need some preliminary results.

Lemma 5.3.2. *$\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy})\rangle$ is non-zero for all λ and μ.*

Proof. We have $s_\lambda(\mathbf{xy}) = \sum_\nu K_{\lambda\nu} m_\nu(\mathbf{xy})$ [19, I 6 p. 101] and $m_\nu(\mathbf{xy}) = \sum_\mu C_{\nu\mu}(\mathbf{y})\, m_\mu(\mathbf{x})$ for some $C_{\nu\mu}(\mathbf{y})$. Hence

$$\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy})\rangle = \sum_\nu K_{\lambda\nu} C_{\nu\mu}(\mathbf{y}). \tag{5.3.2}$$

For any set of variables $\mathbf{xy} = \{x_i y_j\}_{1\leq i, 1\leq j}$ we have

$$C_{\nu\mu}(\mathbf{y}) = \sum m_{\rho^1}(\mathbf{y}) \cdots m_{\rho^r}(\mathbf{y}), \tag{5.3.3}$$

where the sum is over all partitions $\rho^1, \ldots, \rho^r$ such that $|\rho^p| = \mu_p$ and $\rho^1 \cup \cdots \cup \rho^r = \nu$. In particular the coefficients of $C_{\nu\mu}(\mathbf{y})$ as power series in q are non-negative. We can take, for example, $\rho^p = (1^{\mu_p})$ and then $\nu = (1^n)$. Since $K_{\lambda\nu} \geq 0$ [19, I (6.4)] for any λ, ν and $K_{\lambda,(1^n)} = n!/h_\lambda$ [19, I 6 Example 2], with $h_\lambda = \prod_{s\in\lambda} h(s)$ the product of the hook lengths, we see that $\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy})\rangle$ is non-zero and our claim follows. □

In particular $\mathcal{A}_{\lambda\boldsymbol{\mu}}$ is non-zero for all λ and $\boldsymbol{\mu}$. Define

$$v(\lambda, \mu) := v_q\left(\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy})\rangle\right). \tag{5.3.4}$$

Lemma 5.3.3. *We have*

$$-v(\lambda) = (2g - 2 + k)n(\lambda) + (g-1)n - \sum_{i=1}^{k} v(\lambda, \mu^i).$$

Proof. Straightforward. □

Lemma 5.3.4. *For $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathcal{P}_n$ we have*

$$v(\lambda, \mu) = \min\{n(\rho^1) + \cdots + n(\rho^r) \mid |\rho^p| = \mu_p,\ \cup_p \rho^p \trianglelefteq \lambda\}. \tag{5.3.5}$$

Proof. For $C_{\nu\mu}(\mathbf{y})$ non-zero let $v_m(\nu,\mu) := v_q\left(C_{\nu\mu}(\mathbf{y})\right)$. When $y_i = q^{i-1}$ we have $v_q(m_\rho(\mathbf{y})) = n(\rho)$ for any partition ρ. Hence by (5.3.3)

$$v_m(\nu,\mu) = \min\{n(\rho^1) + \cdots + n(\rho^r) \mid |\rho^p| = \mu_p,\ \cup_p \rho^p = \nu\}.$$

Since $K_{\lambda\nu} \geq 0$ for any λ, ν, $K_{\lambda\nu} > 0$ if and only if $\nu \trianglelefteq \lambda$ [5, Example 2, p. 26], and the coefficients of $C_{\nu\mu}(\mathbf{y})$ are non-negative, our claim follows from (5.3.2). □

For example, if $\lambda = (1^n)$ then necessarily $\rho^p = (1^{\mu_p})$ and hence $\rho^1 \cup \cdots \cup \rho^r = \lambda$. We have then

$$v\left((1^n),\mu\right) = \sum_{p=1}^{r} \binom{\mu_p}{2} = -\frac{1}{2}n + \frac{1}{2}\sum_{p=1}^{r} \mu_p^2. \tag{5.3.6}$$

Similarly,

$$v(\lambda,(n)) = n(\lambda) \tag{5.3.7}$$

by the next lemma.

Lemma 5.3.5. *If $\beta \trianglelefteq \alpha$ then $n(\alpha) \leq n(\beta)$ with equality if and only if $\alpha = \beta$.*

Proof. We will use the raising operators R_{ij}; see [19, I p. 8]. Consider vectors w with coefficients in $\mathbb{Z}$ and extend the function n to them in the natural way

$$n(w) := \sum_{i\geq 1}(i-1)w_i.$$

Applying a raising operator R_{ij}, where $i < j$, has the effect

$$n(R_{ij}w) = n(w) + i - j.$$

Hence for any product R of raising operators we have $n(Rw) < n(w)$ with equality if and only if R is the identity operator. Now the claim follows from the fact that $\beta \trianglelefteq \alpha$ implies there exist such an R with $\alpha = R\beta$. □

Recall [19, (1.6)] that for any partition λ we have $\langle\lambda,\lambda\rangle = 2n(\lambda) + |\lambda| = \sum_i (\lambda_i')^2$, where $\lambda' = (\lambda_1', \lambda_2', \ldots)$ is the dual partition. Note also that $(\lambda \cup \mu)' = \lambda' + \mu'$. Define

$$\|\lambda\| := \sqrt{\langle\lambda',\lambda'\rangle} = \sqrt{\sum_i \lambda_i^2}.$$

The following inequality is a particular case of the theorem of Section 6.

Lemma 5.3.6. *Fix $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathcal{P}_n$. Then for every $(\nu^1, \ldots, \nu^r) \in \mathcal{P}_{\mu_1} \times \cdots \times \mathcal{P}_{\mu_r}$ we have*

$$\mu_1 \left\| \sum_p \nu^p \right\|^2 - n \sum_p \left\|\nu^p\right\|^2 \leq \mu_1 n^2 - n\,\|\mu\|^2. \tag{5.3.8}$$

Moreover, equality holds in (5.3.8) if and only if either:

(i) *the partition μ is rectangular and all partitions ν^p are equal;*
or

(ii) *for each* $p = 1, 2, \ldots, r$ *we have* $\nu^p = (\mu_p)$.

Proof. Our claim is a consequence of the theorem of Section 6. Taking $x_{ps} = \nu_s^p$ we have $c_p := \sum_s x_{ps} = \sum_s \nu_s^p = \mu_p$ and $c := \max_p c_p = \mu_1$. □

The following fact will be crucial for the proof of connectedness.

Proposition 5.3.7. *For a fixed* $\mu = (\mu_1, \mu_2, \ldots, \mu_r) \in \mathcal{P}_n$ *we have*

$$\mu_1 n(\lambda) - n v(\lambda, \mu) \leq \mu_1 n^2 - n \|\mu\|^2, \quad \lambda \in \mathcal{P}_n.$$

Equality holds only at $\lambda = (1^n)$ *unless* μ *is rectangular* $\mu = (t^{n/t})$*, in which case it also holds when* λ *is the union of* n/t *copies of any* $\lambda_0 \in \mathcal{P}_t$.

Proof. Given $\nu \trianglelefteq \lambda$ write $\mu_1 n(\lambda) - n v(\lambda, \mu)$ as

$$\mu_1 n(\lambda) - n v(\lambda, \mu) = \mu_1 (n(\lambda) - n(\nu)) + \mu_1 n(\nu) - n v(\lambda, \mu). \tag{5.3.9}$$

By Lemma 5.3.5 the first term is non-negative. Hence

$$\mu_1 n(\lambda) - n v(\lambda, \mu) \leq \mu_1 n(\nu) - n v(\lambda, \mu), \quad \nu \trianglelefteq \lambda.$$

Combining this with (5.3.5) yields

$$\max_{|\lambda|=n} [\mu_1 n(\lambda) - n v(\lambda, \mu)] \leq \max_{|\rho^p|=\mu_p} \Big[\mu_1 n(\rho^1 \cup \rho^2 \cup \cdots \cup \rho^r) - (n(\rho^1) + \cdots + n(\rho^r))n\Big]. \tag{5.3.10}$$

Take ν^p to be the dual of ρ^p for $p = 1, 2, \ldots, r$. Then the right hand side of (5.3.10) is precisely

$$\mu_1 \left\| \sum_p \nu^p \right\|^2 - n \sum_p \|\nu^p\|^2,$$

which by Lemma 5.3.6 is bounded above by $\mu_1 n^2 - n \|\mu\|^2$ with equality only where either $\rho^p = (1^{\mu_p})$ (case (ii)) or all ρ^p are equal and $\mu = (t^{n/t})$ for some t (case (i)).

Combining this with Lemma 5.3.5 we see that to obtain the maximum of the left hand side of (5.3.10) we must also have $\rho^1 \cup \cdots \cup \rho^r = \lambda$. In case (i), λ is the union of n/t copies of λ_0, the common value of ρ^p, and in case (ii), $\lambda = (1^n)$. □

Proof of Theorem 5.3.1. We first prove (ii). Using Lemma 5.3.3 we have

$$\begin{aligned} -v(\lambda) &= (2g - 2 + k)n(\lambda) + (g-1)n - \sum_{i=1}^k v(\lambda, \mu^i) \\ &= \frac{\delta}{n} n(\lambda) + (g-1)n + \frac{1}{n} \sum_{i=1}^k \Big[\mu_1^i n(\lambda) - n v(\lambda, \mu^i)\Big]. \end{aligned} \tag{5.3.11}$$

The terms $n(\lambda)$ and $\sum_{i=1}^n \left[\mu_1^i n(\lambda) - n v(\lambda, \mu^i)\right]$ are all maximal at $\lambda = (1^n)$ (the last by Proposition 5.3.7). Hence $-v(\lambda)$ is also maximal at (1^n), since $\delta \geq 0$. Now $n(\lambda)$ has a unique maximum at (1^n) by Lemma 5.3.5; hence $-v(\lambda)$ reaches its maximum at other partitions if and only if $\delta = 0$ and for each i we have $\mu^i = (t_i^{n/t_i})$ for some positive integer $t_i \mid n$ (again by

Proposition 5.3.7). In this case the maximum occurs only for λ the union of n/t copies of a partition $\lambda_0 \in \mathcal{P}_t$, where $t = \gcd t_i$. Now (ii) follows from Proposition 5.2.6.

To prove (i) we use Lemma 5.3.3 and (5.3.6) and find that $v((1^n)) = -\Delta(\boldsymbol{\mu})$ as claimed. □

Lemma 5.3.8. *Let* $\boldsymbol{\mu} = (\mu^1, \mu^2, \dots, \mu^k) \in \mathcal{P}_n{}^k$ *with* $\delta(\boldsymbol{\mu}) \geq 0$. *Suppose that* $v(\lambda)$ *is minimal. Then the coefficient of* $q^{v(\lambda)}$ *in* $\mathcal{A}_{\lambda\boldsymbol{\mu}}$ *is* 1.

Proof. We use the notation of the proof of Lemma 5.3.4. Note that the coefficient of the lowest power of q in $\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) \left(q^{-n(\lambda)} H_\lambda(q)\right)^k$ is 1 (see (2.1.10)). Also, the coefficient of the lowest power of q in each $m_\lambda(\mathbf{y})$ is always 1; hence so is the coefficient of the lowest power of q in $C_{\nu\mu}(\mathbf{y})$.

In the course of the proof of Proposition 5.3.7 we found that when $v(\lambda)$ is minimal, and $\rho^1, \dots, \rho^r$ achieve the minimum on the right hand side of (5.3.5), then $\lambda = \rho^1 \cup \dots \cup \rho^r$. Hence by Lemma 5.3.4, the coefficient of the lowest power of q in $\langle h_\mu(\mathbf{x}), s_\lambda(\mathbf{xy})\rangle = \sum_{\nu \trianglelefteq \lambda} K_{\lambda\nu} C_{\nu\mu}(\mathbf{y})$ equals the coefficient of the lowest power of q in $K_{\lambda\lambda} C_{\lambda\mu}(\mathbf{y}) = C_{\lambda\mu}(\mathbf{y})$ which we just saw is 1. This completes the proof. □

5.3.2. Leading terms of Log Ω

We now proceed to the second step in the proof of connectedness where we analyze the smallest power of q in the coefficients of Log $\left(\Omega\left(\sqrt{q}, 1/\sqrt{q}\right)\right)$. Write

$$\Omega\left(\sqrt{q}, 1/\sqrt{q}\right) = \sum_{\boldsymbol{\mu}} P_{\boldsymbol{\mu}}(q)\, m_{\boldsymbol{\mu}} \tag{5.3.12}$$

with $P_{\boldsymbol{\mu}}(q) := \sum_\lambda \mathcal{A}_{\lambda\boldsymbol{\mu}}$ and $\mathcal{A}_{\lambda\boldsymbol{\mu}}$ as in (5.3.1).

Then by Lemma 2.1.4 we have

$$\operatorname{Log}\left(\Omega\left(\sqrt{q}, 1/\sqrt{q}\right)\right) = \sum_{\boldsymbol{\omega}} C^0_{\boldsymbol{\omega}} P_{\boldsymbol{\omega}}(q)\, m_{\boldsymbol{\omega}}(q)$$

where $\boldsymbol{\omega}$ runs over *multi-types* $(d_1, \boldsymbol{\omega}^1) \cdots (d_s, \boldsymbol{\omega}^s)$ with $\boldsymbol{\omega}^p \in (\mathcal{P}_{n_p})^k$ and $P_{\boldsymbol{\omega}}(q) := \prod_p P_{\boldsymbol{\omega}^p}$ (q^{d_p}), $m_{\boldsymbol{\omega}}(\mathbf{x}) := \prod_p m_{\boldsymbol{\omega}^p}(\mathbf{x}^{d_p})$.

Now if we let $\gamma_{\boldsymbol{\mu\omega}} := \langle m_{\boldsymbol{\omega}}, h_{\boldsymbol{\mu}}\rangle$ then we have

$$\mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, 1/\sqrt{q}\right) = \frac{(q-1)^2}{q}\left(\sum_{\boldsymbol{\omega}\in\mathbf{T}^k} C^0_{\boldsymbol{\omega}} P_{\boldsymbol{\omega}}(q) \gamma_{\boldsymbol{\mu\omega}}\right).$$

By Theorem 5.3.1, $v_q\left(P_{\boldsymbol{\omega}}(q)\right) = -d \sum_{p=1}^s \Delta(\boldsymbol{\omega}^p)$ for a multi-type $\boldsymbol{\omega} = (d, \boldsymbol{\omega}^1) \cdots (d, \boldsymbol{\omega}^s)$.

Lemma 5.3.9. *Let* $\nu^1, \dots, \nu^s$ *be partitions. Then*

$$\langle m_{\nu^1} \cdots m_{\nu^s}, h_\mu\rangle \neq 0$$

if and only if $\mu = \nu^1 + \dots + \nu^s$ *up to permutation of the parts of each* ν^p *for* $p = 1, \dots, s$.

Proof. It follows immediately from the definition of the monomial symmetric function. □

Let $\mathbf{v}$ be the dimension vector associated to $\boldsymbol{\mu}$.

Theorem 5.3.10. *If* $\mathbf{v}$ *is in the fundamental set of imaginary roots of* Γ *then the character variety* $\mathcal{M}_{\boldsymbol{\mu}}$ *is non-empty and connected.*

Proof. Assume $\mathbf{v}$ is in the fundamental set of roots of Γ. By Lemma 5.2.3 this is equivalent to $\delta(\boldsymbol{\mu}) \geq 0$.

Note that $m_\nu(\mathbf{x}^d) = m_{d\nu}(\mathbf{x})$ for any partition ν and positive integer d. Suppose $\boldsymbol{\omega} = (d, \boldsymbol{\omega}^1) \cdots (d, \boldsymbol{\omega}^s)$ is a multi-type for which $\gamma_{\boldsymbol{\mu}\boldsymbol{\omega}}$ is non-zero. Let $\boldsymbol{\nu}^p = d\boldsymbol{\omega}^p$ for $p = 1, \ldots, s$ (scale every part by d). These multi-partitions are then exactly in the hypothesis of Proposition 5.2.7 by Lemma 5.3.9. Hence

$$d \sum_{p=1}^{s} \Delta(\boldsymbol{\omega}^p) \leq d^2 \sum_{p=1}^{s} \Delta(\boldsymbol{\omega}^p) = \sum_{p=1}^{s} \Delta(\boldsymbol{\nu}^p) \leq \Delta(\boldsymbol{\mu}). \tag{5.3.13}$$

Suppose Γ is not affine. Then by Proposition 5.2.7 we have equality of the endpoints in (5.3.13) if and only if $s = 1$, $\boldsymbol{\nu}^1 = \boldsymbol{\mu}$ and $d = 1$, in other words, if and only if $\boldsymbol{\omega} = (1, \boldsymbol{\mu})$. Hence, since $C^0_{(1,\boldsymbol{\mu})} = 1$, the coefficient of the lowest power of q in $\mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, 1/\sqrt{q}\right)$ equals the coefficient of the lowest power of q in $P_{\boldsymbol{\mu}}(q)$ which is 1 by Lemma 5.3.8 and Theorem 5.3.1, Case II. This proves our claim in this case.

Suppose now Γ is affine. Then by Proposition 5.2.7 we have equality of the endpoints in (5.3.13) if and only if $\boldsymbol{\mu} = t\boldsymbol{\mu}^*$ and $\boldsymbol{\omega} = (1, t_1\boldsymbol{\mu}^*), \ldots, (1, t_s\boldsymbol{\mu}^*)$ for a partition $(t_1, t_2, \ldots, t_s)$ of t and $d = 1$. Combining this with Lemma 5.3.8 and Theorem 5.3.1, Case I we see that the lowest order terms in q in $\operatorname{Log}\left(\Omega\left(\sqrt{q}, 1/\sqrt{q}\right)\right)$ are

$$L := \sum C^0_{\boldsymbol{\omega}} p(t_1) \cdots p(t_s)\, m_{t\boldsymbol{\mu}^*},$$

where the sum is over types $\boldsymbol{\omega}$ as above. Comparison with Euler's formula

$$\operatorname{Log}\left(\sum_{n\geq 0} p(n)\, T^n\right) = \sum_{n\geq 1} T^n,$$

shows that L reduces to $\sum_{t\geq 1} m_{t\boldsymbol{\mu}^*}$. Hence the coefficient of the lowest power of q in $\mathbb{H}_{\boldsymbol{\mu}}\left(\sqrt{q}, 1/\sqrt{q}\right)$ is also 1 in this case finishing the proof. □

Proof of Theorem 5.1.1. If $g \geq 1$, the dimension vector $\mathbf{v}$ is always in the fundamental set of imaginary roots of Γ. If $g = 0$ the character variety if not empty if and only if $\mathbf{v}$ is a strict root of Γ and if $\mathbf{v}$ is real then $\mathcal{M}_{\boldsymbol{\mu}}$ is a point [1, Theorem 8.3]. If $\mathbf{v}$ is imaginary then it can be taken by the Weyl group to some $\mathbf{v}'$ in the fundamental set and the two corresponding varieties $\mathcal{M}_{\boldsymbol{\mu}}$ and $\mathcal{M}_{\boldsymbol{\mu}'}$ are isomorphic for appropriate choices of conjugacy classes [1, Theorem 3.2, Lemma 4.3(ii)]; hence Theorem 5.1.1. □

6. Appendix by Gergely Harcos

Theorem 6.0.11. *Let n, r be positive integers, and let x_{ik} $(1 \leq i \leq n, 1 \leq k \leq r)$ be arbitrary nonnegative numbers. Let $c_i := \sum_k x_{ik}$ and $c := \max_i c_i$. Then we have*

$$c\sum_k\left(\sum_i x_{ik}\right)^2 - \left(\sum_i c_i\right)\left(\sum_{i,k} x_{ik}^2\right) \leq c\left(\sum_i c_i\right)^2 - \left(\sum_i c_i\right)\left(\sum_i c_i^2\right).$$

Assuming $\min_i c_i > 0$, equality holds if and only if we are in one of the following situations

(i) $x_{ik} = x_{jk}$ *for all* i, j, k,
(ii) *there exists some* l *such that* $x_{ik} = 0$ *for all* i *and all* $k \neq l$.

Remark 6.0.12. The assumption $\min_i c_i > 0$ does not result in any loss of generality, because the values i with $c_i = 0$ can be omitted without altering any of the sums.

Proof. Without loss of generality we can assume $c = c_1 \geq \cdots \geq c_n$, then the inequality can be rewritten as

$$\left(\sum_i c_i\right)\left(\sum_j \sum_{k,l} x_{jk}x_{jl} - \sum_{j,k} x_{jk}^2\right) \leq c\left(\sum_{i,j}\sum_{k,l} x_{ik}x_{jl} - \sum_{i,j}\sum_k x_{ik}x_{jk}\right).$$

Here and later i, j will take values from $\{1, \ldots, n\}$ and k, l, m will take values from $\{1, \ldots, r\}$. We simplify the above as

$$\left(\sum_i c_i\right)\left(\sum_j \sum_{\substack{k,l\\k\neq l}} x_{jk}x_{jl}\right) \leq c\left(\sum_{i,j}\sum_{\substack{k,l\\k\neq l}} x_{ik}x_{jl}\right),$$

then we factor out and also utilize the symmetry in k, l to arrive at the equivalent form

$$\sum_{i,j} c_i \sum_{\substack{k,l\\k<l}} x_{jk}x_{jl} \leq \sum_{i,j} c \sum_{\substack{k,l\\k<l}} x_{ik}x_{jl}.$$

We distribute the terms in i, j on both sides as follows:

$$\sum_i c_i \sum_{\substack{k,l\\k<l}} x_{ik}x_{il} + \sum_{\substack{i,j\\i<j}}\left(c_i \sum_{\substack{k,l\\k<l}} x_{jk}x_{jl} + c_j \sum_{\substack{k,l\\k<l}} x_{ik}x_{il}\right)$$
$$\leq \sum_i c \sum_{\substack{k,l\\k<l}} x_{ik}x_{il} + \sum_{\substack{i,j\\i<j}} c \sum_{\substack{k,l\\k<l}} (x_{ik}x_{jl} + x_{jk}x_{il}).$$

It is clear that

$$c_i \sum_{\substack{k,l\\k<l}} x_{ik}x_{il} \leq c \sum_{\substack{k,l\\k<l}} x_{ik}x_{il}, \quad 1 \leq i \leq n;$$

therefore it suffices to show that

$$c_i \sum_{\substack{k,l\\k<l}} x_{jk}x_{jl} + c_j \sum_{\substack{k,l\\k<l}} x_{ik}x_{il} \leq c \sum_{\substack{k,l\\k<l}} (x_{ik}x_{jl} + x_{jk}x_{il}), \quad 1 \leq i < j \leq n.$$

We will prove this in the stronger form

$$c_i \sum_{\substack{k,l\\k<l}} x_{jk}x_{jl} + c_j \sum_{\substack{k,l\\k<l}} x_{ik}x_{il} \leq c_i \sum_{\substack{k,l\\k<l}} (x_{ik}x_{jl} + x_{jk}x_{il}), \quad 1 \leq i < j \leq n.$$

We now fix $1 \leq i < j \leq n$ and introduce $x_k := x_{ik}$, $x'_k := x_{jk}$. Then the previous inequality reads

$$\left(\sum_m x_m\right)\left(\sum_{\substack{k,l\\k<l}} x'_k x'_l\right) + \left(\sum_m x'_m\right)\left(\sum_{\substack{k,l\\k<l}} x_k x_l\right) \leq \left(\sum_m x_m\right) \sum_{\substack{k,l\\k<l}} (x_k x'_l + x'_k x_l),$$

that is,

$$\sum_{\substack{k,l,m\\k<l}} (x_m x'_k x'_l + x_k x_l x'_m) \leq \sum_{\substack{k,l,m\\k<l}} (x_k x_m x'_l + x_l x_m x'_k).$$

The right hand side equals

$$\begin{aligned}\sum_{\substack{k,l,m\\k<l}}(x_kx_mx'_l+x_lx_mx'_k) &= \sum_{\substack{k,l,m\\l\neq k}} x_kx_mx'_l = \sum_{\substack{k,l,m\\m\neq k}} x_kx_lx'_m = \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,l,m\\l\neq k\\m\neq k}} x_kx_lx'_m\\
&= \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,m\\m\neq k}} x_kx_mx'_m + \sum_{\substack{k,l,m\\l\neq k\\m\neq k,l}} x_kx_lx'_m\\
&= \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,m\\k<m}} x_kx_mx'_m + \sum_{\substack{k,m\\m<k}} x_kx_mx'_m + 2\sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_lx'_m\\
&= \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,l\\k<l}} x_kx_lx'_l + \sum_{\substack{k,l\\k<l}} x_kx_lx'_k + 2\sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_lx'_m\\
&= \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,l,m\\k<l}} x_kx_lx'_m + \sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_lx'_m;\end{aligned}$$

therefore it suffices to prove

$$\sum_{\substack{k,l,m\\k<l}} x_mx'_kx'_l \le \sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_lx'_m.$$

This is trivial if $x'_m = 0$ for all m. Otherwise $\sum_m x'_m > 0$; hence $c_i \ge c_j$ yields

$$\lambda := \left(\sum_m x_m\right)\left(\sum_m x'_m\right)^{-1} \ge 1.$$

Clearly, we are done if we can prove

$$\lambda^2\sum_{\substack{k,l,m\\k<l}} x_mx'_kx'_l \le \lambda\sum_{\substack{k,m\\m\neq k}} x_k^2x'_m + \lambda\sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_lx'_m.$$

We introduce $\tilde{x}_m := \lambda x'_m$; then

$$\sum_m \tilde{x}_m = \sum_m x_m,$$

and the last inequality reads

$$\sum_{\substack{k,l,m\\k<l}} x_m\tilde{x}_k\tilde{x}_l \le \sum_{\substack{k,m\\m\neq k}} x_k^2\tilde{x}_m + \sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_l\tilde{x}_m.$$

By adding equal sums to both sides this becomes

$$\sum_{\substack{k,l,m\\k<l}} x_m\tilde{x}_k\tilde{x}_l + \sum_{\substack{k,l,m\\k<l}} x_kx_l\tilde{x}_m \le \sum_{\substack{k,m\\m\neq k}} x_k^2\tilde{x}_m + \sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_kx_l\tilde{x}_m + \sum_{\substack{k,l,m\\k<l}} x_kx_l\tilde{x}_m,$$

which can also be written as

$$\left(\sum_m x_m\right)\left(\sum_{\substack{k,l\\k<l}} \tilde{x}_k\tilde{x}_l\right) + \left(\sum_m \tilde{x}_m\right)\left(\sum_{\substack{k,l\\k<l}} x_kx_l\right)$$

$$\leq \sum_{k} x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m \right) + \sum_{\substack{k,l \\ k<l}} x_k x_l \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m + \sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right).$$

The right hand side equals

$$\begin{aligned}
&\sum_{k} x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m \right) + \sum_{\substack{k,l \\ k<l}} x_k x_l \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m + \sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right) \\
&\quad = \sum_{k} x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m \right) + \sum_{\substack{k,l \\ l<k}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right) + \sum_{\substack{k,l \\ k<l}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right) \\
&\quad = \sum_{k} x_k^2 \left(\sum_{\substack{m \\ m \neq k}} \tilde{x}_m \right) + \sum_{\substack{k,l \\ k \neq l}} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right) \\
&\quad = \sum_{k,l} x_k x_l \left(\sum_{\substack{m \\ m \neq l}} \tilde{x}_m \right) = \left(\sum_{k} x_k \right) \left(\sum_{\substack{m,l \\ m \neq l}} x_l \tilde{x}_m \right);
\end{aligned}$$

hence the previous inequality is the same as

$$\left(\sum_{m} x_m \right) \left(\sum_{\substack{k,l \\ k<l}} \tilde{x}_k \tilde{x}_l \right) + \left(\sum_{m} \tilde{x}_m \right) \left(\sum_{\substack{k,l \\ k<l}} x_k x_l \right) \leq \left(\sum_{k} x_k \right) \left(\sum_{\substack{m,l \\ m \neq l}} x_l \tilde{x}_m \right).$$

The first factors are equal and positive; hence after renaming m, l to k, l when $m < l$ and to l, k when $m > l$ on the right hand side we are left with proving

$$\sum_{\substack{k,l \\ k<l}} (\tilde{x}_k \tilde{x}_l + x_k x_l) \leq \sum_{\substack{k,l \\ k<l}} (\tilde{x}_k x_l + x_k \tilde{x}_l).$$

This can be written in the elegant form

$$\sum_{\substack{k,l \\ k<l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) \leq 0.$$

However,

$$\begin{aligned}
0 = \left(\sum_{k} (\tilde{x}_k - x_k) \right)^2 &= \sum_{k,l} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) = \sum_{k} (\tilde{x}_k - x_k)^2 \\
&\quad + 2 \sum_{\substack{k,l \\ k<l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l),
\end{aligned}$$

so that

$$\sum_{\substack{k,l \\ k<l}} (\tilde{x}_k - x_k)(\tilde{x}_l - x_l) = -\frac{1}{2} \sum_{k} (\tilde{x}_k - x_k)^2 \leq 0$$

as required.

We now verify, under the assumption $\min_i c_i > 0$, that equation in the theorem holds if and only if $x_{ik} = x_{jk}$ for all i, j, k or there exists some l such that $x_{ik} = 0$ for all i and all $k \neq l$. The "if" part is easy, so we focus on the "only if" part. Inspecting the above argument carefully, we can see that equation can hold only if for any $1 \le i < j \le n$ the numbers $x_k := x_{ik}, x'_k := x_{jk}$ satisfy

$$\lambda \sum_{\substack{k,l,m\\k<l}} x_m x'_k x'_l = \sum_{\substack{k,l,m\\k<l}} x_m x'_k x'_l = \sum_{\substack{k,m\\m\neq k}} x_k^2 x'_m + \sum_{\substack{k,l,m\\k<l\\m\neq k,l}} x_k x_l x'_m,$$

where λ is as before. If $x'_k x'_l = 0$ for all $k < l$, then $x_k^2 x'_m = 0$ for all $k \neq m$, i.e. $x_k x'_l = 0$ for all $k \neq l$. Otherwise $\lambda = 1$ and $x_k = \tilde{x}_k = x'_k$ for all k by the above argument. In other words, equation in the theorem can hold only if for any $i \neq j$ we have $x_{ik} x_{jl} = 0$ for all $k \neq l$ or we have $x_{ik} = x_{jk}$ for all k. If there exist j, l such that $x_{jk} = 0$ for all $k \neq l$, then $x_{jl} > 0$ and for any $i \neq j$ both alternatives imply $x_{ik} = 0$ for all $k \neq l$, hence we are done. Otherwise the first alternative cannot hold for any $i \neq j$, so we are again done. $\square$

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