ARITHMETIC HARMONIC ANALYSIS ON CHARACTER AND QUIVER VARIETIES

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Abstract

We propose a general conjecture for the mixed Hodge polynomial of the generic character varieties of representations of the fundamental group of a Riemann surface of genus g to $\mathrm{GL}_n(\mathbb{C})$ with fixed generic semisimple conjugacy classes at k punctures. This conjecture generalizes the Cauchy identity for Macdonald polynomials and is a common generalization of two formulas that we prove in this paper. The first is a formula for the E-polynomial of these character varieties which we obtain using the character table of $\mathrm{GL}_n(\mathbb{F}_q)$. We use this formula to compute the Euler characteristic of character varieties. The second formula gives the Poincaré polynomial of certain associated quiver varieties which we obtain using the character table of $\mathfrak{gl}_n(\mathbb{F}_q)$. In the last main result we prove that the Poincaré polynomials of the quiver varieties equal certain multiplicities in the tensor product of irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$. As a consequence we find a curious connection between Kac-Moody algebras associated with comet-shaped, and typically wild, quivers and the representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$.

Contents

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1. Introduction

1.1. Cauchy identity for Macdonald polynomials

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ and $\mathbf{y} = \{y_1, y_2, \ldots\}$ be two infinite sets of variables, and let $\Lambda(\mathbf{x})$ and $\Lambda(\mathbf{y})$ be the corresponding rings of symmetric functions. For a partition λ, let $\tilde{H}_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbf{x}) \otimes_{\mathbb{Z}} \mathbb{Q}(q,t)$ be the *Macdonald symmetric function* defined in [14, I.11]. These functions satisfy the Cauchy identity (in a form equivalent to [14, Theorem 3.3])

$$\operatorname{Exp}\Big(\frac{m_{(1)}(\mathbf{x})m_{(1)}(\mathbf{y})}{(q-1)(1-t)}\Big) = \sum_{\lambda \in \mathcal{P}} \frac{\tilde{H}_\lambda(\mathbf{x};q,t)\tilde{H}_\lambda(\mathbf{y};q,t)}{\prod (q^{a+1}-t^l)(q^a - t^{l+1})} \tag{1.1.1}$$

where Exp is the plethystic exponential (see, e.g., [22, Section 2.5]; we recall the formalism of Exp and its inverse Log in Section 2.3.3), $\mathcal{P}$ is the set of all partitions, $m_\lambda \in \Lambda$ are the monomial symmetric functions, and the product in the denominator on the right-hand side is over the cells of λ with a and l their arm and leg lengths, respectively.

In this paper we will think of (1.1.1) as the special case $g = 0, k = 2$ of a formula pertaining to a genus g Riemann surface with k punctures. Fix integers $g \geq 0$ and $k > 0$. Let $\mathbf{x}_1 = \{x_{1,1}, x_{1,2}, \ldots\}, \ldots, \mathbf{x}_k = \{x_{k,1}, x_{k,2}, \ldots\}$ be k sets of infinitely many independent variables, and let $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ be the ring of functions separately symmetric in each of the set of variables. When there is no risk of confusion of what variables are involved we will simply write Λ for this ring.

Define the *k-point genus g Cauchy function* (throughout the paper k will denote a positive integer)

$$\Omega(z,w) := \sum_{\lambda \in \mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2), \tag{1.1.2}$$

with coefficients in $\mathbb{Q}(z,w) \otimes_{\mathbb{Z}} \Lambda$, where

$$\mathcal{H}_\lambda(z,w) := \prod \frac{(z^{2a+1} - w^{2l+1})^{2g}}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})}$$

is a (z,w)-deformation of the $(2g-2)$th power of the standard hook polynomial. Thus in particular $\Omega(\sqrt{q}, \sqrt{t})$ equals the right-hand side of (1.1.1) for $g = 0, k = 2$.

For $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in \mathcal{P}^k$, let

$$\mathbb{H}_{\boldsymbol{\mu}}(z, w) := (z^2 - 1)(1 - w^2)\langle \mathrm{Log}\, \Omega(z, w), h_{\boldsymbol{\mu}} \rangle. \tag{1.1.3}$$

Here $h_{\boldsymbol{\mu}} := h_{\mu^1}(\mathbf{x}_1) \cdots h_{\mu^k}(\mathbf{x}_k) \in \Lambda$ are the complete symmetric functions and $\langle \cdot, \cdot \rangle$ is the extended Hall pairing defined in (2.3.1). Recall that $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual bases with respect to the Hall pairing, and we may hence recover $\Omega(z, w)$ from the $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$'s by the formula

$$\Omega(z, w) = \mathrm{Exp}\Big(\sum_{\boldsymbol{\mu} \in \mathcal{P}^k} \frac{\mathbb{H}_{\boldsymbol{\mu}}(z, w)}{(z^2 - 1)(1 - w^2)} m_{\boldsymbol{\mu}} \Big).$$

Note that $\mathbb{H}_{\boldsymbol{\mu}} = 0$ unless $|\mu^1| = \cdots = |\mu^k|$.

With this notation, (1.1.1) is equivalent to

$$\mathbb{H}_{\boldsymbol{\mu}}(z, w) = \begin{cases} 1 & \text{if } \boldsymbol{\mu} = ((1), (1)), \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.4}$$

when $g = 0$ and $k = 2$.

1.2. Character varieties

Fix $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in \mathcal{P}_n{}^k$ for the rest of this introduction where $\mu^i = (\mu^i_1, \mu^i_2, \dots, \mu^i_{r_i})$ and $r_i := \ell(\mu^i)$ is the length of μ^i. ($\mathcal{P}_n$ denotes the set of partitions of n.) Let $\mathcal{M}_{\boldsymbol{\mu}}$ be a $\mathrm{GL}_n(\mathbb{C})$ character variety of a k-punctured genus g Riemann surface, with generic semisimple conjugacy classes of type $\boldsymbol{\mu}$ at the punctures. In other words, fix semisimple conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_k \subset \mathrm{GL}_n(\mathbb{C})$, which are generic in the sense of Definition 2.1.1 and have type $\mu^1, \dots, \mu^k$; that is, $\{\mu^i_1, \mu^i_2, \dots\}$ are the multiplicities of the eigenvalues of any matrix in $\mathcal{C}_i$. (We prove in Lemma 2.1.2 that there always exist generic semisimple conjugacy classes for every $\boldsymbol{\mu}$.) The variety depends on the actual choice of eigenvalues, but for simplicity we drop this choice from the notation.

Concretely, we have

$$\mathcal{M}_{\boldsymbol{\mu}} := \Big\{ A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}_n(\mathbb{C}), X_1 \in \mathcal{C}_1, \dots, X_k \in \mathcal{C}_k \,\Big|\, (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n \Big\} /\!/ \mathrm{GL}_n(\mathbb{C}),$$

an affine geometric invariant theory (GIT) quotient by the conjugation action of $\mathrm{GL}_n(\mathbb{C})$ where, for two matrices $A, B \in \mathrm{GL}_n(\mathbb{C})$, we put $(A, B) = ABA^{-1}B^{-1}$ and I_n is the identity matrix. We prove in Theorem 2.1.5 that $\mathcal{M}_{\boldsymbol{\mu}}$, if nonempty, is a nonsingular variety of dimension

$$d_{\boldsymbol{\mu}} := n^2(2g - 2 + k) - \sum_{i,j} (\mu^i_j)^2 + 2. \tag{1.2.1}$$

For example, if $k = 1$ and $\boldsymbol{\mu} = ((n))$, then $\mathcal{M}_{\boldsymbol{\mu}}$ is just the variety $\mathcal{M}_n$ of [22] and $\mathbb{H}_{\boldsymbol{\mu}}$ is the polynomial $\bar{H}_n$ (see Section 1.5.2 for more details).

1.2.1. Mixed Hodge polynomial: The conjectures

As a natural continuation of [22] here we study the compactly supported *mixed Hodge polynomials*

$$H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t) := \sum h_c^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}}) x^i y^j t^k,$$

where $h_c^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}})$ are the compactly supported mixed Hodge numbers of [6] and [7]. For any variety $X/\mathbb{C}$ the polynomial $H_c(X; x, y, t)$ is a common deformation of its compactly supported *Poincaré polynomial* $P_c(X; t) = H_c(X; 1, 1, t)$ and its so-called *E-polynomial* $E(X; x, y) = H_c(X; x, y, -1)$.

We define the *pure part* of H_c as the polynomial

$$PH_c(X; x, y) := \sum_{i,j} h_c^{i,j;i+j}(X) x^i y^j.$$

If $h_c^{i,j;k}(X) = 0$ unless $i = j$, we will simplify the notation and write $H_c(X; q, t) := H_c(X; \sqrt{q}, \sqrt{q}, t)$, $PH_c(X; q) := PH_c(X; \sqrt{q}, \sqrt{q})$, and $E(X; q) := E(X; \sqrt{q}, \sqrt{q})$.

CONJECTURE 1.2.1

We have the following.

(i) *The rational function $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ defined in (1.1.3) is a polynomial. It has degree $d_{\boldsymbol{\mu}}$ in each variable, and $\mathbb{H}_{\boldsymbol{\mu}}(-z, w)$ has nonnegative integer coefficients.*

(ii) *The mixed Hodge polynomial $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t)$ is a polynomial in xy and t and is independent of the choice of generic eigenvalues of multiplicities $\boldsymbol{\mu}$.*

(iii) *Moreover,**

$$H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t) = (t\sqrt{q})^{d_\mu} \mathbb{H}_{\boldsymbol{\mu}}\Big(-\frac{1}{\sqrt{q}}, t\sqrt{q}\Big).$$

(iv) *In particular, the pure part of $H_c(\mathcal{M}_{\boldsymbol{\mu}}; q, t)$ is*

$$PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{d_\mu/2} \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}).$$

In this paper we will present several consistency checks and prove several implications of this conjecture. For example, we show in Section 5.1 that, although $\mathcal{M}_{\boldsymbol{\mu}}$ itself

*Warning: our use of the variables q, t in the Hodge polynomial context is different from the standard one in the theory of Macdonald polynomials. It should always be clear from the context which is in use.

depends on the choice of eigenvalues, $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t)$ is constant on a dense subset (in the analytic topology) of generic eigenvalues of multiplicities $\boldsymbol{\mu}$. This is consistent with (ii) of Conjecture 1.2.1.

Due to the known symmetry $\tilde{H}_\lambda(\mathbf{x}_i; q, t) = \tilde{H}_{\lambda'}(\mathbf{x}_i; t, q)$ of Macdonald polynomials (2.3.12), the right-hand side of (1.1.3) is invariant both under changing (z, w) to (w, z) and under changing (z, w) to $(-z, -w)$. Hence the same holds for $\mathbb{H}_\mu(z, w)$, and Conjecture 1.2.1 implies the following.

CONJECTURE 1.2.2 (Curious Poincaré duality)
We have

$$H_c\Big(\mathcal{M}_\mu; \frac{1}{qt^2}, t\Big) = (qt)^{-d_\mu} H_c(\mathcal{M}_\mu; q, t).$$

1.2.2. E-polynomial

THEOREM 1.2.3
The polynomial $E(\mathcal{M}_{\boldsymbol{\mu}}; x, y)$ depends only on xy and

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{(1/2)d_\mu} \mathbb{H}_{\boldsymbol{\mu}}\Big(\sqrt{q}, \frac{1}{\sqrt{q}}\Big).$$

In other words, Conjecture 1.2.1 is true under the specialization $(q, t) \mapsto (q, -1)$. We prove this in Section 5.2.

The calculation of $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$ follows the same route as in [22]. We prove that $\mathcal{M}_{\boldsymbol{\mu}}$ is polynomial count, and hence by Katz's theorem (see [22, Theorem 6.1.2.3]), $E(\mathcal{M}_{\boldsymbol{\mu}}; q) = \#\mathcal{M}_{\boldsymbol{\mu}}(\mathbb{F}_q)$. To count the points of $\mathcal{M}_{\boldsymbol{\mu}}$ over a finite field we use the mass formula

$$\#\mathcal{M}_{\boldsymbol{\mu}}(\mathbb{F}_q) = \sum_{\mathcal{X} \in \mathrm{Irr}(\mathrm{GL}_n(\mathbb{F}_q))} \frac{|\mathrm{GL}_n(\mathbb{F}_q)|^{2g-2}(q-1)}{\mathcal{X}(1)^{2g-2}} \prod_i \frac{\mathcal{X}(C_i)}{\mathcal{X}(1)} |C_i| \qquad (1.2.2)$$

originally due to Frobenius [12] for $g = 0$. The evaluation of the formula is facilitated by the combinatorial understanding of the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ first obtained in [16].

COROLLARY 1.2.4
The E-polynomial is palindromic; that is, it satisfies

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{d_\mu} E(\mathcal{M}_{\boldsymbol{\mu}}; q^{-1}).$$

In a forthcoming paper [21] we use our formula (1.2.3) for its E-polynomial to prove that $\mathcal{M}_{\boldsymbol{\mu}}$ is connected (as announced in [20]).

1.2.3. Euler characteristic

The $2g$-dimensional torus $(\mathbb{C}^\times)^{2g}$ acts on $\mathcal{M}_{\boldsymbol{\mu}}$ by scalar multiplication on the first $2g$-coordinates. We let $\tilde{\mathcal{M}}_{\boldsymbol{\mu}} := \mathcal{M}_{\boldsymbol{\mu}}//(\mathbb{C}^\times)^{2g}$. As a second application of Theorem 1.2.3, we compute the Euler characteristic $E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) := E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}; 1)$ of $\tilde{\mathcal{M}}_{\boldsymbol{\mu}}$ when $g > 0$, using $E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = E(\mathcal{M}_{\boldsymbol{\mu}})/(q-1)^{2g}$ (see Section 5.3). We obtain the following.

THEOREM 1.2.5
Assume that $g > 1$; then

$$E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = \begin{cases} \mu(n) n^{2g-3} & \text{if } \boldsymbol{\mu} = ((n), \ldots, (n)), \\ 0 & \text{otherwise,} \end{cases}$$

where μ is the ordinary Möbius function.

THEOREM 1.2.6
For $g = 1$,

$$E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = \frac{1}{n} \sum_{d | \gcd(\mu_i^j)} \sigma(n/d) \mu(d) \frac{((n/d)!)^k}{\prod_{i,j} (\mu_i^j/d)!},$$

where $\sigma(m) = \sum_{d|m} d$.

For the proofs of these theorems see Section 5.3.

1.3. Quiver varieties

For $i = 1, \ldots, k$, let $\mathcal{O}_i \subset \mathfrak{gl}_n(\mathbb{C})$ be a semisimple adjoint orbit in the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of type μ^i; as before, this means that $\{\mu_1^i, \mu_2^i, \ldots\}$ are the multiplicities of the eigenvalues of any matrix in $\mathcal{O}_i$. We will call the collection $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ *generic* if certain linear equations among the eigenvalues of the conjugacy classes are not satisfied (see Definition 2.2.1). There exists a generic collection of conjugacy classes of type $\boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is indivisible (i.e., $\gcd(\{\mu_j^i\}) = 1$). For a generic $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ we define

$$\mathcal{Q}_{\boldsymbol{\mu}} := \Big\{ A_1, B_1, \ldots, A_g, B_g \in \mathfrak{gl}_n(\mathbb{C}), C_1 \in \mathcal{O}_1, \ldots, C_k \in \mathcal{O}_k \Big| \\ [A_1, B_1] + \cdots + [A_g, B_g] + C_1 \cdots + C_k = 0 \Big\} // \mathrm{GL}_n(\mathbb{C}),$$

an affine GIT quotient by the conjugation action of $\mathrm{GL}_n(\mathbb{C})$, where $[\cdot, \cdot]$ is the Lie bracket in $\mathfrak{gl}_n(\mathbb{C})$. We prove in Theorem 2.2.4 that $\mathcal{Q}_{\boldsymbol{\mu}}$ is a smooth variety of dimension $d_{\boldsymbol{\mu}}$. It is a quiver variety in the sense of Nakajima and Crawley-Boevey associated to the comet-shaped quiver Γ described in Section 2.2.

THEOREM 1.3.1

For $\boldsymbol{\mu}$ indivisible the mixed Hodge structure on $H_c^*(\mathcal{Q}_{\boldsymbol{\mu}})$ is pure; in other words, $h^{i,j;k}(\mathcal{Q}_{\boldsymbol{\mu}}) = 0$ unless $i + j = k$, and $E(\mathcal{Q}_{\boldsymbol{\mu}}; x, y)$ only depends on the product xy. Moreover,

$$P_c(\mathcal{Q}_{\boldsymbol{\mu}}; \sqrt{q}) = E(\mathcal{Q}_{\boldsymbol{\mu}}; q) = q^{(1/2)d_\mu} \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}), \tag{1.3.1}$$

where $P_c(\mathcal{Q}_{\boldsymbol{\mu}}, t)$ is the compactly supported Poincaré polynomial of $\mathcal{Q}_{\boldsymbol{\mu}}$.

As in the multiplicative case, Katz's theorem [22, Theorem 6.1.2.3] implies that $E(\mathcal{Q}_{\boldsymbol{\mu}}; q) = \#\mathcal{Q}_{\boldsymbol{\mu}}(\mathbb{F}_q)$. The calculation of the number of points on the right is performed using the mass formula

$$\#\mathcal{Q}_{\boldsymbol{\mu}}(\mathbb{F}_q) = \frac{|\mathfrak{gl}_n(\mathbb{F}_q)|^{g-1}}{|\mathrm{PGL}_n(\mathbb{F}_q)|} \sum_{x \in \mathfrak{gl}_n(\mathbb{F}_q)} |C_{\mathfrak{gl}_n(\mathbb{F}_q)}(x)|^g \prod_{i=1}^{k} \mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}_i})(x), \tag{1.3.2}$$

where $C_{\mathfrak{gl}_n(\mathbb{F}_q)}(x)$ denotes the centralizer of x in $\mathfrak{gl}_n(\mathbb{F}_q)$ and $\mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}_i})$ is the Fourier transform (2.5.3) of the characteristic function of $\mathcal{O}_i$. The evaluation of this sum is based on a combinatorial understanding of the formulas in [35] in the case of $\mathfrak{gl}_n(\mathbb{F}_q)$. The proof of Theorem 1.3.1 is given in Section 6.2.

Remark 1.3.2

The *purity conjecture* of [19] claims that $PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q) = E(\mathcal{Q}_{\boldsymbol{\mu}}; q)$. Combined with Conjecture 1.2.1 it implies that the right-hand side of (1.3.1) should equal $PH_c(\mathcal{M}_{\boldsymbol{\mu}}; q)$. By extension, we call the *pure part* of a function of z, w its specialization $z = 0, w = \sqrt{q}$. For example, the pure part of the Macdonald polynomial is $\tilde{H}_\lambda(\mathbf{x}; w) := \tilde{H}_\lambda(\mathbf{x}; 0, w)$ a (transformed version of) the Hall-Littlewood polynomial (see Section 2.3.4). In particular, Theorem 1.3.1 shows that the E-polynomials of the quiver varieties $\mathcal{Q}_{\boldsymbol{\mu}}$ are closely related to the generalized Cauchy formula for Hall-Littlewood functions.

Remark 1.3.3

Let $A_{\mathbf{v}}(q)$ be the number of absolutely indecomposable representations of a quiver of dimension $\mathbf{v}$ over the finite field $\mathbb{F}_q$ (up to isomorphism). Kac [26] proved that $A_{\mathbf{v}}(q)$ is a polynomial in q with integer coefficients. He conjectured that these coefficients are nonnegative [26, Conjecture 2]. Crawley-Boevey and Van den Bergh [5] proved this conjecture for $\mathbf{v}$ indivisible by giving a cohomological interpretation for $A_{\mathbf{v}}(q)$. In our case, writing $A_{\boldsymbol{\mu}}$ for $A_{\mathbf{v}}$, their result says that for $\boldsymbol{\mu}$ indivisible,

$$E(\mathcal{Q}_{\boldsymbol{\mu}}; q) = \#\mathcal{Q}_{\boldsymbol{\mu}}(\mathbb{F}_q) = q^{(1/2)d_\mu} A_{\boldsymbol{\mu}}(q). \tag{1.3.3}$$

In particular, for $\boldsymbol{\mu}$ indivisible, $A_{\boldsymbol{\mu}}$ is the pure part of $\mathbb{H}_{\boldsymbol{\mu}}$; that is,

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \tag{1.3.4}$$

In fact, we give in [21] an independent proof of (1.3.4) for *any* $\boldsymbol{\mu}$ using Hua's formula [25]. Conjecture 1.2.1 would then give a cohomological interpretation of $A_{\boldsymbol{\mu}}(q)$ for a comet-shaped quiver, which, in particular, would imply Kac's conjecture on the nonnegativity of the coefficients of $A_{\boldsymbol{\mu}}(q)$ for such quivers for all dimension vectors (see also Remark 1.4.3).

1.4. Multiplicities

For our third main theorem we need to introduce some complex irreducible characters of $G := \mathrm{GL}_n(\mathbb{F}_q)$. Pick distinct linear characters $\alpha_{i,1}, \dots, \alpha_{i,r_i}$ of $\mathbb{F}_q^\times$ for each i. Consider the subgroup $L_i := \prod_{j=1}^{r_i} \mathrm{GL}_{\mu^i_j}(\mathbb{F}_q)$ of G and the linear character $\tilde{\alpha}_i := \prod_{j=1}^{r_i}(\alpha_{i,j} \circ \det)$ of L_i. We get an irreducible character of G by taking the Harish-Chandra induction $R_{L_i}^G(\tilde{\alpha}_i)$. We assume now that the $\alpha_{i,j}$'s are chosen such that the k-tuple $(R_{L_1}^G(\tilde{\alpha}_1), \dots, R_{L_k}^G(\tilde{\alpha}_k))$ is *generic* in the sense of Definition 4.2.2. (Such a choice is always possible for every $\boldsymbol{\mu}$ assuming that $\mathrm{char}(\mathbb{F}_q)$ and q are large enough.) To simplify the notation we let

$$R_{\boldsymbol{\mu}} := \bigotimes_{i=1}^{k} R_{L_i}^G(\tilde{\alpha}_i).$$

Let $\Lambda : G \to \mathbb{C}$ be defined by $x \mapsto q^{g \dim C_G(x)}$, where $C_G(x)$ is the centralizer of x in G. If $g = 1$, it is the character of the permutation representation where G acts on the finite set $\mathfrak{gl}_n(\mathbb{F}_q)$ by conjugation.

THEOREM 1.4.1
The identity

$$\mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}) = \langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1 \rangle \tag{1.4.1}$$

holds where $\langle \cdot, \cdot \rangle$ is the usual scalar product of characters.

The proof of this theorem can be found in Section 6.1.

For a finite group H, let $\mathcal{R}(H)$ be the character ring of H (i.e., the Grothendieck ring of the category of $\mathbb{C}[G]$-modules). The irreducible characters $\mathcal{X}_1, \dots, \mathcal{X}_k$ form a natural basis $\mathcal{B}$ of $\mathcal{R}_H$. It is an important and difficult problem to compute the fusion rules of $\mathcal{R}(H)$ with respect to $\mathcal{B}$, that is, to compute $N_{i,j}^r := \langle \mathcal{X}_i \otimes \mathcal{X}_j, \mathcal{X}_r \rangle$ for all i, j, r. The character ring of $\mathrm{GL}_n(\mathbb{F}_q)$ does not seem to have been studied in the literature, although the character table of $\mathrm{GL}_n(\mathbb{F}_q)$ was computed 50 years ago (see [16]).

Our Theorem 1.4.1 (with $g = 0$ and $k = 3$) gives a formula for the multiplicities $N^r_{i,j}$ when $(\mathcal{X}_i, \mathcal{X}_j, \mathcal{X}_r)$ is a generic triple of semisimple irreducible characters. This suggest an interesting connection between the character ring of $\mathrm{GL}_n(\mathbb{F}_q)$, Kac-Moody algebras, and quiver representations that we discuss further in Section 6.1.

By formulas (1.3.1) and (1.4.1) we have the following.

COROLLARY 1.4.2
For $\boldsymbol{\mu}$ indivisible the following are equivalent:
(a) $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle = 0$;
(b) *The quiver variety $\mathcal{Q}_{\boldsymbol{\mu}}$ is empty.*

In the genus $g = 0$ case, the problem of deciding whether $\mathcal{Q}_{\boldsymbol{\mu}}$ is empty was solved by Kostov [29], [30]. Later on, Crawley-Boevey [3] reformulated Kostov's answer in terms of roots. Namely, he proved that $\mathcal{Q}_{\boldsymbol{\mu}}$ is nonempty if and only if $\mathbf{v}$, the dimension vector for Γ with dimension $n - \sum_{j=1}^{l} \mu^i_j$ at the lth vertex on the ith leg, is a root of the Kac-Moody algebra associated to Γ.

Remark 1.4.3
Combining (1.3.3) with Theorems 1.3.1 and 1.4.1 we find that

$$A_{\boldsymbol{\mu}}(q) = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}) = \langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle \tag{1.4.2}$$

when $\boldsymbol{\mu}$ is indivisible. In [21] we prove the equality (1.4.2) for *any* $\boldsymbol{\mu}$. Assuming Conjecture 1.2.1, this gives a cohomological interpretation of $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle$ (see also Remark 1.3.3).

1.5. Examples

When the associated comet-shaped quiver (see Section 2.2 for a description) is finite or tame, our main conjecture (Conjecture 1.2.1) reduces to purely combinatorial formulas, some of which are known. We illustrate this in a few examples.

1.5.1. Cases related to Garsia-Haiman's formulas

For $g = 0$ and $k = 1$ (resp., $k = 2$) we have

$$\mathcal{M}_{\boldsymbol{\mu}} := \begin{cases} \text{point} & \text{if } \boldsymbol{\mu} = (1) \text{ (resp., } \boldsymbol{\mu} = ((1), (1))), \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence for $g = 0$ and $k = 1$ the formula (cf. [14, Corollary 3.3])

$$\mathrm{Exp}\Big(\frac{m_{(1)}(\mathbf{x})}{(q-1)(1-t)}\Big) = \sum_{\lambda} \frac{\tilde{H}_\lambda(\mathbf{x}; q, t)}{\prod (q^{a+1} - t^l)(q^a - t^{l+1})}$$

implies Conjecture 1.2.1, and for $g = 0$ and $k = 2$, the conjecture follows from the Cauchy formula (1.1.1) or, equivalently, (1.1.4).

1.5.2. Comet-shaped quivers with $k = 1$ and $l(\mu^1) = 1$

As mentioned at the end of Section 1.2, in this case we have $\mathcal{M}_{\boldsymbol{\mu}} = \mathcal{M}_n$ and $\mathbb{H}_{\boldsymbol{\mu}} = \bar{H}_n$ in the notation of [22]. The point is that if we are only interested in partitions μ^i of length at most l we can, without loss of generality, set all variables x^i_j with $j > l$ to zero (see Section 2.3.6). If $l = 1$ this means we may specialize to $\mathbf{x}^i = (T, 0, \ldots)$ for some variable T. Since $\tilde{H}_\lambda(T, 0, \ldots) = T^{|\lambda|}$, we see that $\Omega(z, w)$ specializes to the corresponding series (see left-hand side of (3.5.8)) in [22].

If in addition $g = 1$, then Conjecture 1.2.1 reduces to the following purely combinatorial identity of generating functions (see [22, Conjecture 4.3.2]).

CONJECTURE 1.5.1

We have

$$\sum_\lambda \prod \frac{(z^{2a+1} - w^{2l+1})^2}{(z^{2a+2} - w^{2l})(z^{2a} - w^{2l+2})} T^{|\lambda|}$$
$$= \prod_{n\geq 1} \prod_{r>0} \prod_{s\geq 0} \frac{(1 - z^{2s+1} w^{-2r+1} T^n)^2}{(1 - z^{2s} w^{-2r+2} T^n)(1 - z^{2s+2} w^{-2r} T^n)}. \qquad (1.5.1)$$

The associated quiver is the Jordan quiver (one loop, one node), which is tame. We know that the Euler specialization $z = \sqrt{q}, w = 1/\sqrt{q}$ of (1.5.1) is true; after taking Log's it amounts to the following easy facts:

$$\sum_{\lambda \in \mathcal{P}} T^{|\lambda|} = \prod_{n \geq 1} (1 - T^n)^{-1}, \qquad \sum_{r>0} \sum_{s \geq 0} q^{r+s} = (q + q^{-1} - 2)^{-1}.$$

1.5.3. Star-shaped quiver with k legs and $l(\mu^i) \leq 2$

Consider the quiver consisting of one central node with no loops ($g = 0$) and k legs of length 1. It is enough (see Section 2.3.6) to consider partitions μ^i of length at most 2 and restrict to these by specializing the variables $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots)$ to, say, $u_0^{1/k}(1, u_i, 0, 0, \ldots)$ for $i = 1, \ldots, k$ for some new independent variables u_i. The variable u_0, corresponding to the central node, keeps track of the degree of the symmetric functions. Multipartitions $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}_n^k$ with $l(\mu^i) \leq 2$ are of the form $\mu^i = (n - n_i, n_i)$ for some $0 \leq n_i \leq n/2$ for $i = 1, \ldots, k$. To simplify somewhat the notation, we extend by symmetry the definition of $\mathbb{H}_{\boldsymbol{\mu}}$ to arbitrary pairs $\mu^i = (n - n_i, n_i)$ with $0 \leq n_i \leq n$ for $i = 1, 2, \ldots, k$. For $\mathbf{v} = (n, n_1, \ldots, n_k)$ we let $\mathbb{H}_{\mathbf{v}} = \mathbb{H}_{\boldsymbol{\mu}}$, where $\boldsymbol{\mu}$ is the corresponding multipartition obtained by appropriate permutation of the entries of each pair $(n - n_i, n_i)$. It then follows easily from the definition that

$$\sum_{\mathbf{v}} \mathbb{H}_{\mathbf{v}}(z,w)u^{\mathbf{v}}$$
$$= (z^2-1)(1-w^2)\,\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}} \frac{\prod_{i=1}^k \tilde{H}_\lambda(1,u_i,0,\ldots;z^2,w^2)}{\prod(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})} u_0^{|\lambda|}\Big), \quad (1.5.2)$$

where the sum is over all nonzero $\mathbf{v}=(n,n_1,\ldots,n_k)$ with $0\le n_i\le n$ and $u^{\mathbf{v}} := u_0^n u_1^{n_1}\cdots u_k^{n_k}$.

Remark 1.5.2
Note that since $g=0$ (see Remark 2.3.7), the right-hand side and hence $\mathbb{H}_{\mathbf{v}}(z,w)$ are actually functions of z^2 and w^2; we exploit this below without further comment.

By work of Crawley-Boevey [4] the character variety $\mathcal{M}_{\boldsymbol{\mu}}$ is empty unless $\mathbf{v}$ is a root of the associated Kac-Moody root system. For $1\le k\le 3$ the corresponding quiver is finite. In particular, the main conjecture implies that the sum on the left-hand side of (1.5.2) is finite in this case. More precisely, we have the following. (For convenience we reverted to the combinatorial variables q,t.)

CONJECTURE 1.5.3
We have

$$(q-1)(1-t)\,\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}} \frac{\prod_{i=1}^3 \tilde{H}_\lambda(1,u_i,0,\ldots;q,t)}{\prod(q^{a+1}-t^l)(q^a-t^{l+1})} u_0^{|\lambda|}\Big)$$
$$= (1+u_1)(1+u_2)(1+u_3)u_0 + u_1u_2u_3u_0^2. \quad (1.5.3)$$

Indeed, for $k=3$ our system is D_4, and hence all roots are real and given by those vectors $\mathbf{v}$ satisfying $\mathbf{v}^t C\mathbf{v}=2$, where C is the Cartan matrix

$$C := \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}.$$

The roots with positive first coordinate are precisely $(1,n_1,n_2,n_3)$ with $n_i=0,1$ and $(2,1,1,1)$. For $\boldsymbol{\mu}$ corresponding to a real root with positive n we actually know that $\mathcal{M}_{\boldsymbol{\mu}}$ is a point (see [4]), and hence its mixed Hodge polynomial is just 1. The cases $k=1,2$ can be obtained from (1.5.3) by setting $u_3=u_2=0$ and $u_3=0$, respectively; a proof for these cases follows by specializing the Cauchy formula (1.1.1).

For $k = 4$ the quiver is tame; it corresponds to the affine system $\tilde{D}_4$. Its Cartan matrix is

$$C := \begin{pmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where the first coordinate corresponds to the central vertex. The positive real roots are the vectors $\mathbf{v} = (v_0, v_1, \ldots, v_4)$ with $v_i \geq 0$ for $i = 0, 1, \ldots, 4$ such that $\mathbf{v}^t C \mathbf{v} = 2$. The positive imaginary roots are the vectors $r\mathbf{v}^*$, where $\mathbf{v}^* := (2, 1, 1, 1, 1)$, with r a positive integer.

The main conjecture now specializes to the following (again expressed in the combinatorial variables q, t).

CONJECTURE 1.5.4
For $u_0, \ldots, u_4$ independent variables we have

$$(q-1)(1-t)\,\mathrm{Log}\Big(\sum_{\lambda \in \mathcal{P}} \frac{\prod_{i=1}^4 \tilde{H}_\lambda(1, u_i, 0, \ldots; q, t)}{\prod (q^{a+1} - t^l)(q^a - t^{l+1})} u_0^{|\lambda|}\Big)$$
$$= \sum_{\mathbf{v}} u^{\mathbf{v}} + (q + 4 + t) \sum_{r \geq 1} u^{r\mathbf{v}^*},$$

where the first sum is over all positive real roots $\mathbf{v} = (v_0, \ldots, v_4)$ with $v_0 > 0$ and $u^{\mathbf{v}} := \prod_{j=0}^4 u_j^{v_j}$.

To see this, note that for $\boldsymbol{\mu} = ((r,r), (r,r), (r,r), (r,r))$, corresponding to the imaginary root rv^*, the variety $\mathcal{M}_{\boldsymbol{\mu}}$ is a smooth affine surface. (By (1.2.1) the dimension $d_{\boldsymbol{\mu}}$ equals 2 for all r.) By a result of [10, Theorem 6.14], $\mathcal{M}_{\boldsymbol{\mu}}$ is isomorphic to $S^0 = S \setminus \Delta \subseteq \mathbb{P}^3$, where S is a smooth cubic surface and Δ is the union of the coordinate axes. A calculation shows that the mixed Hodge polynomials of S^0 are $H(S^0; q, t) = (qt)^2 + 4qt^2 + 1$ and $H_c(S^0; q, t) = t^2 + 4t^2 q + t^4 q^2$. We should then have $\mathbb{H}_{\boldsymbol{\mu}}(z, w) = z^2 + 4 + w^2$.

In fact, for $n = 2$ the connection to cubic surfaces goes back to Fricke and Klein [11, Section II.2, p. 285]. It boils down to the following *Fricke relation* (see [41, p. 93]). Given three matrices $A_i \in \mathrm{SL}_2(\mathbb{C})$ for $i = 1, 2, 3$, let $a_i := \mathrm{Tr}(A_i)$, $x_i := \mathrm{Tr}(A_j A_k)$, where Tr denotes the trace and $\{i, j, k\} = \{1, 2, 3\}$. Then

$$0 = x_1 x_2 x_3 + \sum_{i=1}^3 (x_i^2 - \theta_i x_i) + \theta_4, \tag{1.5.4}$$

where, with the same convention on indices,

$$a_4 := \mathrm{Tr}(A_1A_2A_3), \qquad \theta_i := a_ia_4 + a_ja_k,$$
$$\theta_4 := a_1 \cdots a_4 + a_1^2 + \cdots + a_4^2 - 4.$$

(Viewed as a quadratic equation in a_4 the other solution to (1.5.4) is $\mathrm{Tr}(A_1A_3A_2)$.)

Remark 1.5.5
A similar form of the conjecture occurs when the quiver is the Dynkin diagram of the affine systems $\tilde{E}_6, \tilde{E}_7$, and $\tilde{E}_8$. The corresponding surfaces are now smooth del Pezzo surfaces of degree $9-s$, where $s = 6, 7$, and 8, respectively, with a nodal $\mathbb{P}^1$ removed. The polynomial corresponding to any positive imaginary root should then be $\mathbb{H}_{\boldsymbol{\mu}}(z,w) = z^2 + s + w^2$.

For $k \geq 5$, the quiver is wild and the main conjecture does not take a particularly simple form. For future reference we record here the first few values of $\mathbb{H}_{\boldsymbol{\mu}}$ for $\boldsymbol{\mu} = ((n-1,1),\ldots,(n-1,1))$ or, equivalently, $\mathbf{v} = (n,1,\ldots,1)$, calculated on a computer. For completeness we include also the case $n = 1$, where $\mathbf{v} = (1,1,\ldots,1)$; the relevant range so that $\mathbf{v}$ is a root is then $1 \leq n \leq k-1$. We should stress the fact that the computed $\mathbb{H}_{\boldsymbol{\mu}}$ turned out to be polynomials with nonnegative coefficients, as predicted, something that is not clear a priori.

To simplify, below we write simply $\mathbb{H}_{n,k}$ for $\mathbb{H}_{\boldsymbol{\mu}}$ and display its coefficients as an array. As mentioned above (see Remark 1.5.2), $\mathbb{H}_{\boldsymbol{\mu}}$ is a function of z^2, w^2, so we record only the even powers. To be sure, for example,

$$\mathbb{H}_{2,4} = \begin{array}{ll} 1 & \\ 4 & 1 \end{array}$$

corresponds to the polynomial $\mathbb{H}_{\boldsymbol{\mu}}(z,w) = z^2 + 4 + w^2$ discussed above for $k = 4$. We have the following:

$$\mathbb{H}_{1,5} = \mathbb{H}_{4,5} = 1, \qquad \mathbb{H}_{2,5} = \mathbb{H}_{3,5} = \begin{array}{lll} 1 & & \\ 5 & 1 & \\ 11 & 5 & 1 \end{array}$$

$$\mathbb{H}_{1,6} = \mathbb{H}_{5,6} = 1, \qquad \mathbb{H}_{2,6} = \mathbb{H}_{4,6} = \begin{array}{llll} 1 & & & \\ 6 & 1 & & \\ 16 & 6 & 1 & \\ 26 & 16 & 6 & 1 \end{array} \qquad \mathbb{H}_{3,6} = \begin{array}{lllll} 1 & & & & \\ 6 & 1 & & & \\ 22 & 7 & 1 & & \\ 51 & 27 & 7 & 1 & \\ 66 & 51 & 22 & 6 & 1 \end{array}$$

$$\mathbb{H}_{1,7}=\mathbb{H}_{6,7}=1, \qquad \mathbb{H}_{2,7}=\mathbb{H}_{5,7}=\begin{matrix}1\\7&1\\22&7&1\\42&22&7&1\\57&42&22&7&1\end{matrix}$$

$$\mathbb{H}_{3,7}=\mathbb{H}_{4,7}=\begin{matrix}1\\7&1\\29&8&1\\85&36&8&1\\190&113&37&8&1\\308&246&113&36&8&1\\302&308&190&85&29&7&1\end{matrix}$$

Remark 1.5.6
The observed symmetry $\mathbb{H}_{n,k}=\mathbb{H}_{k-n,n}$ should be a consequence of the action of the reflection associated to the central vertex (by [4], dimension vectors in the same orbit of the Weyl group of the quiver yield isomorphic varities); at the level of the generating functions, though, this symmetry is far from evident.

Remark 1.5.7
The attentive reader may have noticed that the constant terms of the $\mathbb{H}_{n,k}$'s are the first Eulerian numbers. Concretely, the polynomials $A_k(t):=\sum_{n=0}^{k-1}\mathbb{H}_{n+1,k+1}(0,0)t^n$ are the Eulerian polynomials:

$$A_3(t)=1+4t+t^2, \qquad A_4(t)=1+11t+11t^2+t^3,$$
$$A_5(t)=1+6t+26t^2+6t^3+t^4,\ldots.$$

This relation will be the subject of a future publication.

1.5.4. Tennis-racquet quiver
For our next example consider the tennis-racquet quiver, consisting of one vertex, one loop, and one leg of length one. We specialize the variables as in the case (i): $\mathbf{x}=u_0(1,u_1,0,\ldots)$,

$$(z^2-1)(1-w^2)\,\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}}\mathcal{H}_\lambda(z,w)\tilde{H}_\lambda(1,u,0,\ldots;z^2,w^2)u_0^{|\lambda|}\Big)=\sum_{\mathbf{v}}\mathbb{H}_{\mathbf{v}}(z,w)u^{\mathbf{v}},$$

where we recall

$$\mathcal{H}_\lambda(z,w)=\prod\frac{(z^{2a+1}-1)(1-w^{2l+1})}{(z^{2a+2}-w^{2l})(z^{2a}-w^{2l+2})}.$$

In the sum $\mathbf{v}$ runs over all nonzero vectors (n, n_1) with $0 \leq n_1 \leq n$ and $u^v := u_0^n u_1^{n_1}$. (We extended the definition of $\mathbb{H}$ to all such $\mathbf{v}$ as in Conjecture 1.2.1(i).)

With a computer we calculated the first few terms of the right-hand side and obtained the following. We list the coefficients of u_0^n for $n = 1, 2, 3$ divided by $(z - w)^2$ (as pointed out in Remark 5.3.1 below, the divisibility of $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ by $(z - w)^{2g}$ is predicted by the geometry):

1 $\quad 1 + u,$

2 $\quad 1 + (1 + z^2 + w^2)u + u^2,$

3 $\quad 1 + (1 + z^2 + w^2 + z^4 + w^4 - 2zw + z^2 w^2)(1 + u) + u^3.$

So concretely, for example, for $g = k = 1$ and $\boldsymbol{\mu} = ((1^2))$ we have $\mathbb{H}_{\boldsymbol{\mu}}(z, w) = (z - w)^2(1 + z^2 + w^2)$. Again, note that the computations confirm that $\mathbb{H}_{\mathbf{v}}(-z, w)$ is a polynomial with nonnegative coefficients.

1.5.5. $\mathrm{GL}_3(\mathbb{C})$-*character varieties*

Consider the comet-shaped quiver with k legs of length two and any g. We show how to compute $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ by hand using the tables of Macdonald polynomials when we take the partition $\mu^i = (1^3)$ at each leg. Similar calculations can be performed with other partitions of 3. We use freely the notation and definitions of Section 2.3.

From formula (2.3.9) we find that

$$\mathbb{H}_{\boldsymbol{\mu}}(z, w) = \sum_{\omega} \mathbb{H}_{\boldsymbol{\mu}}^{\omega}(z, w) \tag{1.5.5}$$

with

$$\mathbb{H}_{\boldsymbol{\mu}}^{\omega}(z, w) := (z^2 - 1)(1 - w^2) C_{\omega}^{o} \mathcal{H}_{\omega}(z, w) \prod_{i=1}^{k} \langle \tilde{H}_{\omega}(\mathbf{x}_i, z^2, w^2), h_{(1^3)}(\mathbf{x}_i) \rangle,$$

where the sum is over all types of size 3. Since $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k)$ with $\mu^i = (1^3)$ for all i, we have

$$\mathbb{H}_{\boldsymbol{\mu}}^{\omega}(z, w) = (z^2 - 1)(1 - w^2) C_{\omega}^{o} \mathcal{H}_{\omega}(z, w) \langle \tilde{H}_{\omega}(\mathbf{x}, z^2, w^2), h_{(1^3)}(\mathbf{x}) \rangle^k.$$

There are eight types of $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r)$, with $d_i \in \mathbb{Z}_{\geq 0}$ and $\omega^i \in \mathcal{P}$, of size 3:

$$(1, 3^1), \qquad (1, 1^3), \qquad (1, 1^1 2^1), \qquad (3, 1), \qquad (1, 2^1)(1, 1),$$
$$(1, 1^2)(1, 1), \qquad (2, 1)(1, 1), \qquad (1, 1)^3,$$

where we wrote $(1, 1)^3$ for $(1, 1)(1, 1)(1, 1)$.

PROPOSITION 1.5.8
We have $\mathbb{H}_{\boldsymbol{\mu}}^{(3,1)}(z,w) = \mathbb{H}_{\boldsymbol{\mu}}^{(2,1)(1,1)}(z,w) = 0$. *The sum (1.5.5) reduces to*

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) = \sum_{i=1}^{6} \frac{1}{\alpha_i} \beta_i^{2g} \gamma_i^k \tag{1.5.6}$$

where α, β, *and* γ *are the following polynomials in* z, w:

α	β	γ
$(z^6-1)(z^4-w^2)(z^4-1)(z^2-w^2)$	$(z^5-w)(z^3-w)(z-w)$	$1+2z^2+2z^4+z^6$
$(z^2-w^4)(w^6-1)(w^4-1)(z^2-w^2)$	$(z-w^5)(z-w^3)(z-w)$	$1+2w^2+2w^4+w^6$
$(z^4-w^2)(z^2-w^4)(z^2-1)(1-w^2)$	$(z^3-w^3)(z-w)^2$	$1+2z^2+2w^2+z^2w^2$
$-(z^4-1)(z^2-w^2)(z^2-1)(1-w^2)$	$(z^3-w)(z-w)^2$	$3(z^2+1)$
$-(z^2-w^2)(1-w^4)(z^2-1)(1-w^2)$	$(z-w^3)(z-w)^2$	$3(w^2+1)$
$3(z^2-1)^2(1-w^2)^2$	$(z-w)^3$	6

Proof
We only compute $\mathbb{H}_{\boldsymbol{\mu}}^{(1,3^1)}(z,w)$ and $\mathbb{H}_{\boldsymbol{\mu}}^{(1,1)^3}(z,w)$, as the other cases are similar. We start with

$$\mathbb{H}_{\boldsymbol{\mu}}^{(1,3^1)}(z,w) = (z^2-1)(1-w^2)C^o_{3^1}\mathcal{H}_{3^1}(z,w)\langle \tilde{H}_{3^1}(\mathbf{x};z^2,w^2), h_{1^3}(\mathbf{x})\rangle^k.$$

From formula (2.3.10) we find that $C^o_{(1,3^1)} = 1$. From the Young diagram of the partition 3^1 we find that

$$\mathcal{H}_{3^1}(z,w) = \frac{((z^5-1)(z^3-1)(z-w))^{2g}}{(z^6-1)(z^4-w^2)(z^4-1)(z^2-w^2)(z^2-1)(1-w^2)}.$$

It remains to compute $\langle \tilde{H}_{3^1}(\mathbf{x};z^2,w^2), h_{1^3}(\mathbf{x})\rangle$. For any partition λ we have

$$\tilde{H}_\lambda(\mathbf{x};q,t) = \sum_\nu \tilde{K}_{\nu\lambda}(q,t)s_\nu(\mathbf{x})$$

where $s_\nu(\mathbf{x})$ is the Schur symmetric function and where $\tilde{K}_{\nu\lambda}(q,t) := t^{n(\lambda)}K_{\nu\lambda}(q, t^{-1})$ are the (q,t)-Kostka polynomials. From the tables in [40, p. 359] we find for $n=3$ the following table for $\{\tilde{K}_{\nu\lambda}(q,t)\}_{\nu,\lambda}$:

	3^1	1^12^1	1^3
3^1	1	1	1
1^12^1	$q+q^2$	$q+t$	$t+t^2$
1^3	q^3	qt	t^3

Hence

$$\tilde{H}_{1^3}(\mathbf{x};z^2,w^2) = s_{3^1}(\mathbf{x}) + (z^2+z^4)s_{1^12^1}(\mathbf{x}) + z^6 s_{1^3}(\mathbf{x}).$$

Since the set of monomial symmetric functions $\{m_\lambda(\mathbf{x})\}_\lambda$ is the dual basis (with respect to the Hall pairing) of the set of complete symmetric functions $\{h_\mu(\mathbf{x})\}$, we need to express the Schur functions in terms of monomial symmetric functions. Using the tables in [40, pp. 101, 111] we find that $s_{31}(\mathbf{x}) = m_{31}(\mathbf{x}) + m_{1^12^1}(\mathbf{x}) + m_{1^3}(\mathbf{x})$, $s_{1^12^1}(\mathbf{x}) = m_{1^12^1}(\mathbf{x}) + 2m_{1^3}(\mathbf{x})$, and $s_{1^3}(\mathbf{x}) = m_{1^3}(\mathbf{x})$. We thus deduce that

$$\langle \tilde{H}_{3^1}(\mathbf{x}; z^2, w^2), h_{1^3}(\mathbf{x})\rangle = 1 + 2z^2 + 2z^4 + z^6.$$

Let us now compute the term

$$\mathbb{H}_{\boldsymbol{\mu}}^{(1,1)^3}(z,w) = (z^2-1)(1-w^2)C^o_{(1,1)^3}\mathcal{H}_{(1,1)^3}(z,w)\langle \tilde{H}_{(1,1)^3}(\mathbf{x}; z^2, w^2), h_{1^3}(\mathbf{x})\rangle^k.$$

We have $C^o_{(1,1)^3} = 1/3$. By definition of $\mathcal{H}_\omega(z,w)$ and $\tilde{H}_\omega(\mathbf{x}; q,t)$ for a type ω (see Section 2.3.2), we have $\mathcal{H}_{(1,1)^3}(z,w) = \mathcal{H}_1(z,w)^3$ and $\tilde{H}_{(1,1)^3}(\mathbf{x}; q,t) = \tilde{H}_1(\mathbf{x}; q,t)^3$.

Hence

$$\mathcal{H}_{(1,1)^3}(z,w) = \frac{(z-w)^{6g}}{(z^2-1)^3(1-w^2)^3}$$

and

$$\tilde{H}_{(1,1)^3}(\mathbf{x}; q,t) = m_{(1)}(\mathbf{x})m_{(1)}(\mathbf{x})m_{(1)}(\mathbf{x}).$$

With $\mathbf{x} = \{x_1, x_2, \ldots\}$, the monomial symmetric function $m_{(1)}(\mathbf{x})$ is written $x_1 + x_2 + x_3 + \cdots$. Hence $m_{(1)}(\mathbf{x})^3$ decomposes as $m_{3^1}(\mathbf{x}) + 3m_{1^12^1}(\mathbf{x}) + 6m_{1^3}(\mathbf{x})$, and so

$$\langle \tilde{H}_{(1,1)^3}(\mathbf{x}; z^2, w^2), h_{1^3}(\mathbf{x})\rangle = 6. \qquad \square$$

COROLLARY 1.5.9
For $\boldsymbol{\mu} = ((1^3), \ldots, (1^3))$ and g arbitrary, $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$ is a polynomial in z, w.

Proof
With the notation of Proposition 1.5.8 consider the following rational function of z, w, u, v, where u, v are two new indeterminates:

$$R := \sum_{i=1}^{6} \frac{\gamma_i}{\alpha_i(1-u\beta_i^2)(1-v\gamma_i)}. \tag{1.5.7}$$

If we expand R as a power series in u, v, then by (1.5.6) the coefficient of $u^g v^k$ equals $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$. A calculation using Maple shows that $R = A/B$ with $A, B \in \mathbb{Z}[z,w,u,v]$ and B a product of polynomials in $1 + u\mathbb{Z}[z,w,u,v]$ or $1 + v\mathbb{Z}[z,w,u,v]$. The claim follows. $\square$

Let us write $H(\mathcal{M}_{\boldsymbol{\mu}};x,y,t)=\sum h^{i,j;k}(\mathcal{M}_{\boldsymbol{\mu}})x^iy^jt^k$ for the mixed Hodge polynomial for ordinary cohomology. Since $\mathcal{M}_{\boldsymbol{\mu}}$ is nonsingular, Poincaré duality gives

$$H(\mathcal{M}_{\boldsymbol{\mu}};x,y,t)=(xyt^2)^{d_\mu}H_c(\mathcal{M}_{\boldsymbol{\mu}};x^{-1},y^{-1},t^{-1}).$$

Hence Conjecture 1.2.1 and Proposition 1.5.8 imply the following.

CONJECTURE 1.5.10
The polynomial $H(\mathcal{M}_{\boldsymbol{\mu}};x,y,t)$ depends only on xy and t. Moreover,

$$H(\mathcal{M}_{\boldsymbol{\mu}};q,t)=(qt^2)^{3k+9g-8}\mathbb{H}_{\boldsymbol{\mu}}\Big(-\sqrt{q},\frac{1}{\sqrt{q}t}\Big).$$

That is,

$$\begin{aligned}
H(\mathcal{M}_{\boldsymbol{\mu}};q,t)
&=\frac{q^{6g-6}t^{12g-12}((q^3t^6)(1+2q+2q^2+q^3))^k((q^3t+1)(q^2t+1)(qt+1))^{2g}}{(q^3t^2-1)(q^3-1)(q^2t^2-1)(q^2-1)}\\
&\quad+\frac{((q^3t^5+1)(q^2t^3+1)(qt+1))^{2g}((qt^2+1)(q^2t^4+qt^2+1))^k}{(q^3t^6-1)(q^3t^4-1)(q^2t^4-1)(q^2t^2-1)}\\
&\quad+\frac{(qt^2)^{4g-4}((q^3t^3+1)(qt+1)^2)^{2g}(q^2t^4(2+q+qt^2+2q^2t^2))^k}{(q^3t^4-1)(q^3t^2-1)(qt^2-1)(q-1)}\\
&\quad-\frac{(qt^2)^{6g-6}((q^2t+1)(qt+1)^2)^{2g}(3q^3t^6(q+1))^k}{(q^2t^2-1)(q^2-1)(qt^2-1)(q-1)}\\
&\quad-\frac{(qt^2)^{4g-4}((q^2t^3+1)(qt+1)^2)^{2g}(3q^2t^4(qt^2+1))^k}{(q^2t^4-1)(q^2t^2-1)(qt^2-1)(q-1)}\\
&\quad+\frac{(qt^2)^{6g-6}(qt+1)^{6g}6^k(qt^2)^{3k}}{3(qt^2-1)^2(q-1)^2}.
\end{aligned}$$

Note that by Corollary 1.5.9 the predicted $H(\mathcal{M}_{\boldsymbol{\mu}};q,t)$ is indeed a polynomial in q,t. Specializing it to $(q,t)\mapsto(1,t)$ gives a conjectural formula for the Poincaré polynomial $P(\mathcal{M}_{\boldsymbol{\mu}};t)$ of $\mathcal{M}_{\boldsymbol{\mu}}$. We have verified that our formula for $P(\mathcal{M}_{\boldsymbol{\mu}};t)$ agrees with those of [13] (cf. [13, Remark 11.3]) for small values of g and k giving support to our main conjecture.

For example, for $g=0$ and $k<3$ we have $\mathbb{H}_{\boldsymbol{\mu}}(z,w)=0$. For $k=3$ we have

$$z^2+(w^2+6),$$

and for $k=4$,

$$\begin{aligned}\mathbb{H}_{\boldsymbol{\mu}}(z,w) = z^8 &+ (w^2+8)z^6 + (w^4+9w^2+33)z^4 \\ &+ (w^6+9w^4+41w^2+93)z^2 \\ &+ (w^8+8w^6+33w^4+93w^2+136).\end{aligned}$$

Hence

$$t^2\mathbb{H}_{\boldsymbol{\mu}}(-1,1/t) = 7t^2+1$$

and

$$t^8\mathbb{H}_{\boldsymbol{\mu}}(-1,1/t) = 271t^8+144t^6+43t^4+9t^2+1,$$

respectively, matching the values of $P(\mathcal{M}_{\boldsymbol{\mu}},t)$ calculated in [13, pp. 62–63]. Note that the case $k=3$ corresponds to the basic imaginary root of the affine $\tilde{E}_6$-quiver that we already encountered (see Remark 1.5.5).

Incidentally, specializing R in (1.5.7) to $z=w=1$ gives the rational function

$$\frac{4v^3(72v^2+57v+2)}{(1-6v)^5} = 8v^3+468v^4+11448v^5+192240v^6+\cdots.$$

Hence Theorem 1.2.3 implies that the Euler characteristic $E(\mathcal{M}_{\boldsymbol{\mu}}) = E(\mathcal{M}_{\boldsymbol{\mu}},1)$ of $\mathcal{M}_{\boldsymbol{\mu}}$ for $g=0$ and $\boldsymbol{\mu}=((1^3),\ldots,(1^3))$ is

$$E(\mathcal{M}_{\boldsymbol{\mu}}) = 2^{-5}\cdot 3^{-3}\cdot(k-1)(k-2)(9k^2-27k+16)\cdot 6^k \tag{1.5.8}$$

(see Remark 5.3.4; for $g=1$ and $\boldsymbol{\mu}=((1^3),\ldots,(1^3))$ a similar calculation yields $E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = 3^{-1}\cdot 4\cdot 6^k$, agreeing with Theorem 1.2.6).

We have also checked that the result of similar calculations for GL_2-character varieties matches those of [1].

1.6. Related work

The present paper has spawned some recent work on the A-polynomial. In [24], Helleloid and Rodriguez-Villegas study A-polynomials of general quivers from a viewpoint motivated by [22] and this paper. Hausel [18] proves a further conjecture of Kac [26, Conjecture 1], claiming that the constant term of the A-polynomial of a quiver is a certain multiplicity in the corresponding Kac-Moody algebra, for any loop-free quiver using Nakajima quiver varieties and techniques closely related to the ones in this paper.

In [36], the second author obtained a generalization of the results of Section 1.4 to arbitrary irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ by computing the Poincaré polynomial (for the intersection cohomology) of quiver varieties associated with the Zariski closure of k arbitrary adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$. As in the semisimple case, it is expected that this Poincaré polynomial coincides with the pure part of the mixed Hodge polynomial (again, for the intersection cohomology) of character varieties with the Zariski closure of conjugacy classes at the punctures. This will be the subject of a future publication.

2. Generalities

2.1. Character varieties

Fix integers $g \geq 0$, $k, n > 0$. We also fix a k-tuple of partitions of n which we denote by $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in (\mathcal{P}_n)^k$, that is, $\mu^i = (\mu^i_1, \mu^i_2, \dots, \mu^i_{r_i})$ such that $\mu^i_1 \geq \mu^i_2 \geq \cdots$ are nonnegative integers and $\sum_j \mu^i_j = n$. Let d be the gcd of $\{\mu^i_j\}_{i,j}$, and let $\mathbb{K}$ be an algebraically closed field such that

$$\mathrm{char}(\mathbb{K}) d. \tag{2.1.1}$$

We now construct a variety whose points parameterize representations of the fundamental group of a k-punctured Riemann surface of genus g into $\mathrm{GL}_n(\mathbb{K})$ with prescribed images in semisimple conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_k$ at the punctures. Assume that

$$\prod_{i=1}^{k} \det C_i = 1 \tag{2.1.2}$$

and that $(C_1, C_2, \dots, C_k)$ has type $\boldsymbol{\mu} = (\mu^1, \mu^2, \dots, \mu^k)$; that is, C_i has type μ^i for each $i = 1, 2, \dots, k$, where the *type* of a semisimple conjugacy class $C \subset \mathrm{GL}_n(\mathbb{K})$ is defined as the partition $\mu = (\mu_1, \mu_2, \dots) \in \mathcal{P}_n$ describing the multiplicities of the eigenvalues of (any matrix in) C.

Definition 2.1.1

The k-tuple $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is *generic* if the following holds. If $V \subseteq \mathbb{K}^n$ is a subspace stable by some $X_i \in C_i$ for each i such that

$$\prod_{i=1}^{k} \det(X_i|_V) = 1, \tag{2.1.3}$$

then either $V = 0$ or $V = \mathbb{K}^n$.

For example, if $k = 1$ and C_1 is of type (n)—that is, consists of the diagonal matrix of eigenvalue ζ (with $\zeta^n = 1$ so that (2.1.2) is satisfied)—then C is generic if and only if ζ is a *primitive* nth root of 1.

LEMMA 2.1.2

There exists a generic k-tuple of semisimple conjugacy classes $(\mathcal{C}_1,\ldots,\mathcal{C}_k)$ of type $\boldsymbol{\mu}$ over $\mathbb{K}$.

Proof

Let r_i be the length of the ith coordinate μ^i of $\boldsymbol{\mu}$. Let $A := \mathbb{G}_m^{r_1} \times \cdots \times \mathbb{G}_m^{r_k}$ over $\mathbb{K}$. For any $\boldsymbol{\nu} = (\nu^1,\ldots,\nu^k) = (\nu^i_j) \in \mathbb{Z}^{r_1} \times \cdots \times \mathbb{Z}^{r_k}$ define the homomorphism

$$\phi_{\boldsymbol{\nu}} : A \to \mathbb{G}_m,$$
$$(a^i_j) \mapsto \prod_{i,j} (a^i_j)^{\nu^i_j},$$

and set $A_{\boldsymbol{\nu}} := \ker \phi_{\boldsymbol{\nu}}$. By hypothesis, $\mathrm{char}(\mathbb{K}) \nmid d$ and hence $\mathbb{K}$ contain a primitive dth root of unity ζ_d. Let A' be defined by

$$A' : \prod_{i,j} (a^i_j)^{\mu^i_j/d} = \zeta_d.$$

Observe that $u := (\mu^i_j/d)_{i,j}$ is a primitive vector in $\mathbb{Z}^{r_1} \times \cdots \times \mathbb{Z}^{r_k}$. Hence we can change coordinates in this lattice so that u is part of a basis. In the corresponding new variables of A the equation defining A' is simply $a_1 = \zeta_d$, and therefore $A' \cong \mathbb{G}_m^{\sum r_i - 1}$, showing it is irreducible. Thus A' is a connected component of $A_{\boldsymbol{\mu}}$.

Now if $A' \subseteq A_{\boldsymbol{\nu}}$, then $A_{\boldsymbol{\mu}} \subseteq A_{\boldsymbol{\nu}}$ as A' generates $A_{\boldsymbol{\mu}}$. But $A_{\boldsymbol{\mu}} \subseteq A_{\boldsymbol{\nu}}$ implies $l\boldsymbol{\mu} = \boldsymbol{\nu}$ for some $l \in \mathbb{Z}_{\geq 0}$, since $\mathrm{char}(\mathbb{K})$ does not divide d. So $A'_{\boldsymbol{\nu}} := A' \cap A_{\boldsymbol{\nu}} \subseteq A'$ is a proper Zariski-closed subset of the irreducible space A' for every $\boldsymbol{\nu} = (\nu^i_j)$ with $0 \leq \nu^i_j \leq \mu^i_j$ different from $\boldsymbol{\mu}$ and $\mathbf{0}$. The same is true for all the subgroups B determined by the equalities $a^i_{j_1} = a^i_{j_2}$ for $j_1 \neq j_2$. Hence the union of all $A'_{\boldsymbol{\nu}}$'s and all B's is not equal to the irreducible A', and the complement contains a $\mathbb{K}$-point. Given such a $\mathbb{K}$-point (a^i_j), define C_i to be the semisimple conjugacy class with eigenvalues a^i_j with multiplicities μ^i_j. Then $(C_1,\ldots,C_k)$ is generic of type $\boldsymbol{\mu}$. □

For a k-tuple of conjugacy classes $(\mathcal{C}_1,\ldots,\mathcal{C}_k)$ of type $\boldsymbol{\mu}$ define $\mathcal{U}_{\boldsymbol{\mu}}$ as the subvariety of $\mathrm{GL}_n(\mathbb{K})^{2g+k}$ of elements $(A_1,\ldots,A_g,B_1,\ldots,B_g,X_1,\ldots,X_k)$ which satisfy

$$(A_1,B_1)\cdots(A_g,B_g)X_1\cdots X_k = I_n, \quad X_i \in \mathcal{C}_i. \tag{2.1.4}$$

Remark 2.1.3

If Σ_g is a compact Riemann surface of genus g with punctures $S = \{s_1,\ldots,s_k\} \subseteq \Sigma_g$, then $\mathcal{U}_{\boldsymbol{\mu}}$ can be identified with the set

$$\left\{\rho \in \mathrm{Hom}(\pi_1(\Sigma_g \setminus S), \mathrm{GL}_n(\mathbb{K})) \;\middle|\; \rho(\gamma_i) \in \mathcal{C}_i\right\}$$

(for some choice of base point, which we omit from the notation). Here we use the standard presentation

$$\pi_1(\Sigma_g \setminus S) = \langle \alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g; \gamma_1, \ldots, \gamma_k \mid (\alpha_1, \beta_1) \cdots (\alpha_g, \beta_g)\gamma_1 \cdots \gamma_k = 1 \rangle$$

(γ_i is the class of a simple loop around s_i with orientation compatible with that of Σ_g).

We have GL_n acting on GL_n^{2g+k} by conjugation. As the center acts trivially, this induces an action of PGL_n. The action also leaves (2.1.4), the defining equations of $\mathcal{U}_{\boldsymbol{\mu}}$, invariant and thus induces an action of PGL_n on $\mathcal{U}_{\boldsymbol{\mu}}$. We call the affine GIT quotient

$$\mathcal{M}_{\boldsymbol{\mu}} := \mathcal{U}_{\boldsymbol{\mu}} // \mathrm{PGL}_n = \mathrm{Spec}(\mathbb{K}[\mathcal{U}_{\boldsymbol{\mu}}]^{\mathrm{PGL}_n})$$

a generic character variety of type $\boldsymbol{\mu}$. We denote by $\pi_{\boldsymbol{\mu}}$ the quotient morphism

$$\pi_{\boldsymbol{\mu}} : \mathcal{M}_{\boldsymbol{\mu}} \to \mathcal{U}_{\boldsymbol{\mu}}.$$

PROPOSITION 2.1.4

If $(C_1, \ldots, C_k)$ is generic of type $\boldsymbol{\mu}$, then the group $\mathrm{PGL}_n(\mathbb{K})$ acts set-theoretically freely on $\mathcal{U}_{\boldsymbol{\mu}}$ and every point of $\mathcal{U}_{\boldsymbol{\mu}}$ corresponds to an irreducible representation of $\pi_1(\Sigma_g \setminus S)$.

Proof

Let $A_1, B_1, \ldots, A_g, B_g \in \mathrm{GL}_n(\mathbb{K})$ and $X_i \in C_i$ satisfy

$$(A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n. \tag{2.1.5}$$

Assume that all the matrices A_i, B_i, and X_j preserve a subspace $V \subseteq \mathbb{K}^n$. Let $A_i' = A_i|_V$, $B_i' = B_i|_V$, and $X_i' = X_i|_V$. Then

$$(A_1', B_1') \cdots (A_g', B_g') X_1' \cdots X_k' = I_V. \tag{2.1.6}$$

Taking determinants of both sides we see that the product of the eigenvalues of the matrices X_i' equals 1. Hence, by the genericity assumption, either $V = 0$ or $V = \mathbb{K}^n$ and the corresponding representation of $\pi_1(\Sigma_g \setminus S)$ is irreducible.

Now suppose that $g \in \mathrm{GL}_n(\mathbb{K})$ commutes with all the matrices A_i, B_i, and X_j. By the irreducibility of the action we just proved, it follows from Schur's lemma that $g \in \mathrm{GL}_n(\mathbb{K})$ is a scalar. Hence $\mathrm{PGL}_n(\mathbb{K})$ acts set-theoretically freely on $\mathcal{U}_{\boldsymbol{\mu}}(\mathbb{K})$. □

Recall (see (1.2.1)) that $d_{\boldsymbol{\mu}} = (2g + k - 2)n^2 - \sum_{i,j} (\mu_j^i)^2 + 2$.

THEOREM 2.1.5

If $(C_1, \ldots, C_k)$ *is a generic* k*-tuple of semisimple conjugacy classes in* $\mathrm{GL}_n(\mathbb{K})$ *of type* $\boldsymbol{\mu}$*, then the quotient* $\pi_\mu : \mathcal{U}_{\boldsymbol{\mu}} \to \mathcal{M}_{\boldsymbol{\mu}}$ *is a geometric quotient and a principal* PGL_n*-bundle. Consequently, when nonempty, the variety* $\mathcal{M}_{\boldsymbol{\mu}}$ *is nonsingular of pure dimension* $d_{\boldsymbol{\mu}}$*; that is, it is the disjoint union of its irreducible components all nonsingular of same dimension* $d_{\boldsymbol{\mu}}$*.*

Proof

If $k = 1$ and C_1 is a central matrix, then this is [22, Theorem 2.2.5]; if $g = 0$ and $\mathbb{K} = \mathbb{C}$, then this is [9, Proposition 5.2.8]. Our proof will combine the proofs of these two results.

Let

$$\rho : \mathrm{GL}_n(\mathbb{K})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \to \mathrm{SL}_n(\mathbb{K})$$

be given by

$$(A_1, B_1, A_2, B_2, \ldots, A_g, B_g, X_1, \ldots, X_k) \mapsto (A_1, B_1) \cdots (A_k, B_k) X_1 \cdots X_k.$$

We have $\mathcal{U}_{\boldsymbol{\mu}} = \rho^{-1}(I_n)$. Combining the calculations in [22, Theorem 2.2.5] and [9, Proposition 5.2.8], it is straightforward, albeit lengthy, to calculate the differential $d_s\rho$; we leave it to the reader. Exactly as in [22, Theorem 2.2.5] and [9, Proposition 5.2.8] we can then argue that $d_s\rho$ is surjective for all $s \in \mathcal{U}_{\boldsymbol{\mu}}$, and so the affine variety $\mathcal{U}_{\boldsymbol{\mu}}$ is nonsingular of dimension

$$\dim\big(\mathrm{GL}_n(\mathbb{K})^{2g} \times C_1 \times \cdots \times C_k\big) - \dim \mathrm{SL}_n(\mathbb{K}) = 2gn^2 + kn^2 - n^2 + 1 - \sum_{i,j} (\mu^i_j)^2.$$

Exactly as in [22, Corollaries 2.2.7, 2.2.8] we can argue that this is a geometric quotient as well as a PGL_n principal bundle, proving that $\mathcal{M}_{\boldsymbol{\mu}}$ is nonsingular of dimension $d_{\boldsymbol{\mu}}$ given by (1.2.1). □

2.2. Quiver varieties

As in Section 2.1 we fix $g, k, n, \boldsymbol{\mu}$. But in this section we take an algebraically closed field $\mathbb{K}$, which satisfies

$$\mathrm{char}(\mathbb{K}) \nmid D! \tag{2.2.1}$$

where $D = \min_i \max_j \mu^i_j$. For $i = 1, \ldots, k$, let $\mathcal{O}_i \subset \mathfrak{gl}_n$ be a semisimple adjoint orbit satisfying

$$\sum_{i=1}^{k} \mathrm{Tr}\, \mathcal{O}_i = 0. \tag{2.2.2}$$

Let $a^i_1, \dots, a^i_{r_i}$ be the distinct eigenvalues of $\mathcal{O}_i$, and let μ^i_j be the multiplicity of a^i_j. We assume that $\mu^i_1 \geq \cdots \geq \mu^i_{r_i}$. As in Section 2.1, we assume that the multiplicities $\{\mu^i_j\}_j$ determine our fixed partitions μ^i of n which is called the type of $\mathcal{O}_i$, and $\boldsymbol{\mu} := (\mu^1, \dots, \mu^k)$ is called the type of $(\mathcal{O}_1, \dots, \mathcal{O}_k)$.

Definition 2.2.1
The k-tuple $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ of semisimple adjoint orbits is *generic* if the following holds. If $V \subseteq \mathbb{K}^n$ is a subspace stable by some $X_i \in \mathcal{O}_i$ for each i such that

$$\sum_{i=1}^{k} \mathrm{Tr}(X_i|_V) = 0, \tag{2.2.3}$$

then either $V = 0$ or $V = \mathbb{K}^n$.

Let $d := \gcd\{\mu^i_j\}$. We have the following.

LEMMA 2.2.2
Assume (2.2.1). If $d > 1$, then generic k-tuples of adjoint orbits of type $\boldsymbol{\mu}$ do not exist. If $d = 1$, in which case we say that $\boldsymbol{\mu}$ is indivisible, they do.

Proof
In terms of eigenvalues, (2.2.2) is equivalent to $\sum_{i,j} a^i_j \mu^i_j = 0$. If $d > 1$, then it is easy to construct for a fixed basis in $\mathbb{K}^n$ diagonal matrices $X_i \in \mathcal{O}_i$ and $V \subset \mathbb{K}^n$ of dimension n/d such that

$$\sum_i \mathrm{Tr}(X_i|_V) = \sum_{i,j} a^i_j \frac{\mu^i_j}{d} = 0.$$

This shows the first part of our lemma.

Phrased in terms of the eigenvalues of a matrix in $\mathcal{O}_i$, in the indivisible case we are looking for a point in the complement of a hyperplane arrangement in $\mathbb{K}^{\sum r_i - 1}$. (The hyperplanes do not degenerate due to the assumption (2.2.1).)As $\mathbb{K}^{\sum r_i - 1}$ is irreducible, such a point exists. (In the present, additive case we do not have the crutch of a dth torsion point as we did in Lemma 2.1.2.) $\square$

For a k-tuple of semisimple adjoint orbits $(\mathcal{O}_1, \dots, \mathcal{O}_k)$ of type $\boldsymbol{\mu}$ define $\mathcal{V}_{\boldsymbol{\mu}}$ as the subvariety of $\mathfrak{gl}_n(\mathbb{K})^{2g+k}$ of matrices $(A_1, \dots, A_g, B_1, \dots, B_g, X_1, \dots, X_k)$ which satisfy

$$[A_1, B_1] + \cdots + [A_g, B_g] + X_1 + \cdots + X_k = 0, \quad X_i \in \mathcal{O}_i, \tag{2.2.4}$$

where $[\cdot,\cdot]$ is the Lie bracket in $\mathfrak{gl}_n(\mathbb{K})$. As explained in Remark A.2 one can define $\mathcal{V}_{\boldsymbol{\mu}}$ by equations showing that it is indeed an affine variety.

PROPOSITION 2.2.3
If $(\mathcal{O}_1,\ldots,\mathcal{O}_k)$ is generic, then $\mathrm{PGL}_n(\mathbb{K})$ acts set-theoretically freely on $\mathcal{V}_{\boldsymbol{\mu}}$, and for any element $(A_1,B_1,\ldots,A_g,B_g,X_1,\ldots,X_k)\in\mathcal{V}_{\boldsymbol{\mu}}$ there is no nonzero proper subspace of $\mathbb{K}^n$ stable by $A_1,B_1,\ldots,A_g,B_g,X_1,\ldots,X_k$.

Proof
The proof is similar to that of Proposition 2.1.4. □

GL_n acts on $\mathcal{V}_{\boldsymbol{\mu}}$ by simultaneously conjugating the matrices in the defining equation (2.2.4) of $\mathcal{V}_{\boldsymbol{\mu}}$. We can thus construct an affine *quiver variety* of type $\boldsymbol{\mu}$ as the affine GIT quotient

$$\mathcal{Q}_{\boldsymbol{\mu}} := \mathcal{V}_{\boldsymbol{\mu}}//\mathrm{PGL}_n = \mathrm{Spec}(\mathbb{K}[\mathcal{V}_{\boldsymbol{\mu}}]^{\mathrm{PGL}_n}).$$

In Theorem 2.2.5 below we will prove that $\mathcal{Q}_{\boldsymbol{\mu}}$ is isomorphic to a quiver variety associated to a certain comet-shaped quiver; hence its name.

THEOREM 2.2.4
If $(\mathcal{O}_1,\ldots,\mathcal{O}_k)$ is generic, then the variety $\mathcal{Q}_{\boldsymbol{\mu}}$ is nonsingular of dimension $d_{\boldsymbol{\mu}}$. Moreover, $\mathcal{V}_{\boldsymbol{\mu}}//\mathrm{PGL}_n(\mathbb{K})$ is a geometric quotient, and the quotient map $\mathcal{V}_{\boldsymbol{\mu}}\to\mathcal{Q}_{\boldsymbol{\mu}}$ is a principal PGL_n-bundle.

Proof
The proof is similar to that of Theorem 2.1.5. □

We now review the connection between $\mathcal{Q}_{\boldsymbol{\mu}}$ and quiver representations due to Crawley-Boevey [3]. Let $\mathbf{s}=(s_1,\ldots,s_k)\in\mathbb{Z}^k_{\geq 0}$. Put $I=\{0\}\cup\{[i,j]\}_{1\leq i\leq k,1\leq j\leq s_i}$, and let Γ be the quiver with g loops on the central vertex represented as in Figure 1.*

A *dimension vector* for Γ is a collection of nonnegative integers $\mathbf{v}=\{v_i\}_{i\in I}\in\mathbb{Z}^I_{\geq 0}$, and a representation of Γ of dimension $\mathbf{v}$ over $\mathbb{K}$ is a collection of $\mathbb{K}$-linear maps $\phi_{i,j}:\mathbb{K}^{v_i}\to\mathbb{K}^{v_j}$ for each arrow $i\to j$ of Γ that we identify with matrices (using the canonical basis of $\mathbb{K}^r$). Let Ω be a set indexing the edges of Γ. For $\gamma\in\Omega$, let $h(\gamma),t(\gamma)\in I$ denote, respectively, the head and the tail of γ. The algebraic group $\prod_{i\in I}\mathrm{GL}_{v_i}(\mathbb{K})$ acts on the space

$$\mathrm{Rep}_{\mathbb{K}}(\Gamma,\mathbf{v}) := \bigoplus_{\gamma\in\Omega}\mathrm{Mat}_{v_{h(\gamma)},v_{t(\gamma)}}(\mathbb{K})$$

*The picture is from [46].

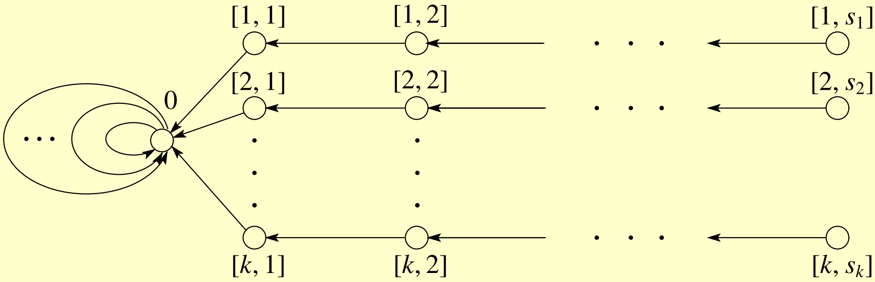

Figure 1.

of representations of dimension $\mathbf{v}$ in the obvious way. As the diagonal center $(\lambda I_{v_i})_{i\in I} \in (\prod_{i\in I} \mathrm{GL}_{v_i}(\mathbb{K}))$ acts trivially, the action reduces to an action of

$$\mathrm{G}_{\mathbf{v}}(\mathbb{K}) := \Big(\prod_{i\in I} \mathrm{GL}_{v_i}(\mathbb{K})\Big)/\mathbb{K}^\times.$$

Clearly two elements of $\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$ are isomorphic if and only if they are $\mathrm{G}_{\mathbf{v}}(\mathbb{K})$-conjugate.

Let $\overline{\Gamma}$ be the *double quiver* of Γ; that is, $\overline{\Gamma}$ has the same vertices as Γ but the edges are given by $\overline{\Omega} := \{\gamma, \gamma^* \mid \gamma \in \Omega\}$ where $h(\gamma^*) = t(\gamma)$ and $t(\gamma^*) = h(\gamma)$. Then via the trace pairing we may identify $\mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v})$ with the cotangent bundle $\mathrm{T}^*\mathrm{Rep}_{\mathbb{K}}(\Gamma, \mathbf{v})$. Define the *moment map*

$$\mu_{\mathbf{v}} : \mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v}) \to M(\mathbf{v}, \mathbb{K})^0, \tag{2.2.5}$$

$$(x_\gamma)_{\gamma\in\overline{\Omega}} \mapsto \sum_{\gamma\in\Omega} [x_\gamma, x_{\gamma^*}], \tag{2.2.6}$$

where

$$M(\mathbf{v}, \mathbb{K})^0 := \Big\{(f_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{gl}_{v_i}(\mathbb{K}) \;\Big|\; \sum_{i\in I} \mathrm{Tr}(f_i) = 0\Big\}$$

is identified with the dual of the Lie algebra of $\mathrm{G}_{\mathbf{v}}(\mathbb{K})$. It is a $\mathrm{G}_{\mathbf{v}}(\mathbb{K})$-equivariant map. We define a bilinear form on $\mathbb{K}^I$ by $\mathbf{a}\cdot\mathbf{b} = \sum_i a_i b_i$. For $\boldsymbol{\xi} = (\xi_i)_i \in \mathbb{K}^I$ such that $\boldsymbol{\xi}\cdot\mathbf{v} = 0$, the element

$$(\xi_i .\mathrm{Id})_i \in \bigoplus_i \mathfrak{gl}_{v_i}(\mathbb{K})$$

is in fact in $M(\mathbf{v}, \mathbb{K})^0$. For such a $\boldsymbol{\xi} \in \mathbb{K}^I$, the affine variety $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})$ is endowed with a $\mathrm{G}_{\mathbf{v}}(\mathbb{K})$-action. We call the affine GIT quotient

$$\mathfrak{M}_{\xi}(\mathbf{v}) := \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})//\mathrm{G}_{\mathbf{v}}(\mathbb{K})$$

the affine *quiver variety*. These and related quiver varieties were considered by many authors including Kronheimer, Lusztig, Nakajima, and Crawley-Boevey (see [31], [38], [42], [2]).

Following [3], we now identify our $\mathcal{Q}_{\boldsymbol{\mu}}$, constructed from a generic k-tuple $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ of type $\boldsymbol{\mu}$, with a certain quiver variety. We define $\mathbf{s}$ as $s_i = l(\mu^i) - 1$ where $l(\lambda)$ denotes the length of a partition λ. Then we define $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$ as $v_0 = n$ and $v_{[i,j]} = n - \sum_{r=1}^j \mu_r^i$ for $[i,j] \in I$. Clearly $n \geq v_{[i,1]} \geq \cdots \geq v_{[i,s_i]}$. We define $\boldsymbol{\xi} \in \mathbb{K}^I$ as $\xi_0 = -\sum_{i=1}^k a_1^i$ and $\xi_{[i,j]} = a_j^i - a_{j+1}^i$. Observe that $\boldsymbol{\xi} \cdot \mathbf{v} = 0$.

For convenience, the symbol $[i,0]$, with $i \in \{1, \ldots, k\}$, will also denote the vertex 0. For a representation $\varphi \in \mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v})$ and an arrow $[i,j] \to [i,j-1] \in \Omega$ with $1 \leq j \leq s_i$, denote by $\varphi_{[i,j]}$ (resp., $\varphi^*_{[i,j]}$) the corresponding linear map $\mathbb{K}^{v_{[i,j]}} \to \mathbb{K}^{v_{[i,j-1]}}$ (resp., $\mathbb{K}^{v_{[i,j-1]}} \to \mathbb{K}^{v_{[i,j]}}$). If $\gamma_1, \ldots, \gamma_g$ are the loops in Ω, then we denote by $\varphi_i : \mathbb{K}^{v_0} \to \mathbb{K}^{v_0}$ the linear map corresponding to γ_i and by φ_i^* the one corresponding to γ_i^*. Following [3, Section 3], we construct a surjective algebraic morphism $\omega : \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi}) \to \mathcal{V}$ which is constant on $\prod_{i \in I - \{0\}} \mathrm{GL}_{v_i}(\mathbb{K})$ orbits. Let $\varphi \in \mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})$. For each $i \in \{1, \ldots, k\}$, define

$$X_i = \varphi_{[i,1]} \varphi^*_{[i,1]} + a_1^i \mathrm{Id} \in \mathrm{Mat}_{v_0}(\mathbb{K}).$$

For $j \in \{1, \ldots, g\}$, put $A_j = \varphi_j$ and $B_j = \varphi_j^*$. We will set

$$\omega(\phi) := (A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k). \tag{2.2.7}$$

To show that $\omega(\phi) \in \mathcal{V}$, recall that μ at the vertex 0 is given by

$$\sum_{j=1}^g [\varphi_j, \varphi_j^*] + \sum_{i=1}^k \varphi_{[i,1]} \varphi^*_{[i,1]} = \xi_0 \mathrm{Id}$$

which gives

$$\sum_{j=1}^g [A_i, B_i] + \sum_{i=1}^k X_i = 0.$$

It is straightforward to see from [3, Section 3] that we have $X_i \in \mathcal{O}_i$ for all $i \in \{1, \ldots, k\}$, from which we deduce that indeed

$$(A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k) \in \mathcal{V}.$$

The map ω induces a bijection between isomorphic classes of simple representations in $\mu_{\mathbf{v}}^{-1}(\boldsymbol{\xi})$ and the $\mathrm{GL}_n(\mathbb{K})$-conjugacy classes of the set of tuples $(A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k) \in \mathcal{V}$; thus we have the following (see [3]).

THEOREM 2.2.5
If $\mathbb{K}=\mathbb{C}$, the bijective morphism $\mathfrak{M}_\xi(\mathbf{v}) \to \mathcal{Q}_{\boldsymbol{\mu}}$ induced by the map ω in (2.2.7) is an isomorphism.

We use this theorem in the proof of the following proposition.

PROPOSITION 2.2.6
Let $\mathbb{K}=\mathbb{C}$. If $(\mathcal{O}_1,\dots,\mathcal{O}_k)$ is generic, then the mixed Hodge structure of the cohomology $H^*(\mathcal{Q}_{\boldsymbol{\mu}})$ of the quiver variety $\mathcal{Q}_{\boldsymbol{\mu}}$ is pure.

Proof
We will construct a nonsingular variety $\mathfrak{M}$ with a smooth map $f:\mathfrak{M}\to\mathbb{C}$ such that for $0\neq\lambda\in\mathbb{C}$ the preimage $f^{-1}(\lambda)\simeq\mathfrak{M}_\xi(\mathbf{v})\simeq\mathcal{Q}_{\boldsymbol{\mu}}$. Moreover, we will define an action of $\mathbb{C}^\times$ on $\mathfrak{M}$ covering the standard action on $\mathbb{C}$ such that $\mathfrak{M}^{\mathbb{C}^\times}$ is projective and the limit point $\lim_{\lambda\to 0}\lambda x$ exists for all $x\in\mathfrak{M}$. Then by Theorem B.1 in Appendix B, $H^*(\mathcal{Q}_{\boldsymbol{\mu}})$ has pure mixed Hodge structure.

Similarly to (2.2.5) we define

$$\mu:\mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma},\mathbf{v})\times\mathbb{C}\to M(\mathbf{v},\mathbb{K})^{\mathbf{0}}$$
$$\big((x_\gamma)_{\gamma\in\overline{\Omega}},z\big)\mapsto\sum_{\gamma\in\Omega}[x_\gamma,x_\gamma^*]-\sum_{i\in I}z\xi_i\,\mathrm{Id}.$$

Now for $\mathbf{n}=(n_i)_{i\in I}\in\mathbb{Z}^I$ satisfying $\sum_{i\in I}n_iv_i=0$ we have a character $\chi_{\mathbf{n}}$ of $\mathrm{G}_{\mathbf{v}}$ given by

$$\chi_{\mathbf{n}}\big((g_i)_{i\in I}\big)=\prod_{i\in I}\det(g_i)^{n_i}.$$

We call

$$\mathbf{n}=(n_i)_{i\in I}\in\mathbb{Z}^I \textit{ generic} \text{ if } \mathbf{n}\cdot\mathbf{v}=0 \text{ and}$$
$$\text{for } \mathbf{v}'\in\mathbb{Z}^I_{\geq 0},\ 0<\mathbf{v}'<\mathbf{v} \text{ implies that } \mathbf{n}\cdot\mathbf{v}'\neq 0. \tag{2.2.8}$$

Because $\boldsymbol{\mu}$ is indivisible we can take a generic $\mathbf{n}$. Now the character $\chi_{\mathbf{n}}$ will give a linearization of the action of $\mathrm{G}_{\mathbf{v}}$ on $\mu^{-1}(0)\times\mathbb{C}$, and so we can consider the GIT quotient

$$\mathfrak{M}:=\mu^{-1}(0)//_{\chi_{\mathbf{n}}}\mathrm{G}_{\mathbf{v}}.$$

We note that $\mathbb{C}^\times$ acts on $\mu^{-1}(0)$ by

$$\lambda\big((x_\gamma)_{\gamma\in\overline{\Omega}},z\big)=\big((\lambda x_\gamma)_{\gamma\in\overline{\Omega}},\lambda^2z\big) \tag{2.2.9}$$

commuting with the $G_{\mathbf{v}}$-action thus descending to an action of $\mathbb{C}^\times$ on $\mathfrak{M}$. Finally, we also have the map $f : \mathfrak{M} \to \mathbb{C}$ given by $f((x_\gamma)_{\gamma\in\overline{\Omega}}, z) = z$. We have the following.

THEOREM 2.2.7
For a generic $\mathbf{n}$ *the variety* $\mathfrak{M}$ *is nonsingular,* f *is a smooth map (in other words a submersion),* $\mathfrak{M}^{\mathbb{C}^\times}$ *is complete, and* $\lim_{\lambda\to 0} \lambda x$ *exists for all* $x \in \mathfrak{M}$.

Proof
The variety $\mathfrak{M}$ is nonsingular because by the Hilbert-Mumford criterion for (semi)stability (see [27]), every semistable point on $\mu^{-1}(0)$ will be stable due to (2.2.8). The map f is a submersion because the derivative $\partial_z \mu = -\sum_{i\in I} \xi_i \mathrm{Id}$ is nonzero.

Construct the affine GIT quotient

$$\mathfrak{M}_0 := \mu^{-1}(0)//_{\chi_0} G_{\mathbf{v}}$$

using the nongeneric $\mathbf{0} \in \mathbb{Z}^I$ weight. Then the natural map $\mathfrak{M} \to \mathfrak{M}_0$ is proper, the $\mathbb{C}^\times$-action (2.2.9) on $\mathfrak{M}_0$ has one fixed point coming from the origin in $\mathrm{Rep}_{\mathbb{K}}(\overline{\Gamma}, \mathbf{v}) \times \mathbb{C}$, and all $\mathbb{C}^\times$ orbits on $\mathfrak{M}_0$ will have this origin in its closure. The remaining statements of the theorem follow. □

To conclude the proof of Proposition 2.2.6 it is enough to note that by the GIT construction we have the natural map $f^{-1}(1) \to \mathfrak{M}_\xi$, which, as a resolution of singularities and $\mathfrak{M}_\xi$ being nonsingular, is an isomorphism. Therefore Theorem B.1 implies the result. □

2.3. *Symmetric functions*

2.3.1. *Partitions and types*

We denote by $\mathscr{P}$ the set of all partitions including the unique partition 0 of 0, by $\mathscr{P}^\times$ the set of nonzero partitions, and by $\mathscr{P}_n$ the set of partitions of n. Partitions λ are denoted by $\lambda = (\lambda_1, \lambda_2, \ldots)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. We will also sometimes write a partition as $(1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, where m_i denotes the multiplicity of i in λ. The *size* of λ is $|\lambda| := \sum_i \lambda_i$; the *length* $l(\lambda)$ of λ is the maximum i with $\lambda_i > 0$.

For two partitions λ and μ, we define $\langle \lambda, \mu\rangle$ as $\sum_i \lambda'_i \mu'_i$, where λ' denotes the dual partition of λ. We put $n(\lambda) := \sum_{i>0}(i-1)\lambda_i$. Then $\langle\lambda,\lambda\rangle = 2n(\lambda) + |\lambda|$. For two partitions $\lambda = (1^{n_1}, 2^{n_2}, \ldots)$ and $\mu = (1^{m_1}, 2^{m_2}, \ldots)$, we denote by $\lambda \cup \mu$ the partition $(1^{n_1+m_1}, 2^{n_2+m_2}, \ldots)$. For a nonnegative integer d and a partition λ, we denote by $d \cdot \lambda$ the partition $(d\lambda_1, d\lambda_2, \ldots)$. The *dominance ordering* for partitions is defined as follows: $\mu \trianglelefteq \lambda$ if and only if $\mu_1 + \cdots + \mu_j \leq \lambda_1 + \cdots + \lambda_j$ for all $j \geq 1$.

For a partition λ, let t_λ in the symmetric group of permutations of $|\lambda|$ letters $\mathcal{S}_{|\lambda|}$ be an element in the conjugacy class of type λ. We denote by z_λ the cardinality of the centralizer of t_λ in $\mathcal{S}_{|\lambda|}$. For two partitions λ, μ such that $|\lambda| = |\mu|$, we denote by χ^λ_μ the value at t_μ of the irreducible character χ^λ of $\mathcal{S}_{|\lambda|}$.

We choose once for all a total order $\geq$ on the set of pairs (d, λ) where $d \in \mathbb{Z}_{>0}$ and $\lambda \in \mathcal{P}^\times$ such that if $d > d'$, then $(d, \lambda) > (d', \mu)$; if $|\lambda| > |\mu|$, then $(d, \lambda) > (d, \mu)$; and if $|\lambda| = |\mu|$, then $(d, \lambda) \geq (d, \mu)$ if λ is larger than μ with respect to the lexicographic order. We denote by $\mathbf{T}$ the set of nonincreasing sequences $\omega = (d_1, \omega^1) \geq (d_2, \omega^2) \geq \cdots \geq (d_r, \omega^r)$, which we will call a *type*. To alleviate the notation we will then omit the symbol $\geq$ and write simply $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r)$. The *size* of a type ω is $|\omega| := \sum_i d_i |\lambda^i|$. We denote by $\mathbf{T}_n$ the set of types of size n. For a type $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r)$, we put $n(\omega) := \sum_i d_i n(\omega^i)$ and $[\omega] := \bigcup_i d_i \cdot \omega^i$, a partition of size $|\omega|$.

As with partitions it is sometimes convenient to consider a type in terms of multiplicities. Given a type ω, let $m_{d,\lambda}(\omega)$ be the multiplicity of (d, λ) in ω, that is, how many times the pair (d, λ) appears in ω. The integers $m_{d,\lambda} \geq 0$ indexed by pairs $(d, \lambda) \in \mathbb{Z}_{>0} \times \mathcal{P}^\times$ determine ω uniquely.

A partition $\lambda = (n_1, \ldots, n_r)$ of n can be seen as the type $\lambda_* := (1, 1^{n_1}) \cdots (1, 1^{n_r}) \in \mathbf{T}_n$, which is the type of a semisimple conjugacy class in the sense of Section 4.1. Similarly, when a multipartition $\boldsymbol{\lambda}$ is considered as a multitype it is denoted by $\boldsymbol{\lambda}_*$.

2.3.2. *Symmetric functions*

Let $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) := \Lambda(\mathbf{x}_1) \otimes_\mathbb{Z} \cdots \otimes_\mathbb{Z} \Lambda(\mathbf{x}_k)$ be the ring of functions separately symmetric in each set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ of infinitely many variables. We will consider elements in $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_\mathbb{Z} \mathbb{Q}(q, t)$, where q and t are two indeterminates or similarly $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_\mathbb{Z} \mathbb{Q}(z, w)$, depending on the situation. To ease the notation we will simply write Λ for the various rings $\Lambda(\mathbf{x}), \Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k), \Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_\mathbb{Z} \mathbb{Q}(q, t)$, $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k) \otimes_\mathbb{Z} \mathbb{Q}(z, w)$, and so on, as long as the context is clear. When considering elements $a_{\boldsymbol{\mu}} \in \Lambda$ indexed by multipartitions $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^r) \in \mathcal{P}^k$, we will always assume that they are homogeneous of degree $(|\mu^1|, \ldots, |\mu^k|)$. Given any family of symmetric functions indexed by partitions $\mu \in \mathcal{P}$ and a multipartition $\boldsymbol{\mu} \in \mathcal{P}^k$ as above, define

$$a_{\boldsymbol{\mu}} := a_{\mu^1}(\mathbf{x}_1) \cdots a_{\mu^k}(\mathbf{x}_k).$$

We will deal with elements of the ring $\Lambda(\mathbf{x}) \otimes_\mathbb{Z} \mathbb{Q}(z, w)$ and their images under two specializations: their *pure part*, $z = 0, w = \sqrt{q}$, and their *Euler specialization*, $z = \sqrt{q}, w = 1/\sqrt{q}$.

Let $\langle \cdot, \cdot \rangle$ be the Hall pairing on $\Lambda(\mathbf{x})$, and extend its definition to $\Lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ by setting

$$\langle a_1(\mathbf{x}_1) \cdots a_k(\mathbf{x}_k), b_1(\mathbf{x}_1) \cdots b_k(\mathbf{x}_k) \rangle = \langle a_1, b_1 \rangle \cdots \langle a_k, b_k \rangle, \tag{2.3.1}$$

for any $a_1, \ldots, a_k; b_1, \ldots, b_k \in \Lambda(\mathbf{x})$ and to formal series by linearity.

Given any family of symmetric functions $A_\lambda(\mathbf{x}_1, \ldots, \mathbf{x}_k; q, t) \in \Lambda$ indexed by partitions with $A_0 = 1$, we extend its definition to types $\omega = (d_1, \omega^1)(d_2, \omega^2) \cdots (d_r, \omega^r) \in \mathbf{T}$ by setting

$$A_\omega(\mathbf{x}_1, \ldots, \mathbf{x}_k; q, t) := \prod_j A_{\omega^j}(\mathbf{x}_1^{d_j}, \ldots, \mathbf{x}_k^{d_j}; q^{d_j}, t^{d_j}).$$

Here $\mathbf{x}^d$ stands for all the variables $x_1, x_2, \ldots$ in $\mathbf{x}$ replaced by $x_1^d, x_2^d, \ldots$. (Technically we are applying the Adams operation ψ_d to A_{ω^j} in the λ-ring Λ.)

We will need the following lemma; $p_\lambda \in \Lambda(\mathbf{x})$ are the *power sums* symmetric functions.

LEMMA 2.3.1
Let $\lambda \in \mathcal{P}_n$, and let d be a positive integer such that $d \mid n$. Then

$$\langle p_{(d^{n/d})}, h_\lambda \rangle = \begin{cases} \frac{(n/d)!}{\prod_i \rho_i!} & \text{if } \lambda = d \cdot \rho \text{ for some } \rho = (\rho_1, \rho_2, \ldots) \in \mathcal{P}_{n/d}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof
For a finite group G, let $\langle \cdot, \cdot \rangle_G$ denote the standard inner product on class functions of G. Using the Frobenius characteristic map [40, Chapter I, Part 7] we have, for any two partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ of size n,

$$\langle p_\mu, h_\lambda \rangle = z_\mu \langle \delta_\mu, \mathrm{Ind}_{\mathcal{S}_\lambda}^{\mathcal{S}_n}(1) \rangle_{\mathcal{S}_n},$$

$\delta_\mu(\sigma) = 1$, if $\sigma \in \mathcal{S}_n$ has cycle type μ and $\delta_\mu(\sigma) = 0$ otherwise, and $\mathcal{S}_\lambda := \mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_r} \subseteq \mathcal{S}_n$.

Hence, by Frobenius reciprocity,

$$\langle p_\mu, h_\lambda \rangle = z_\mu \langle \mathrm{Res}_{\mathcal{S}_\lambda}^{\mathcal{S}_n} \delta_\mu, 1 \rangle_{\mathcal{S}_\lambda}.$$

The only nonzero terms contributing to the sum implicit in the right-hand side are those elements of $\mathcal{S}_\lambda$ with cycle type $(\mu^1, \ldots, \mu^r)$ with $|\mu^i| = \lambda_i$ and $\bigcup_i \mu^i = \mu$. If $\mu = (d^{n/d})$, this forces $d \mid \lambda_i$ and $\mu^i = (d^{\rho_i})$, where $\rho_i := \lambda_i / d$, and the claim follows. □

2.3.3. Exp *and* Log
We will use the maps Exp and Log of [22] extended to Λ. The general context is that of λ-rings (see [15]), but the following discussion will suffice for us. For $V \in T\Lambda[[T]]$ let

$$\mathrm{Exp} : T\Lambda[[T]] \to 1 + T\Lambda[[T]], \tag{2.3.2}$$

$$V \mapsto \exp\Big(\sum_{d\ge 1}\frac{1}{d}V(\mathbf{x}_1^d,\dots,\mathbf{x}_k^d,q^d,t^d,T^d)\Big). \tag{2.3.3}$$

The map Exp is related to the Cauchy kernel

$$C(\mathbf{x}) := \prod_i (1-x_i)^{-1} \tag{2.3.4}$$

by

$$\operatorname{Exp}(X) = C(\mathbf{x}), \quad X := x_1 + x_2 + \cdots = m_{(1)}(\mathbf{x})$$

($m_\lambda(\mathbf{x}) \in \Lambda(\mathbf{x})$ is the monomial symmetric function). It has an inverse Log defined as follows. Given $F \in 1 + T\Lambda[[T]]$, let $U_n \in \Lambda$ be the coefficients in the expansion

$$\log(F) =: \sum_{n\ge 1} U_n(\mathbf{x}_1,\dots,\mathbf{x}_k;q,t)\frac{T^n}{n}.$$

Define

$$V_n(\mathbf{x}_1,\dots,\mathbf{x}_k;q,t) := \frac{1}{n}\sum_{d\mid n}\mu(d)U_{n/d}(\mathbf{x}_1^d,\dots,\mathbf{x}_k^d;q^d,t^d), \tag{2.3.5}$$

where μ is the ordinary Möbius function; then

$$\operatorname{Log}(F) := \sum_{n\ge 1} V_n(\mathbf{x}_1,\dots,\mathbf{x}_k;q,t)T^n.$$

To simplify the discussion we now restrict to the case of $k=1$, but everything extends easily to the general case. Suppose that $A_\lambda(\mathbf{x};q,t) \in \Lambda$ is a sequence of symmetric functions indexed by partitions with $A_0 = 1$. We want an expression for $V_n \in \Lambda$ in

$$\sum_{n\ge 1} V_n T^n := \operatorname{Log}\Big(\sum_{\lambda\in\mathcal{P}} A_\lambda T^{|\lambda|}\Big).$$

We first compute

$$\sum_{n\ge 1} U_n \frac{T^n}{n} := \log\Big(\sum_{\lambda\in\mathcal{P}} A_\lambda T^{|\lambda|}\Big),$$

where U_n and V_n are related by (2.3.5). By the multinomial theorem we have

$$\frac{U_n}{n} = \sum_{m_\lambda}(-1)^{m-1}(m-1)!\prod_\lambda \frac{A_\lambda^{m_\lambda}}{m_\lambda!}, \tag{2.3.6}$$

where $m := \sum_\lambda m_\lambda$ and the sum is over all sequences $\{m_\lambda\}_{\lambda\in\mathcal{P}^\times}$ of nonnegative integers such that

$$\sum_\lambda m_\lambda|\lambda| = n.$$

We find then

$$V_n = \sum \frac{\mu(d)}{d}(-1)^{m_d-1}(m_d-1)!\prod_\lambda \frac{A_\lambda(\mathbf{x}_1^d,\ldots,\mathbf{x}_k^d;q^d,t^d)^{m_{d,\lambda}}}{m_{d,\lambda}!},$$

where the sum is over all sequences of nonnegative integers $m_{d,\lambda}$ indexed by pairs $(d,\lambda)\in\mathbb{Z}_{>0}\times\mathcal{P}^\times$ satisfying

$$\sum_\lambda m_{d,\lambda}d|\lambda| = n, \quad m_d := \sum_\lambda m_{d,\lambda}.$$

Alternatively, we may consider not collecting equal terms when expanding the logarithm to obtain

$$V_n = \sum \frac{\mu(d)}{d}\frac{(-1)^{r-1}}{r}A_{\lambda^1}(q^d)\cdots A_{\lambda^r}(q^d), \tag{2.3.7}$$

where the sum is over $\lambda^1,\lambda^2,\ldots\in\mathcal{P}^\times$ and $d\in\mathbb{Z}_{>0}$ such that

$$n = d\sum_j|\lambda^j|.$$

Finally, we may also rewrite the expression for V_n as a sum over types ω:

$$V_n = \sum_{|\omega|=n} C_\omega^0 A_\omega, \tag{2.3.8}$$

so that

$$\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}} A_\lambda T^{|\lambda|}\Big) = \sum_\omega C_\omega^0 A_\omega T^{|\omega|}, \tag{2.3.9}$$

where $C_\omega^0 = 0$ unless ω is concentrated in some degree d; that is, $\omega = (d,\omega^1)(d,\omega^2)\cdots(d,\omega^r)$, in which case,

$$C_\omega^0 = \frac{\mu(d)}{d}(-1)^{r-1}\frac{(r-1)!}{\prod_\lambda m_{d,\lambda}(\omega)!}. \tag{2.3.10}$$

Remark 2.3.2

The formal power series $\sum_{n\geq 0} a_n T^n$ with $a_n\in\Lambda$ that we will consider in what follows will all have a_n homogeneous of degree n. Hence we will typically scale the variables of Λ by $1/T$ and eliminate T altogether.

Remark 2.3.3
Note also the following useful fact. If we write

$$\log\Big(\sum_{\lambda\in\mathcal{P}} A_\lambda(\mathbf{x})T^{|\lambda|}\Big) = \sum_\mu U_\mu(q,t)m_\mu(\mathbf{x}),$$
$$\operatorname{Log}\Big(\sum_{\lambda\in\mathcal{P}} A_\lambda(\mathbf{x})T^{|\lambda|}\Big) = \sum_\mu V_\mu(q,t)m_\mu(\mathbf{x}),$$

where $m_\mu(\mathbf{x})$ are the monomial symmetric functions, then it is easy to check that

$$V_\mu(q,t) := \frac{1}{n}\sum_{d|\mu}\mu(d)U_{\mu/d}(q^d,t^d), \tag{2.3.11}$$

where $d \mid \mu$ means that d divides every part μ_i of μ and $\mu/d := (\mu_1/d, \mu_2/d, \ldots)$. In particular, if μ is indivisible, then the sum on the right-hand side consists of only the $d = 1$ term and $U_\mu = V_\mu/n$. (This is particularly useful for computations.)

2.3.4. Macdonald and Hall-Littlewood symmetric functions: Green polynomials
For a partition λ, let $\tilde{H}_\lambda(\mathbf{x};q,t) \in \Lambda(\mathbf{x})\otimes_{\mathbb{Z}}\mathbb{Q}(q,t)$ be the *Macdonald symmetric function* defined in [14, Chapter I.11]. We collect in this section some basic properties of these functions that we will need.

We have the duality

$$\tilde{H}_\lambda(\mathbf{x};q,t) = \tilde{H}_{\lambda'}(\mathbf{x};t,q) \tag{2.3.12}$$

(see [14, Corollary 3.2]). We define the (transformed) *Hall-Littlewood symmetric function* as

$$\tilde{H}_\lambda(\mathbf{x};q) := \tilde{H}_\lambda(\mathbf{x};0,q). \tag{2.3.13}$$

In the notation just introduced, $\tilde{H}_\lambda(\mathbf{x};q)$ is then the pure part of $\tilde{H}_\lambda(\mathbf{x};z^2,w^2)$.

Define the (q,t)*-Kotska polynomials* $\tilde{K}_{\nu\lambda}(q,t)$ by

$$\tilde{H}_\lambda(\mathbf{x};q,t) = \sum_\nu \tilde{K}_{\nu\lambda}(q,t)s_\nu(\mathbf{x}), \tag{2.3.14}$$

where s_ν are the Schur symmetric functions. These are (q,t)-generalizations of the $\tilde{K}_{\nu\lambda}(q)$ Kostka-Foulkes polynomial (see [40, Chapter III, (7.11)]), which are obtained as $q^{n(\lambda)}K_{\nu\lambda}(q^{-1}) = \tilde{K}_{\nu\lambda}(q) = \tilde{K}_{\nu\lambda}(0,q)$, that is, by taking their pure part. In particular,

$$\tilde{H}_\lambda(\mathbf{x};q) = \sum_\nu \tilde{K}_{\nu\lambda}(q)s_\nu(\mathbf{x}). \tag{2.3.15}$$

For partitions λ, τ we define the *Green polynomial*

$$Q_\lambda^\tau(q) = \sum_\nu \chi_\lambda^\nu \tilde{K}_{\nu\tau}(q), \tag{2.3.16}$$

where $\tilde{K}_{\nu\tau}(q)$ is the Kostka-Foulkes polynomial (2.3.14).

For two partitions $\nu, \lambda \in \mathcal{P}_n$, we have (see [40, p. 363]) the Euler specialization

$$\tilde{K}_{\nu\lambda}(q, q^{-1}) = q^{-n(\lambda)} K_{\nu\lambda}(q, q) = q^{-n(\lambda)} H_\lambda(q) \sum_\rho \frac{\chi_\rho^\nu \chi_\rho^\lambda}{z_\rho \prod_i (1 - q^{\rho_i})}, \tag{2.3.17}$$

where $H_\lambda(q) := \prod_{s\in\lambda}(1 - q^{h(s)})$ is the *hook polynomial* (see [40, Chapter I, Part 3, Example 2]).

If $\mathbf{y} = \{y_1, y_2, \ldots\}, \mathbf{x} = \{x_1, x_2, \ldots\}$ are two sets of infinitely many variables, we denote by $\mathbf{xy}$ the set of variables $\{x_i y_j\}_{i,j}$.

LEMMA 2.3.4
Under the Euler specialization,

$$\tilde{H}_\lambda(\mathbf{x}; q, q^{-1}) = q^{-n(\lambda)} H_\lambda(q) s_\lambda(\mathbf{xy}),$$

where $y_i = q^{i-1}$.

Proof
With the specialization $y_i = q^{i-1}$ we get $p_\rho(\mathbf{y}) = \prod_i (1 - q^{\rho_i})^{-1}$. Hence by (2.3.17)

$$\begin{aligned}
\tilde{H}_\lambda(\mathbf{x}; q, q^{-1}) &= q^{-n(\lambda)} H_\lambda(q) \sum_{\rho,\nu} \frac{\chi_\rho^\nu \chi_\rho^\lambda}{z_\rho} p_\rho(\mathbf{y}) s_\nu(\mathbf{x}) \\
&= q^{-n(\lambda)} H_\lambda(q) \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho(\mathbf{y}) \sum_\nu \chi_\rho^\nu s_\nu(\mathbf{x}) \\
&= q^{-n(\lambda)} H_\lambda(q) \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho(\mathbf{y}) p_\rho(\mathbf{x}) \\
&= q^{-n(\lambda)} H_\lambda(q) \sum_\rho z_\rho^{-1} \chi_\rho^\lambda p_\rho(\mathbf{xy}) \\
&= q^{-n(\lambda)} H_\lambda(q) s_\lambda(\mathbf{xy}).
\end{aligned}$$

□

For two types $\omega = (d_1, \omega^1) \cdots (d_r, \omega^r)$ and $\tau = (\delta_1, \tau^1) \cdots (\delta_s, \tau^s)$, write $\omega \sim \tau$ if $r = s$ and for each $i = 1, 2, \ldots, r$, $d_i = \delta_i$ and $|\omega^i| = |\tau^i|$.

For two types ω and τ, put
$\chi_\tau^\omega := \prod_i \chi_{\tau^i}^{\omega^i}$ if $\omega \sim \tau$, and $\chi_\tau^\omega = 0$ otherwise,

$Q^{\omega}_{\tau}(q) := \prod_i Q^{\omega^i}_{\tau^i}(q^{d_i})$ if $\omega \sim \tau$, and $Q^{\omega}_{\tau}(q) = 0$ otherwise,
$\tilde{K}_{\tau\omega}(q) := \prod_i \tilde{K}_{\tau^i\omega^i}(q^{d_i})$ if $\omega \sim \tau$, and $\tilde{K}_{\tau\omega}(q) = 0$ otherwise.

Note that formulas (2.3.16) and (2.3.15) extend to types, namely, $Q^{\omega}_{\tau}(q) = \sum_{\nu} \chi^{\nu}_{\tau} \tilde{K}_{\nu\omega}(q)$ and $\tilde{H}_{\omega}(\mathbf{x}; q) = \sum_{\tau} \tilde{K}_{\tau\omega}(q) s_{\tau}(\mathbf{x})$, where $\tau, \omega, \nu \in \mathbf{T}$.

LEMMA 2.3.5
For $\alpha, \beta \in \mathbf{T}$, put

$$A(\alpha, \beta) := \sum_{\tau} \frac{z_{[\tau]} \chi^{\alpha}_{\tau}}{z_{\tau}} \sum_{\{\nu | [\nu] = [\tau]\}} \frac{Q^{\beta}_{\nu}(q)}{z_{\nu}},$$

where the sums are over types. Then

$$A(\alpha, \beta) = \langle s_{\alpha}(\mathbf{x}), \tilde{H}_{\beta}(\mathbf{x}; q) \rangle,$$

where for a partition λ, $s_{\lambda}(\mathbf{x}) \in \Lambda(\mathbf{x})$ is the Schur symmetric function and $\tilde{H}_{\lambda}(\mathbf{x}; q)$ the transformed Hall-Littlewood symmetric function (2.3.15).

Proof
For $\omega \in \mathbf{T}$, define

$$a_{\omega}(\mathbf{x}) := \sum_{\tau} \chi^{\omega}_{\tau} \frac{p_{\tau}(\mathbf{x})}{z_{\tau}}$$

and

$$b_{\omega}(\mathbf{x}) := \sum_{\nu} Q^{\omega}_{\nu}(q) \frac{p_{\nu}(\mathbf{x})}{z_{\nu}},$$

where $\{p_{\lambda}(\mathbf{x})\}_{\lambda \in \mathcal{P}}$ is the family of power symmetric functions which satisfies for two partitions $\lambda, \tau \in \mathcal{P}$,

$$\langle p_{\lambda}(\mathbf{x}), p_{\tau}(\mathbf{x}) \rangle = \delta_{\lambda,\tau} z_{\tau}.$$

For a type $\omega \in \mathbf{T}$, we have $p_{\omega}(\mathbf{x}) := \prod_i p_{\omega^i}(\mathbf{x}^{d_i}) = p_{[\omega]}(\mathbf{x})$. Therefore, for $\alpha, \beta \in \mathbf{T}$, we have $\langle p_{\alpha}(\mathbf{x}), p_{\beta}(\mathbf{x}) \rangle = \delta_{[\alpha],[\beta]} z_{[\alpha]}$. Hence

$$\begin{aligned} \langle a_{\alpha}(\mathbf{x}), b_{\beta}(\mathbf{x}) \rangle &= \sum_{\tau} \sum_{\nu} \chi^{\alpha}_{\tau} Q^{\beta}_{\nu}(q) \frac{\langle p_{\tau}(\mathbf{x}), p_{\nu}(\mathbf{x}) \rangle}{z_{\tau} z_{\nu}} \\ &= \sum_{\tau} \sum_{\nu} \chi^{\alpha}_{\tau} Q^{\beta}_{\nu}(q) \delta_{[\tau],[\nu]} \frac{z_{[\tau]}}{z_{\tau} z_{\nu}} \\ &= A(\alpha, \beta). \end{aligned}$$

Recall that for a partition $\lambda \in \mathcal{P}$, we have

$$s_\lambda(\mathbf{x}) = \sum_\tau \chi^\lambda_\tau \frac{p_\tau(\mathbf{x})}{z_\tau}.$$

Hence for a type $\omega \in \mathbf{T}$, we have

$$s_\omega(\mathbf{x}) = \sum_\tau \chi^\omega_\tau \frac{p_\tau(\mathbf{x})}{z_\tau} = a_\omega(\mathbf{x}).$$

Hence we may write

$$\begin{aligned} b_\omega(\mathbf{x}) &= \sum_\nu \sum_\tau \chi^\tau_\nu \tilde{K}_{\tau\omega}(q) \frac{p_\nu(\mathbf{x})}{z_\nu} \\ &= \sum_\tau \sum_\nu \chi^\tau_\nu \tilde{K}_{\tau\omega}(q) \frac{p_\nu(\mathbf{x})}{z_\nu} \\ &= \sum_\tau \tilde{K}_{\tau\omega}(q) s_\tau(\mathbf{x}) = \tilde{H}_\omega(\mathbf{x}; q). \end{aligned}$$

□

LEMMA 2.3.6
Let $\lambda \in \mathcal{P}$. With the specialization $y_i = q^{i-1}$, we have

$$h_\lambda(\mathbf{xy}) = (-1)^{|\lambda|} q^{n(\lambda_*)} \mathcal{H}^0_{\lambda_*}(0, \sqrt{q})\tilde{H}_{\lambda_*}(\mathbf{x}; q), \tag{2.3.18}$$

where $\mathcal{H}^0_\lambda(z, w)$ is the genus 0 hook function.

Proof
We need to prove that for $m \in \mathbb{Z}_{>0}$,

$$h_m(\mathbf{xy}) = (-1)^m q^{n(1^m)} \mathcal{H}^0_{(1^m)}(0, \sqrt{q})\tilde{H}_{(1^m)}(\mathbf{x}; q).$$

In the language of plethystic substitution (we use the notation of [14]), the transformed Hall-Littlewood (2.3.13) $\tilde{H}_\mu(\mathbf{x}; q)$ equals

$$\tilde{H}_\mu(\mathbf{x}; q) = q^{n(\mu)} b_\mu(q^{-1}) P_\mu\Big[\frac{X}{1-q^{-1}}; q^{-1}\Big]$$

where $P_\mu(\mathbf{x}; q)$ is the Hall-Littlewood symmetric function defined in [40]. Since $\mathcal{H}^0_\mu(0, \sqrt{q}) = q^{-\langle\mu,\mu\rangle} b_\mu(q^{-1})^{-1}$, we have

$$(-1)^{|\mu|} q^{n(\mu)} \mathcal{H}^0_\mu(0, \sqrt{q})\tilde{H}_\mu(\mathbf{x}; q) = (-1)^{|\mu|} q^{-|\mu|} P_\mu\Big[\frac{X}{1-q^{-1}}; q^{-1}\Big] \tag{2.3.19}$$

$$= (-q^{-1})^{|\mu|} P_\mu\Big[-\frac{qX}{1-q}; q^{-1}\Big]. \tag{2.3.20}$$

On the other hand, from [40, Chapter VI, (4.8)] we have

$$(-q^{-1})^m P_{(1^m)}[-qX; q^{-1}] \\ = (-q^{-1})^m e_m[-qX] = (-q^{-1})^m s_{(1^m)}[-qX] = s_{(m^1)}(\mathbf{x}) = h_m(\mathbf{x}).$$

Since for any symmetric function u, we have $u(\mathbf{xy}) = u[X/(1-q)]$, we deduce that

$$h_m(\mathbf{xy}) = (-q^{-1})^m P_{(1^m)}\Big[-\frac{qX}{1-q}; q^{-1}\Big].$$

The lemma follows thus from formula (2.3.19). □

2.3.5. Genus g hook function

Given a partition $\lambda \in \mathcal{P}_n$ we define the genus g *hook function* $\mathcal{H}_\lambda(z,w)$ by

$$\mathcal{H}_\lambda(z,w) := \prod_{s\in\lambda} \frac{(z^{2a(s)+1} - w^{2l(s)+1})^{2g}}{(z^{2a(s)+2} - w^{2l(s)})(z^{2a(s)} - w^{2l(s)+2})},$$

where the product is over all cells s of λ with $a(s)$ and $l(s)$ its arm and leg length, respectively. For details on the hook function we refer the reader to [22].

Remark 2.3.7

Note that $\mathcal{H}_\lambda(z,w)$ is a rational function of z^2 and w^2 when $g = 0$.

We have

$$\mathcal{H}_\lambda(z,w) = \mathcal{H}_{\lambda'}(w,z) \qquad \text{and} \qquad \mathcal{H}_\lambda(-z,-w) = \mathcal{H}_\lambda(z,w). \tag{2.3.21}$$

The pure part of $\mathcal{H}_\lambda$ is

$$\mathcal{H}_\lambda(0,\sqrt{q}) = \prod_{a=0} \frac{q^{g(2l+1)}}{q^l(q^{l+1}-1)} \prod_{a\neq 0} q^{(g-1)(2l+1)} \\ = \frac{q^{(g-1)(2n(\lambda)+|\lambda|)}}{\prod_{i\geq 1}(1-1/q)(1-1/q^2)\cdots(1-1/q^{m_i})},$$

where m_i is the multiplicity of i in λ. Hence

$$\mathcal{H}_\lambda(0,\sqrt{q}) = \frac{q^{g\langle\lambda,\lambda\rangle}}{a_\lambda(q)}, \tag{2.3.22}$$

where $a_\lambda(q)$ is the cardinality of the centralizer of a unipotent element of $\mathrm{GL}_n(\mathbb{F}_q)$ with Jordan form of type λ (see [40, Chapter IV, (2.7)]). In particular, when $g = 0$, we have $\mathcal{H}_{(1^n)}(0,\sqrt{q}) = 1/|\mathrm{GL}_n(\mathbb{F}_q)|$.

It is also not difficult to verify that the Euler specialization of $\mathcal{H}_\lambda$ is

$$\mathcal{H}_\lambda(\sqrt{q}, 1/\sqrt{q}) = \left(q^{-\frac{1}{2}\langle\lambda,\lambda\rangle} H_\lambda(q)\right)^{2g-2}. \tag{2.3.23}$$

2.3.6. Cauchy functions

As in the introduction let

$$\Omega(z,w) := \sum_{\lambda\in\mathcal{P}} \mathcal{H}_\lambda(z,w) \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2).$$

By (2.3.12) and (2.3.21), we have

$$\Omega(z,w) = \Omega(w,z) \qquad \text{and} \qquad \Omega(-z,-w) = \Omega(z,w). \tag{2.3.24}$$

LEMMA 2.3.8

With the specialization $y_i = q^{i-1}$, we have

$$\Omega\Big(\sqrt{q}, \frac{1}{\sqrt{q}}\Big) = \sum_{\lambda\in\mathcal{P}} q^{(1-g)|\lambda|}\left(q^{-n(\lambda)} H_\lambda(q)\right)^{2g+k-2} \prod_{i=1}^{k} s_\lambda(\mathbf{x}_i\mathbf{y}).$$

Proof

The proof follows from Lemma 2.3.4 and (2.3.23). □

For $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^k) \in \mathcal{P}^k$, we let

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) := (z^2-1)(1-w^2)\langle \operatorname{Log}\Omega(z,w), h_{\boldsymbol{\mu}}\rangle. \tag{2.3.25}$$

By (2.3.24), we have

$$\mathbb{H}_{\boldsymbol{\mu}}(z,w) = \mathbb{H}_{\boldsymbol{\mu}}(w,z) \qquad \text{and} \qquad \mathbb{H}_{\boldsymbol{\mu}}(-z,-w) = \mathbb{H}_{\boldsymbol{\mu}}(z,w). \tag{2.3.26}$$

We may recover $\Omega(z,w)$ from the $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$'s by the formula

$$\Omega(z,w) = \operatorname{Exp}\Big(\sum_{\boldsymbol{\mu}\in\mathcal{P}^k} \frac{\mathbb{H}_{\boldsymbol{\mu}}(z,w)}{(z^2-1)(1-w^2)} m_{\boldsymbol{\mu}}\Big). \tag{2.3.27}$$

If we want to work with partitions of length at most $l_1, \ldots, l_k$, we can specialize the variables $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots)$ in formula (2.3.27) to say $(u_{i,1}, u_{i,2}, \ldots, u_{i,l_i}, 0, 0, \ldots)$ for some new independent variables $u_{i,j}$. Indeed, this specialization takes any $m_{\boldsymbol{\mu}}$ with $l(\mu^i) > l_i$ for some i to zero.

For instance, if $k=1$ the specialization $\mathbf{x}=(x_1,x_2,\dots)$ to $(T,0,0,\dots)$ in formula (2.3.27) gives

$$\sum_{\lambda} \mathcal{H}_{\lambda}(z,w)T^{|\lambda|} = \operatorname{Exp}\Big(\sum_{n\geq 1} \frac{\mathbb{H}_{(n)}(z,w)}{(z^2-1)(1-w^2)}T^n\Big), \tag{2.3.28}$$

since for a partition μ of n, we have

$$m_{\mu}(T,0,0,\dots) = \begin{cases} T^n & \text{if } \mu=(n), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{H}_{\mu}(T,0,0,\dots;q,t) = \tilde{K}_{(n)\mu}(q,t)T^n = T^n.$$

The identity $\tilde{K}_{(n)\mu}(q,t)=1$ follows from [14, formula (16)]. Comparing with the left-hand side of [22, (3.5.8)] we see that, in the notation of that paper, $\mathbb{H}_{(n)}=\bar{H}_n$.

2.4. *Mixed Hodge polynomials and polynomial count varieties*

We refer the reader to [22] for details on this section. For a complex quasi-projective algebraic variety X we let $H(X;x,y,z)$ and $H_c(X;x,y,z)$ be its mixed Hodge polynomial and compactly supported mixed Hodge polynomial, respectively. They satisfy the following properties. The specialization $H(X;1,1,z)$ is the Poincaré polynomial $P(X;z):=\sum_k \dim H^k(X,\mathbb{C})z^k$ and similarly with H_c and P_c. The E-polynomial of X is $E(X;x,y)=H_c(X;x,y,-1)=\sum_{i,j,k}(-1)^k h_c^{i,j;k}(X)x^iy^j$. The value $E(X;1,1)$ is the compact Euler characteristic $\sum_i(-1)^i \dim H_c^i(X,\mathbb{C})$, which is equal to the ordinary Euler characteristic by [32]. We denote it by $E(X)$.

If X is nonsingular of pure dimension d, that is, if X is the disjoint union of its irreducible components all nonsingular of same dimension d, then Poincaré duality implies that

$$h_c^{d-i,d-j;2d-k}(X) = h^{i,j;k}(X), \quad \text{for all } i,j,k,$$

or, equivalently,

$$H_c(X;x,y,t) = (xyt^2)^d H(X;x^{-1},y^{-1},t^{-1}). \tag{2.4.1}$$

We recall the result of Katz given in the appendix to [22].

THEOREM 2.4.1
Assume that $X/\mathbb{C}$ is polynomial-count with counting polynomial $P_X\in\mathbb{Z}[t]$. Then

$$E(X;x,y) = P_X(xy).$$

If X is polynomial-count, then we put $E(X;q) := E(X;\sqrt{q},\sqrt{q})$ and just call it the E-polynomial of X to simplify. Note that, in this case, $\sum_k (-1)^k h_c^{i,j;k}(X) = 0$ if $i \neq j$.

PROPOSITION 2.4.2
Assume that X is polynomial-count and that the mixed Hodge structure on the compactly supported cohomology $H_c^*(X)$ is pure. Then

$$E(X;q) = P_c(X;\sqrt{q}).$$

Proof
By the above remark we have $\sum_k (-1)^k h_c^{i,j;k}(X) = 0$ if $i \neq j$. Since the only nonzero term of this sum is when $k = i + j$, by the purity assumption, we get that $(-1)^{i+j} \times h_c^{i,j;i+j}(X) = 0$ if $i \neq j$. Hence the nonzero mixed Hodge numbers are all of the form $h_c^{i,i;2i}(X)$ and $E(X;q) = \sum_i h_c^{i,i;2i}(X) q^i$. □

2.5. Complex characters of $\mathrm{GL}_n(\mathbb{F}_q)$ *and* $\mathfrak{gl}_n(\mathbb{F}_q)$
Here we recall how to construct the irreducible characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathfrak{gl}_n(\mathbb{F}_q)$ using the Deligne-Lusztig theory. We choose a prime ℓ which is invertible in the finite field $\mathbb{F}_q$. Since Deligne-Lusztig theory uses ℓ-adic cohomology it will be more convenient to work with $\overline{\mathbb{Q}}_\ell$-characters instead of complex characters. Note that there is a noncanonical isomorphism over $\mathbb{Q}$ between the two fields $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$. The counting formulas (1.2.2) and (1.3.2), which involve character values, do not depend on the choice of such an isomorphism.

For a finite group H, we denote by $\mathrm{Irr}(H)$ the set of irreducible complex characters of H.

2.5.1. Generalities
Let $n \in \mathbb{Z}_{>0}$, and we put $\mathrm{GL}_n = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $\mathfrak{gl}_n = \mathfrak{gl}_n(\overline{\mathbb{F}}_q)$. Unless specified, here the letter G will always denote a Levi subgroup of a parabolic subgroup of GL_n, that is, a subgroup of GL_n which is GL_n-conjugate to some $H = \prod_{i=1}^r \mathrm{GL}_{n_i}$ where $\sum_{i=1}^r n_i = n$. For short we will say that G is a *Levi subgroup* of GL_n. If $n_i = 1$ for all i, then G is a maximal torus of GL_n. The Lie algebra of G is isomorphic to the Lie algebra $\mathscr{H} = \bigoplus_i \mathfrak{gl}_{n_i}$ of H. Let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation: we have $\mathrm{Ad}(g)x = gxg^{-1}$ for $g \in G$ and $x \in \mathfrak{g}$. For $g \in G$, we denote by g_s the semisimple part of g and by g_u the unipotent part of g; we have $g = g_s g_u = g_u g_s$. If $x \in \mathfrak{g}$, we denote by x_s and x_n, respectively, the semisimple part of x and the nilpotent part of x. We then have $x = x_s + x_n$ with $[x_s, x_n] = 0$. Let $x \in \mathfrak{g}$, and let K be a subgroup of G; we denote by $C_K(x)$ the centralizer of x in K with respect to the adjoint action. If $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, we denote by $C_{\mathfrak{k}}(x)$ the centralizer of

x in $\mathfrak{k}$; that is, $C_{\mathfrak{k}}(x) = \{y \in \mathfrak{k} | [x, y] = 0\}$. We denote by Z_G the center of G and by $z(\mathfrak{g})$ the center of $\mathfrak{g}$, respectively. If L is a Levi subgroup of G (i.e., a Levi subgroup of GL_n which is contained in G), then we denote by $W_G(L)$ the finite group $N_G(L)/L$ where $N_G(L)$ denotes the normalizer of L in G.

Finally, we denote by G_{uni} (resp., $\mathfrak{g}_{\mathrm{nil}}$) the subvariety of unipotent elements of G (resp., the subvariety of nilpotent elements of $\mathfrak{g}$).

2.5.2. *Frobenius endomorphisms*

We denote by $F : \mathrm{GL}_n \to \mathrm{GL}_n$ and by $F : \mathfrak{gl}_n \to \mathfrak{gl}_n$ the *standard* Frobenius endomorphisms $(a_{ij}) \mapsto (a_{ij}^q)$. Assume that G is F-stable. Then $\mathfrak{g} \subset \mathfrak{gl}_n$ is F-stable, and the restrictions $F : G \to G$, $F : \mathfrak{g} \to \mathfrak{g}$ are Frobenius endomorphisms on G and $\mathfrak{g}$. We also have $F(\mathrm{Ad}(g)x) = \mathrm{Ad}(F(g))F(x)$; therefore, Ad induces an action of the finite group G^F on the finite Lie algebras $\mathfrak{g}^F$. Since G is conjugate to H, the Frobenius endomorphism $F : G \to G$ corresponds to some $F' : H \to H$, for which we write $(G, F) \simeq (H, F')$. We then have $G^F \simeq H^{F'}$. The Frobenius endomorphism F' is of the form $wF : H \to H$, $h \mapsto wF(h)w^{-1}$ for some $w \in N_{\mathrm{GL}_n}(H)$. We say that an F-stable maximal torus $T \subset G$ of rank n is *split* if there exists an isomorphism $T \simeq (\overline{\mathbb{F}}_q^{\times})^n$ defined over $\mathbb{F}_q$. The $\mathbb{F}_q$*-rank* of an F-stable maximal torus of G is defined to be the rank of its maximal split subtori. An F-stable maximal torus of G is said to be G*-split* if it is maximally split in G. The G-split F-stable maximal tori of G are those which are contained in some F-stable Borel subgroup of G.

2.5.3. *F-conjugacy classes*

Let T be an F-stable maximal torus of G. The Frobenius F acts on the finite group $W_G(T)$, and we say that two elements $w, v \in W_G(T)$ are F-conjugate if there exists $h \in W_G(T)$ such that $w = hv(F(h))^{-1}$. Then we can parameterize the G^F-conjugacy classes of the F-stable maximal tori of G by the F-conjugacy classes of $W_G(T)$ as follows. Let T' be an F-stable maximal torus of G. Then there exists $g \in G$ such that $T' = gTg^{-1}$; that is, $g^{-1}F(g) \in N_G(T)$. There is a well-defined map which sends the G^F-conjugacy class of T' to the F-conjugacy class of the image w of $g^{-1}F(g)$ in $W_G(T)$; moreover, this map is bijective. This parameterization depends only on the G^F-conjugacy class of T. If $w \in W_G(T)$, then we will denote by T_w an arbitrary F-stable maximal torus of G which is in the G^F-conjugacy class corresponding to the F-conjugacy class of w in $W_G(T)$, and we will denote by $\mathfrak{t}_w$ its Lie algebra. Under the isomorphism $T \to T'$, $h \mapsto ghg^{-1}$, the Frobenius $F : T' \to T'$ corresponds to $F' = wF : T \to T, h \mapsto \dot{w}F(h)\dot{w}^{-1}$ where w is the image in $W_G(T)$ of $\dot{w} := g^{-1}F(g) \in N_G(T)$.

Unless specified, we will always consider parameterizations with respect to G-split, F-stable maximal tori of G, in which case we will write W_G instead of $W_G(T)$.

Example
Let $n=2$, let $x\in\mathbb{F}_{q^2}-\mathbb{F}_q$, and let

$$T=\left\{\begin{pmatrix}a&0\\0&b\end{pmatrix}\,\middle|\,a,b\in\overline{\mathbb{F}}_q^\times\right\},$$

$$T'=\left\{\frac{1}{x^q-x}\begin{pmatrix}ax^q-bx&-a+b\\(a-b)xx^q&-ax+bx^q\end{pmatrix}\,\middle|\,a,b\in\overline{\mathbb{F}}_q^\times\right\}.$$

Then T' is F-stable, $T'=gTg^{-1}$ where $g=\left(\begin{smallmatrix}1&1\\x&x^q\end{smallmatrix}\right)$, and $g^{-1}F(g)=\sigma:=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Therefore, $(T',F)\simeq(T,\sigma F)$, and we have $T^F\simeq\mathbb{F}_q^\times\times\mathbb{F}_q^\times$ and $T'^F\simeq T^{\sigma F}\simeq\mathbb{F}_{q^2}^\times$. Since $|W_{\mathrm{GL}_2}(T)|=2$, any F-stable maximal torus of GL_2 is GL_2^F-conjugate either to T or T'.

2.5.4. Lusztig induction
Let $\ell\nmid q$ be a prime. Let L be an F-stable Levi subgroup of a (possibly non-F-stable) parabolic subgroup P of G. Following [8] and [37] we construct a virtual $\overline{\mathbb{Q}}_\ell[G^F]$-module $R_L^G(M)$ for any $\overline{\mathbb{Q}}_\ell[L^F]$-module M as follows. Let U_P be the unipotent radical of P, and let $\mathcal{L}_G:G\to G, g\mapsto g^{-1}F(g)$ be the Lang map. The variety $\mathcal{L}_G^{-1}(U_P)$ is endowed with a left action of G^F by left multiplication and with a right action of L^F by right multiplication. These actions induce actions on the ℓ-adic cohomology $H_c^i(\mathcal{L}_G^{-1}(U_P),\overline{\mathbb{Q}}_\ell)$. The virtual $\overline{\mathbb{Q}}_\ell$-vector space $H_c^*(\mathcal{L}_G^{-1}(U_P)):=\sum_i(-1)^iH_c^i(\mathcal{L}_G^{-1}(U_P),\overline{\mathbb{Q}}_\ell)$ is thus a virtual $\overline{\mathbb{Q}}_\ell[G^F]$-module-$\overline{\mathbb{Q}}_\ell[L^F]$. We put $R_L^G(M):=H_c^*(\mathcal{L}_G^{-1}(U_P))\otimes_{\overline{\mathbb{Q}}_\ell[L^F]}M$.

Let $C(G^F)$ be the $\overline{\mathbb{Q}}_\ell$-vector space of all functions $G^F\to\overline{\mathbb{Q}}_\ell$ which are constant on conjugacy classes of G^F. If C is a conjugacy class of G^F and $x\in C$, we denote either by 1_C or 1_x^G the characteristic function of C that takes the value 1 on C and 0 elsewhere.

The Lusztig functor R_L^G defines a $\mathbb{Z}$-linear map $\mathbb{Z}(\mathrm{Irr}(L^F))\to\mathbb{Z}(\mathrm{Irr}(G^F))$, which by linearity extension leads to the Deligne-Lusztig induction $R_L^G:C(L^F)\to C(G^F)$.

For an F-stable maximal torus T of G, let $Q_T^G:G_{\mathrm{uni}}^F\to\overline{\mathbb{Q}}_\ell$ be the restriction to G_{uni}^F of the function $R_T^G(\mathrm{Id})$. The function Q_T^G is called a *Green function* and its values are products of the Green polynomials defined in [40, Chapter III, (7.8)] (see (2.3.16)). The following formula (see [8, Theorem 4.2]) reduces the computation of the values of $R_T^G(\theta)$ to the computation of Green polynomials:

$$R_T^G(\theta)(g)=|C_G(g_s)^F|^{-1}\sum_{\{h\in G^F|g_s\in hTh^{-1}\}}Q_{hTh^{-1}}^{C_G(g_s)}(g_u)\theta(h^{-1}g_sh)\qquad(2.5.1)$$

where $\theta\in C(T^F)$, $g\in G^F$.

2.5.5. Characters of $\mathrm{GL}_n(\mathbb{F}_q)$

The character table of $\mathrm{GL}_n(\mathbb{F}_q)$ was first computed by Green [16]. Here we recall how to construct it from the point of view of Deligne-Lusztig theory (see [39]).

Here we assume that $G = \mathrm{GL}_n$. Let L be an F-stable Levi subgroup of G, and let φ be an F-stable irreducible character of W_L. Then there is an extension $\tilde{\varphi}$ of φ to the semidirect product $W_L \rtimes \langle F \rangle$ such that the function $\mathcal{X}_\varphi^L : L^F \to \overline{\mathbb{Q}}_\ell$ defined by

$$\mathcal{X}_\varphi^L = |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^L(\mathrm{Id}_{T_w})$$

is an irreducible character of L^F. The characters $\mathcal{X}_\varphi^L$ are called the *unipotent characters* of L^F.

For $g \in G^F$ and $\theta \in \mathrm{Irr}(L^F)$, let ${}^g\theta \in \mathrm{Irr}(gL^Fg^{-1})$ be defined by ${}^g\theta(glg^{-1}) = \theta(l)$. We say that a linear character $\theta : L^F \to \overline{\mathbb{Q}}_\ell^\times$ is *regular* if for $n \in N_{G^F}(L)$, we have ${}^n\theta = \theta$ only if $n \in L^F$. We denote by $\mathrm{Irr}_{\mathrm{reg}}(L^F)$ the set of regular linear characters of L^F. Put $\epsilon_L = (-1)^{\mathbb{F}_q-\mathrm{rank}(L)}$. Then for $\theta^L \in \mathrm{Irr}_{\mathrm{reg}}(L^F)$, the virtual character

$$\mathcal{X} := \epsilon_G \epsilon_L R_L^G(\theta^L \cdot \mathcal{X}_\varphi^L) = \epsilon_G \epsilon_L |W_L|^{-1} \sum_{w \in W_L} \tilde{\varphi}(wF) R_{T_w}^G(\theta^{T_w}), \tag{2.5.2}$$

where $\theta^{T_w} := \theta^L|_{T_w}$, is an irreducible true character of G^F, and any irreducible character of G^F is obtained in this way (see [39]). An irreducible character of G^F is thus completely determined by the G^F-conjugacy class of a datum (L, θ^L, φ) with L an F-stable Levi subgroup of G, $\theta^L \in \mathrm{Irr}_{\mathrm{reg}}(L^F)$ and $\varphi \in \mathrm{Irr}(W_L)^F$. The irreducible characters corresponding to the data $(L, \theta^L, 1)$ are called *semisimple* characters of G^F. This process of decomposing the irreducible characters is sometimes called *Lusztig-Jordan decomposition*. By analogy with Jordan decomposition of conjugacy classes, the *semisimple part* of $\mathcal{X}$ would be θ^L and the unipotent part would be $\mathcal{X}_\varphi^L$. It is indeed well known that if C is a conjugacy class of G^F, $x \in C$, $L = C_G(x_s)$, then $R_L^G(1_{x_s}^L * 1_{x_u}^L) = 1_C$ where $*$ is the usual convolution product on $C(G^F)$ defined by $(f * h)(g) = \sum_{y \in G^F} f(y)h(gy^{-1})$.

2.5.6. Characters of $\mathfrak{gl}_n(\mathbb{F}_q)$

The characters of $\mathfrak{gl}_n(\mathbb{F}_q)$ were first studied by Springer [45].

We denote by $\mathrm{Fun}(\mathfrak{g}^F)$ the $\overline{\mathbb{Q}}_\ell$-vector space of all functions $\mathfrak{g}^F \to \overline{\mathbb{Q}}_\ell$ and by $\mathcal{C}(\mathfrak{g}^F)$ the subspace of all functions $f : \mathfrak{g}^F \to \overline{\mathbb{Q}}_\ell$ which are G^F-invariant; that is, for any $h \in G^F$ and any $x \in \mathfrak{g}^F$, $f(\mathrm{Ad}(h)x) = f(x)$. If $\mathcal{O}$ is a G^F-orbit of $\mathfrak{g}^F$ and $\sigma \in \mathcal{O}$, then we denote either by $1_\mathcal{O}$ or $1_\sigma^G \in \mathcal{C}(\mathfrak{g}^F)$ the characteristic function of $\mathcal{O}$; that is, $1_\sigma^G(x) = 1$ if $x \in \mathcal{O}$, and $1_\sigma^G(x) = 0$ otherwise. We are interested in the characters (nonnecessarily irreducible) of the abelian group $(\mathfrak{g}^F, +)$ which are

G^F-invariant, that is, which are in $\mathcal{C}(\mathfrak{g}^F)$. We call them the *invariant characters* of $\mathfrak{g}^F$. We say that an invariant character of $\mathfrak{g}^F$ is *irreducible* if it can not be written as a sum of two invariant characters. We denote by $\mathrm{Irr}_{G^F}(\mathfrak{g}^F)$ the set of irreducible invariant characters of $\mathfrak{g}^F$. We now describe them in terms of Fourier transforms.

We fix once for all a nontrivial additive character $\Psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times$, and we denote by $\mu : \mathfrak{g}\times\mathfrak{g} \to \overline{\mathbb{F}}_q$ the trace map $(a,b) \mapsto \mathrm{Trace}(ab)$. It is a nondegenerate G-invariant symmetric bilinear form defined over $\mathbb{F}_q$. We define the Fourier transform $\mathcal{F}^\mathfrak{g}$: $\mathrm{Fun}(\mathfrak{g}^F) \to \mathrm{Fun}(\mathfrak{g}^F)$ with respect to (Ψ, μ) by

$$\mathcal{F}^\mathfrak{g}(f)(x) = \sum_{y\in\mathfrak{g}^F} \Psi\big(\mu(x,y)\big) f(y). \tag{2.5.3}$$

Note that for $\sigma, x \in \mathfrak{g}^F$,

$$\mathcal{F}^\mathfrak{g}(1_\sigma^G)(x) = \sum_{y\in\mathcal{O}_\sigma^{G^F}} \Psi\big(\mu(x,y)\big).$$

For a fixed $y \in \mathfrak{g}^F$, the map $\mathfrak{g}^F \to \overline{\mathbb{Q}}_\ell$, $x \mapsto \Psi(\langle x,y\rangle)$ is an irreducible character of the abelian finite group $(\mathfrak{g}^F,+)$. Therefore $\mathcal{F}^\mathfrak{g}(1_\sigma^G)$, being a sum of characters of $(\mathfrak{g}^F,+)$, is a character of $(\mathfrak{g}^F,+)$. Since the sum is over a single adjoint orbit it is clearly an irreducible invariant character; that is, $\mathcal{F}^\mathfrak{g}(1_\sigma^G) \in \mathrm{Irr}_{G^F}(\mathfrak{g}^F)$.

Let L be an F-stable Levi subgroup of G, and let $\mathfrak{l}$ be its Lie algebra. We also have a Deligne-Lusztig induction $\mathcal{C}(\mathfrak{l}^F) \to \mathcal{C}(\mathfrak{g}^F)$ defined in [34]. Let $\omega : \mathfrak{g}_{\mathrm{nil}} \to G_{\mathrm{uni}}$ be the G-equivariant isomorphism given by $v \mapsto v+1$. For an F-stable maximal torus T of G with $\mathfrak{t} := \mathrm{Lie}(T)$, the Deligne-Lusztig induction $\mathcal{R}_\mathfrak{t}^\mathfrak{g}$ is defined by the following character formula:

$$\mathcal{R}_\mathfrak{t}^\mathfrak{g}(\theta)(x) = |C_G(x_s)^F|^{-1} \sum_{\{h\in G^F \mid x_s\in\mathrm{Ad}(h)\mathfrak{t}\}} Q_{hTh^{-1}}^{C_G(x_s)}\big(\omega(x_n)\big)\theta\big(\mathrm{Ad}(h^{-1})x_s\big), \tag{2.5.4}$$

where $\theta \in \mathcal{C}(\mathfrak{t}^F)$ and $x \in \mathfrak{g}^F$. Note that $C_G(x_s)$ is a Levi subgroup of G. For any semisimple element $\sigma \in \mathfrak{g}^F$, we have the following character formula (see [35, (7.3.3)]):

$$\mathcal{F}^\mathfrak{g}(1_\sigma^G) = \epsilon_G\epsilon_L|W_L|^{-1} \sum_{w\in W_L} q^{d_L/2}\mathcal{R}_{\mathfrak{t}_w}^\mathfrak{g}\big(\mathcal{F}^{\mathfrak{t}_w}(1_\sigma^{T_w})\big), \tag{2.5.5}$$

where $L = C_G(\sigma)$, $d_L = \dim G - \dim L$.

Note that if $\mathcal{X}$ is a semisimple character of $\mathrm{GL}_n(\mathbb{F}_q)$, then it is given by formula (2.5.2) with $\tilde{\varphi} = 1$. Hence the character formulas for semisimple characters of $\mathrm{GL}_n(\mathbb{F}_q)$ and $\mathfrak{gl}_n(\mathbb{F}_q)$ are similar and can be computed in the same way.

3. Counting with Fourier transforms

Let K be an algebraically closed field isomorphic to $\mathbb{C}$. Fixing such an isomorphism gives us an involution $K \to K$, $x \mapsto \overline{x}$ such that $\overline{\zeta} = \zeta^{-1}$ for any root of unity ζ in K.

3.1. Group Fourier transform

Let G be a finite group. We construct an analogue of the Fourier transform for class functions of G. For convenience we introduce the following notation. Let $G_\bullet$ be the measure space consisting of G with its Haar measure, that is, such that the measure of $\{g\}$ for $g \in G$ is $1/|G|$. Clearly, the total mass of $G_\bullet$ is 1. Let $C(G_\bullet)$ be the K-vector space of class functions on G. (A *class function* on G is a function which is constant on conjugacy classes.)

Similarly, let $G^\bullet$ be the measure space on the set of irreducible characters of G with its Plancherel measure, that is, such that the measure of the set $\{\chi\}$ for an irreducible character χ of G is $\chi(1)^2/|G|$. Again, the total mass of $G^\bullet$ is 1. Let $C(G^\bullet)$ be the K-vector space of functions on $G^\bullet$.

We now define maps $\mathcal{F}_\bullet$ and $\mathcal{F}^\bullet$ which are analogues of the Fourier transform for G. We describe some of their formal properties, leaving their proofs to the reader.

Define $\mathcal{F}_\bullet : C(G_\bullet) \to C(G^\bullet)$ by

$$\mathcal{F}_\bullet(f)(\chi) := |G| \int_{G_\bullet} f(g) \frac{\chi(g)}{\chi(1)} \, dg = \sum_g f(g) \frac{\chi(g)}{\chi(1)},$$

and define $\mathcal{F}^\bullet : C(G^\bullet) \to C(G_\bullet)$ by

$$\mathcal{F}^\bullet(F)(g) := |G| \int_{G^\bullet} F(\chi) \overline{\Big(\frac{\chi(g)}{\chi(1)}\Big)} \, d\chi = \sum_\chi F(\chi)\chi(1)\overline{\chi}(g).$$

Up to a factor of $|G|$ these maps are mutual inverses of each other. More precisely,

$$\mathcal{F}^\bullet \circ \mathcal{F}_\bullet = |G| \cdot 1_{G_\bullet}, \qquad \mathcal{F}_\bullet \circ \mathcal{F}^\bullet = |G| \cdot 1_{G^\bullet}. \tag{3.1.1}$$

Consider the algebra structures on $C(G_\bullet)$ and $C(G^\bullet)$ defined by convolution and pointwise multiplication, respectively; that is,

$$(f_1 * f_2)(g) := \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2), \quad f_1, f_2 \in C(G_\bullet)$$

and

$$(F_1 \cdot F_2)(\chi) := F_1(\chi) F_2(\chi), \quad F_1, F_2 \in C(G^\bullet).$$

(It is easy to check that $f_1 * f_2$ is indeed a class function and hence belongs to $C(G_\bullet)$.)

The maps $\mathcal{F}_\bullet$ and $\mathcal{F}^\bullet$ preserve these operations:

$$\mathcal{F}_\bullet(f_1)\cdot\mathcal{F}_\bullet(f_2)=\mathcal{F}_\bullet(f_1 * f_2),\quad f_1,f_2\in C(G_\bullet)$$

and

$$\mathcal{F}^\bullet(F_1)*\mathcal{F}^\bullet(F_2)=|G|\cdot\mathcal{F}^\bullet(F_1\cdot F_2),\quad F_1,F_2\in C(G^\bullet).$$

PROPOSITION 3.1.1
For $f\in C(G_\bullet)$ we have

$$f(1)=\int_{G^\bullet}\mathcal{F}_\bullet(f)(\chi)\,d\chi.$$

Proof
This is just a special case of Fourier inversion (3.1.1) as both sides equal $1/|G|\times\mathcal{F}^\bullet(\mathcal{F}_\bullet(f))(1)$. □

Given a word $w\in F_r$, where $F_r=\langle X_1,\dots,X_r\rangle$ is the free group in generators $X_1,\dots,X_r$, we let $n(w)$ be the function on G defined by

$$n(w)(z):=\#\big\{(x_1,\dots,x_r)\in G^r \;\big|\; w(x_1,\dots,x_r)=z\big\},$$

where $w(x_1,\dots,x_r)$ is a shorthand for $\phi(w)\in G$ with $\phi: F_r\to G$ the homomorphism mapping each X_i to x_i.

Since $w(x_1,\dots,x_r)=z$ implies $w(ux_1u^{-1},\dots,ux_ru^{-1})=uzu^{-1}$ for any $u\in G$, it is clear that $n(w)$ is a class function. For convenience we define

$$N(w):=\mathcal{F}_\bullet\big(n(w)\big)\in C(G^\bullet).$$

The following lemma is straightforward, and we omit its proof.

LEMMA 3.1.2
(1) *For a word $w\in F_r$,*

$$n(w)(1)=\int_{G^\bullet}N(w)(\chi)\,d\chi.$$

(2) *If w_1,w_2 are two words in separate sets of variables, then*

$$n(w_1w_2)=n(w_1)*n(w_2).$$

(3) *Let $C_1,\dots,C_k$ be conjugacy classes in G, and let $w\in F_r$. For $z\in G$ the number of solutions to*

$$w(x_1,\dots,x_r)y_1\cdots y_k=z,\quad x_i\in G,\ y_j\in C_j,$$

is given by

$$n(w) * 1_{C_1} * \cdots * 1_{C_k}(z),$$

where for any conjugacy class C we denote by $1_C \in C(G_\bullet)$ its characteristic function.

A proof of the following result can be found in [22, Section 3.2].

LEMMA 3.1.3
For $w = X_1 X_2 X_1^{-1} X_2^{-1} \in F_2$, we have

$$N(w)(\chi) = \Big(\frac{|G|}{\chi(1)}\Big)^2.$$

Finally, putting all the pieces together we have the following result.

PROPOSITION 3.1.4
Let $C_1, \ldots, C_k$ be conjugacy classes in G. The number of solutions to

$$[x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k = 1, \quad x_i, y_i \in G, z_j \in C_j,$$

equals

$$\int_{G^\bullet} \mathcal{L}(\chi)^g f_\chi(C_1) \cdots f_\chi(C_k)\, d\chi, \tag{3.1.2}$$

where

$$\mathcal{L}(\chi) := \Big(\frac{|G|}{\chi(1)}\Big)^2$$

and for any conjugacy class C,

$$f_\chi(C) := \mathcal{F}_\bullet(1_C)(\chi) = \frac{|C|\chi(C)}{\chi(1)}.$$

Remark 3.1.5
The proposition, as well as the introduction of the functions f_χ, is due to Frobenius. Proofs can be found in many places in the literature since then. The purpose of reproving it here is to draw as close a parallel as possible with the additive version of the next section.

3.2. *Equivariant Fourier transform*

If the group G of the previous section is abelian, then what we have is the usual Fourier transform. Here we consider the situation of an abelian group, which we now denote by A, together with an action of another group G. We will describe a Fourier transform on A, which is equivariant with respect to the action of G and parallels the one in Section 3.1. We will apply this to our main example: $A = \mathfrak{gl}_n$ and $G = \mathrm{GL}_n$ acting via the adjoint action.

Let $A_\bullet$ be as in Section 3.1, and let $X := \mathrm{Hom}(A, K^\times)$. We have a natural action of G on X as follows:

$$(g \cdot \phi)(a) := \phi(g^{-1} \cdot a).$$

Given a G-orbit $\mathcal{X}$ in X we let

$$\chi := \sum_{\phi \in \mathcal{X}} \phi.$$

It is a G-invariant character of the group A. We let $A^\bullet$ be the measure space on the set of such χ's where the measure of $\{\chi\}$ is $\chi(0)/|A|$. The total measure of $A^\bullet$ is 1 as $\chi(0) = \#\mathcal{X}$.

In analogy with Section 3.1 we let $C(A_\bullet)$ be the K-vector space of functions on A which are G-invariant and let $C(A^\bullet)$ be the K-vector space of functions on $A^\bullet$.

Define $\mathcal{F}_\bullet : C(A_\bullet) \to C(A^\bullet)$ by

$$\mathcal{F}_\bullet(f)(\chi) := |A| \int_{A_\bullet} f(a) \frac{\chi(a)}{\chi(0)}\, da,$$

and define $\mathcal{F}^\bullet : C(A^\bullet) \to C(A_\bullet)$ by

$$\mathcal{F}^\bullet(F)(a) := |A| \int_{A^\bullet} F(\chi) \overline{\left(\frac{\chi(a)}{\chi(0)}\right)}\, d\chi.$$

Note that if $f : A \to K$ is constant on G-orbits and $\phi \in X$, then

$$\sum_{a \in A} f(a)\phi(a)$$

is constant on the G-orbit $\mathcal{X}$ of ϕ. Hence we can write this sum as

$$\sum_{a \in A} f(a) \frac{\chi(a)}{\chi(0)},$$

where χ corresponds to $\mathcal{X}$. In other words, $\mathcal{F}_\bullet$ is (up to scaling) just the usual Fourier transform restricted to G-invariant functions on A. Similarly, $\mathcal{F}^\bullet$ is the usual inverse

Fourier transform (up to scaling) restricted to G-stable characters of A. It follows that all the formal properties of the previous section also hold here. In particular, we have the following.

PROPOSITION 3.2.1
For $f \in C(A_\bullet)$ we have

$$f(0) = \int_{A^\bullet} \mathcal{F}_\bullet(f)(\chi)\, d\chi.$$

Now let $A = \mathfrak{gl}_n(\mathbb{F}_q)$ and $G = \mathrm{GL}_n(\mathbb{F}_q)$ acting via the adjoint action on A. We consider the additive analogue of Proposition 3.1.4. For $x, y \in A$, let $[x, y] := xy - yx$. For fixed $\phi \in X$ and $y \in A$, the map $x \mapsto \phi([x, y])$ is in X. Let $C_A(\phi)$ be the subgroup of $y \in A$ for which this character is trivial. Its cardinality depends only on the G-orbit $\mathcal{X}$ of ϕ, and the order of $A/C_A(\phi)$ is a square since it carries the nondegenerate pairing induced from $\phi([\cdot, \cdot])$. Define $c(\chi) := |A/C_A(\phi)|^{1/2}$, where $\chi = \sum_{\phi \in \mathcal{X}} \phi \in A^\bullet$ is associated to $\mathcal{X}$.

PROPOSITION 3.2.2
Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be G-orbits in A. The number of solutions to

$$[x_1, y_1] + \cdots + [x_g, y_g] + z_1 + \cdots + z_k = 0, \quad x_i, y_i \in A, z_j \in \mathcal{O}_j,$$

equals

$$\int_{A^\bullet} \mathcal{L}(\chi)^g f_\chi(\mathcal{O}_1) \cdots f_\chi(\mathcal{O}_k)\, d\chi,$$

where

$$\mathcal{L}(\chi) := \Big(\frac{|A|}{c(\chi)}\Big)^2$$

and

$$f_\chi(\mathcal{O}) := \mathcal{F}_\bullet(1_{\mathcal{O}})(\chi) = \frac{|\mathcal{O}|\chi(\mathcal{O})}{\chi(0)}.$$

Proof
We may proceed exactly as with the proof of Proposition 3.1.4 thanks to the formal properties of the Fourier transform. The analogue of Lemma 3.1.3 is the following calculation. Let $n \in C(A_\bullet)$ be the function whose value at $a \in A$ is the number of solutions $x, y \in A$ of $[x, y] = a$. Then, with our previous notation,

$$\mathcal{F}_\bullet(n)(\chi) = \sum_{a \in A} n(a)\phi(a) = \sum_{x,y \in A} \phi([x, y]).$$

The sum $\sum_{x\in A}\phi([x,y])$ vanishes unless $y \in C_A(\phi)$, in which case it equals $|A|$. Hence $\mathcal{F}_\bullet(n)(\chi) = |A||C_A(\phi)|$. □

Remark 3.2.3
With the notation of Section 2.5.6, the $\mathrm{GL}_n(\mathbb{F}_q)$-invariant characters of $\mathfrak{gl}_n(\mathbb{F}_q)$ are the functions $\mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}})$ where $\mathcal{O}$ describes the set of adjoint orbits. Then note that

$$\mathcal{F}_\bullet(1_{\mathcal{O}'})\big(\mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}})\big) = \mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}'})(\mathcal{O})$$

and $c(\mathcal{F}^{\mathfrak{g}}(1_{\mathcal{O}})) = (|\mathfrak{gl}_n(\mathbb{F}_q)|q^{-\dim C_G(x)})^{\frac{1}{2}}$ where $x \in \mathcal{O}$.

4. Sums of character values

In this section we obtain a formula which is used, together with the results of Section 3, to compute the number of $\mathbb{F}_q$-rational points of character and quiver varieties over $\overline{\mathbb{F}}_q$. Here $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$.

4.1. Types of conjugacy classes, irreducible characters, and Levi subgroups

Let C be a conjugacy class of G^F. The Frobenius $f : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, x \mapsto x^q$ acts on the set of eigenvalues of C; therefore we may write the set of eigenvalues of C as a union of $\langle f\rangle$-orbits

$$\{\gamma_1, \gamma_1^q, \dots\} \coprod \{\gamma_2, \gamma_2^q, \dots\} \coprod \cdots \coprod \{\gamma_s, \gamma_s^q, \dots\}.$$

Put $d_i = \#\gamma_i, \gamma_i^q, \dots$, and let m_i be the multiplicity of γ_i. Clearly $\sum_i m_i d_i = n$. The unipotent part of an element of C defines a unique partition λ^i of m_i given by the Jordan blocks. Then $\lambda = (d_1, \lambda^1)\cdots(d_s, \lambda^s) \in \mathbf{T}_n$ is called the *type* of C. When $q \geq n$, any type $\omega \in \mathbf{T}_n$ arises as the type of some conjugacy class of G^F. The types of the semisimple conjugacy classes are of the form $(d_1, 1^{n_1})\cdots(d_r, 1^{n_r})$ where $n_1, \dots, n_r$ are the multiplicities of the eigenvalues and (1^{n_i}) is the trivial partition $(1, \dots, 1)$ of n_i.

LEMMA 4.1.1
Let $\omega \in \mathbf{T}_n$, and let $\sigma \in G^F$ be an element of type ω. Then

$$\mathcal{H}_\omega(0, \sqrt{q}) = \frac{q^{g \dim C_G(\sigma)}}{|C_{G^F}(\sigma)|}$$

where $\mathcal{H}_\omega(z, w) = \prod_i \mathcal{H}_{\omega^i}(z^{d_i}, w^{d_i})$ for $\omega = (d_1, \omega^1)\cdots(d_r, \omega^r)$.

Proof
This follows from formula (2.3.22) and the identities $\dim C_G(\sigma) = \sum_{i=1}^r d_i \langle\omega^i, \omega^i\rangle$ and $|C_{G^F}(\sigma)| = \prod_{i=1}^r a_{\omega^i}(q^{d_i})$, which are well known. □

Recall (see Section 2.5) that an irreducible character $\mathcal{X}$ of G^F arises from a datum (L, θ^L, φ). There exist positive integers d_i, n_i, $i \in \{1, \dots, s\}$, such that

$$L \simeq \prod_{i=1}^{s} \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}.$$

We choose the indexing such that $d_1 \geq d_2 \geq \cdots \geq d_s$, and $n_i \geq n_j$ if $i > j$ and $d_i = d_j$. Let $\mathcal{S}_n$ be the symmetric group in n letters, and let $\nu \in \mathcal{S}_n \simeq W_G$, where W_G is the Weyl group of G (with respect to some split F-stable maximal torus), be such that the map $z \mapsto \nu z \nu^{-1}$ acts on each component of $\prod_{i=1}^{s} \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}$ by circular permutation of the d_i blocks of length n_i. Then

$$(L, F) \simeq \Big(\prod_{i=1}^{s} \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}, \nu F\Big), \tag{4.1.1}$$

and so L^F is isomorphic to $\prod_{i=1}^{s} \mathrm{GL}_{n_i}(\mathbb{F}_{q^{d_i}})$. Moreover,

$$(W_L, F) \simeq \Big(\prod_{i=1}^{s} (\mathcal{S}_{n_i})^{d_i}, \nu\Big).$$

The F-conjugacy classes of W_L are thus parameterized by the conjugacy classes of $\prod_i \mathcal{S}_{n_i}$, that is, by the set $\mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_s}$. The set of F-stable irreducible characters of W_L is in bijection with $\mathrm{Irr}(\mathcal{S}_{n_1}) \times \cdots \times \mathrm{Irr}(\mathcal{S}_{n_s})$ which, by the Springer correspondence, is parameterized by $\mathcal{P}_{n_1} \times \cdots \times \mathcal{P}_{n_s}$ in such a way that the trivial character corresponds to the multipartition $((n_1), \dots, (n_s))$. Hence $\varphi \in \mathrm{Irr}(W_L)^F$ defines a partition $\lambda^i \in \mathcal{P}_{n_i}$ for all $i \in \{1, \dots, s\}$. The type $(d_1, \lambda^1)(d_2, \lambda^2) \cdots (d_s, \lambda^s) \in \mathbf{T}_n$ is called the *type* of the irreducible character $\mathcal{X}$ of G^F. Note that when $q \geq n$, any type in $\mathbf{T}_n$ arises as the type of some irreducible character of G^F. The type of the semisimple irreducible characters of G^F are of the form $(d_1, (n_1))(d_2, (n_2)) \cdots (d_s, (n_s))$.

It will be convenient to introduce the set $\hat{\mathbf{T}}_n$ of nonincreasing sequences $(d_1, n_1) \cdots (d_r, n_r)$ with $d_i, n_i \in \mathbb{Z}_{>0}$ and $\sum_i d_i n_i = n$ where $(d, k) > (d', k')$ if $d > d'$, or $d = d'$ and $k > k'$.

The types of the semisimple conjugacy classes are in bijection with $\hat{\mathbf{T}}_n$ by

$$(d_1, 1^{n_1}) \cdots (d_r, 1^{n_r}) \mapsto (d_1, n_1) \cdots (d_r, n_r).$$

Similarly, $\hat{\mathbf{T}}_n$ parameterizes the types of the semisimple irreducible characters of G^F by

$$\big(d_1, (n_1)\big) \cdots \big(d_r, (n_r)\big) \mapsto (d_1, n_1) \cdots (d_r, n_r).$$

The map which assigns to a semisimple element of G the Levi subgroup $C_G(\sigma)$ gives a natural bijection between the types of the semisimple conjugacy classes of G^F and the G^F-conjugacy classes of the F-stable Levi subgroups of G. We will use the set $\hat{\mathbf{T}}_n$ to parameterize the G^F-conjugacy classes of the F-stable Levi subgroups of G. Namely, if $\lambda = (d_1, n_1) \cdots (d_r, n_r) \in \hat{\mathbf{T}}_n$, then a representative L of the corresponding G^F-conjugacy class will satisfy (4.1.1). In this case we say that L is of *type* λ.

4.2. *Generic characters and generic conjugacy classes*

Let L be an F-stable Levi subgroup of G. We say that a linear character Γ of Z_L^F is *generic* if its restriction to Z_G^F is trivial and its restriction to Z_M^F is nontrivial for any F-stable proper Levi subgroup M of G such that $L \subset M$. We put

$$(Z_L)_{\mathrm{reg}} := \left\{ x \in Z_L \;\middle|\; C_G(x) = L \right\}.$$

We have the following proposition.

PROPOSITION 4.2.1
Assume that L *is of type* $\omega = (d_1, n_1)(d_2, n_2) \cdots (d_r, n_r) \in \hat{\mathbf{T}}_n$ *and that* Γ *is a generic linear character of* Z_L^F. *Then*

$$\sum_{z \in (Z_L)_{\mathrm{reg}}^F} \Gamma(z) = (q-1) K_\omega^o$$

with

$$K_\omega^o = \begin{cases} (-1)^{r-1} d^{r-1} \mu(d)(r-1)! & \textit{if for all } i, d_i = d, \\ 0 & \textit{otherwise,} \end{cases}$$

where μ *is the ordinary Möbius function.*

Proof
Let ν_ω be an element of $\mathcal{S}_n$ such that the map $z \mapsto \nu_\omega z \nu_\omega^{-1}$ induces an action on each component of $M := \prod_i \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}$ by circular permutation of the d_i blocks of length n_i. Then $(L, F) \simeq (M, F_\omega)$ where F_ω is the Frobenius on G defined by $F_\omega(g) = \nu_\omega F(g) \nu_\omega^{-1}$. Then the character Γ can be transferred to a generic character Γ_M of $Z_M^{F_\omega}$. Its restriction to $Z_G^{F_\omega}$ is also trivial. Then $\sum_{h \in (Z_M)_{\mathrm{reg}}^{F_\omega}} \Gamma_M(h) = \sum_{h \in (Z_L)_{\mathrm{reg}}^F} \Gamma(h)$. We denote by $P(\omega)$ the set of Levi subgroups H of G such that $M \subset H \subset G$ and $P(\omega)^{F_\omega}$, and we denote by $P(\omega)^F$ the set of elements of $P(\omega)$ fixed by F_ω. We have the following partition $Z_M = \coprod_{H \in P(\omega)} (Z_H)_{\mathrm{reg}}$. Indeed, if $z \in Z_M$, then $C_G(z)$ is a Levi subgroup H of G and clearly $z \in (Z_H)_{\mathrm{reg}}$. If $H \in P(\omega)$, then $F_\omega(H) \in P(\omega)$, and $(Z_H)_{\mathrm{reg}} \cap (Z_{F_\omega(H)})_{\mathrm{reg}} = \emptyset$ unless $H \in P(\omega)^{F_\omega}$. Therefore F_ω preserves the above partition, and $Z_M^{F_\omega} = \coprod_{H \in P(\omega)} (Z_H)_{\mathrm{reg}}^{F_\omega}$. We define a partial order on $P(\omega)$ by $H_1 \leq H_2$ if $Z_{H_1} \subset Z_{H_2}$ (i.e., if $H_2 \subset H_1$). Then G

is the unique minimal element and M is the unique maximal element. We have a map $\epsilon : P(\omega)^{F_\omega} \to \overline{\mathbb{Q}}_\ell$ that sends $H \in P(\omega)^{F_\omega}$ to $\sum_{z \in Z_H^{F_\omega}} \Gamma_M(z)$ and a map $\epsilon' : P(\omega)^{F_\omega} \to \overline{\mathbb{Q}}_\ell$ that sends $H \in P(\omega)^{F_\omega}$ to $\sum_{z \in (Z_H)_{\text{reg}}^{F_\omega}} \Gamma_M(z)$. Since $Z_H^{F_\omega} = \coprod_{E \leq H} (Z_E)_{\text{reg}}^{F_\omega}$ for all $H \in P(\omega)^{F_\omega}$, we have $\epsilon(H) = \sum_{E \leq H} \epsilon'(E)$ for all $H \in P(\omega)^{F_\omega}$. Then by the inclusion-exclusion principle, we have $\epsilon'(H) = \sum_{E \leq H} \mu_\omega(E, H)\epsilon(E)$ for all $H \in P(\omega)^{F_\omega}$ where μ_ω is the Möbius function on the poset $P(\omega)^{F_\omega}$. In particular

$$\sum_{z \in (Z_M)_{\text{reg}}^{F_\omega}} \Gamma_M(z) = \sum_{H \leq M} \mu_\omega(H, M) \sum_{z \in Z_H^{F_\omega}} \Gamma_M(z).$$

Using the assumption on Γ, we deduce that

$$\sum_{z \in (Z_M)_{\text{reg}}^{F_\omega}} \Gamma_M(z) = (q-1)\mu_\omega(G, M).$$

Let us compute $\mu_\omega(G, M)$. An element of Z_M is a diagonal matrix $A \in \prod_{i=1}^r \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)^{d_i}$ such that each component of A in $\mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)$ is central. We identify Z_M with $\prod_{i=1}^r (\overline{\mathbb{F}}_q^\times)^{d_i}$ in the obvious way. Then the elements of $(Z_M)_{\text{reg}}$ correspond to the elements of the form $(a_{k,s})_{1 \leq k \leq r, 1 \leq s \leq d_k} \in \prod_{i=1}^r (\overline{\mathbb{F}}_q^\times)^{d_i}$ where $a_{i,j} \neq a_{k,l}$ if $(i, j) \neq (k, l)$. Let $I = \{i_{1,1}, \ldots, i_{1,d_1}, i_{2,1}, \ldots, i_{2,d_2}, \ldots, i_{r,1}, \ldots, i_{r,d_r}\}$ be a set whose elements are indexed by the pairs (k, s) with $1 \leq k \leq r$ and $1 \leq s \leq d_k$. Then the partition $Z_M = \coprod_{H \in P(\omega)} (Z_H)_{\text{reg}}$ is indexed by the partitions of the set I. The part $(Z_M)_{\text{reg}}$ corresponds to the unique partition of I which has $|I|$ parts, that is, to $\{i_{1,1}\}, \{i_{1,2}\}, \ldots, \{i_{r,d_r}\}$, and the part $Z_G = (Z_G)_{\text{reg}}$, which is the set of diagonal matrices with exactly one eigenvalue, corresponds to the unique partition of I which has one part. By abuse of notation we denote by $\nu_\omega \in \mathcal{S}_{|I|}$ the element which acts by circular permutation on each subset $\{i_{k,1}, \ldots, i_{k,d_k}\}$ of I. Then it induces an action on the set $P(I)$ of partitions of I which corresponds via the bijection $P(I) \simeq P(\omega)$ to the action of $F_\omega = \nu_\omega F$ on $P(\omega)$. We denote by O the minimal element of $P(I)^{\nu_\omega}$ and by 1 the unique maximal element of $P(I)^{\nu_\omega}$. Then $\mu_\omega(G, M) = \mu'_\omega(0, 1)$ where μ'_ω is the Möbius function on the poset $P(I)^{\nu_\omega}$. Now $\mu'_\omega(0, 1)$ was computed by Hanlon [17], and we find that $\mu'_\omega(0, 1) = K_\omega^o$. □

Definition 4.2.2

Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be k-irreducible characters of G^F. For each i, let $(L_i, \theta_i, \varphi_i)$ be a datum defining $\mathcal{X}_i$. We say that the tuple $(\mathcal{X}_1, \ldots, \mathcal{X}_k)$ is *generic* if $\prod_{i=1}^k ({}^{g_i}\theta_i)|_{Z_M}$ is a generic character of Z_M^F for any F-stable Levi subgroup M of G which satisfies the following condition: for all $i \in \{1, \ldots, k\}$, there exists $g_i \in G^F$ such that $Z_M \subset g_i L_i g_i^{-1}$.

Let $C_1, \dots, C_k$ be k-conjugacy classes of G^F. For each $i \in \{1, \dots, k\}$, let s_i be the semisimple part of an element of C_i. Let $\tilde{\mathbf{C}}_i$ be the conjugacy class of s_i in G. We say that the tuple $(C_1, \dots, C_k)$ is generic if $(\tilde{\mathbf{C}}_1, \dots, \tilde{\mathbf{C}}_k)$ is generic in the sense of Definition 2.1.1.

The proof of the following proposition is similar to that of Proposition 4.2.1.

PROPOSITION 4.2.3
Let $(C_1, \dots, C_k)$ be a generic tuple of semisimple conjugacy classes of G^F, let $s_i \in C_i$, and put $L_i = C_G(s_i)$. Assume that M is an F-stable Levi subgroup of G of type $\omega \in \hat{\mathbf{T}}_n$ which satisfies the following condition: for all $i \in \{1, \dots, k\}$ there exists $g_i \in G^F$ such that $Z_M \subset g_i L_i g_i^{-1}$. Then

$$\sum_{\theta \in \mathrm{Irr}_{\mathrm{reg}}(M^F)} \prod_{i=1}^{k} \theta(g_i s_i g_i^{-1}) = (q-1) K^o_\omega.$$

Note that $g_i s_i g_i^{-1}$ is in the center of $g_i L_i g_i^{-1}$ and so commutes with the elements of Z_M, that is, $g_i s_i g_i^{-1} \in C_G(Z_M) = M$. Therefore it makes sense to evaluate θ at $g_i s_i g_i^{-1}$ in the above formula.

4.3. Calculation of sums of character values
For a partition ν, put $T_\nu := T_{t_\nu}$, where $t_\nu \in \mathcal{S}_{|\nu|}$ is an element in the conjugacy class of type ν. If u_τ is a unipotent element of G^F whose Jordan form is given by the partition τ, then the Green polynomial (see (2.3.16)) $Q^\tau_\nu(q)$ is the value $Q^G_{T_\nu}(u_\tau)$ of the Green function $Q^G_{T_\nu}$ of Deligne and Lusztig defined in Section 2.5.4.

For $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in (\mathbf{T}_n)^k$ and $\omega \in \mathbf{T}_n$, define

$$\mathbf{H}^{\boldsymbol{\mu}}_\omega(q) := \frac{(q-1)K^o_\omega}{|W(\omega)|} \prod_{i=1}^{k} (-1)^{n+f(\mu_i)} \sum_\tau \frac{z_{[\tau]} \chi^{\mu_i}_\tau}{z_\tau} \sum_{\{\nu | [\nu] = [\tau]\}} \frac{Q^\omega_\nu(q)}{z_\nu},$$

$$\hat{\mathbf{H}}^{\boldsymbol{\mu}}_\omega(q) := \frac{(q-1)K^o_\omega}{|W(\omega)|} \prod_{i=1}^{k} (-1)^{n+f(\omega)} \sum_\tau \frac{z_{[\tau]} \chi^{\omega}_\tau}{z_\tau} \sum_{\{\nu | [\nu] = [\tau]\}} \frac{Q^{\mu_i}_\nu(q)}{z_\nu}$$

where $K^o_\omega := K^o_{\pi(\omega)}$, for any type $\tau = (d_1, \tau^1) \cdots (d_r, \tau^r)$, $f(\tau) := \sum_j |\tau^j|$; and if we write $\omega = \{m_{d,\lambda}\}_{(d,\lambda)}$, then $W(\omega) = \prod_{(d,\lambda) \in \mathbb{Z}_{>0} \times \mathcal{P}^*} (\mathbb{Z}/d\mathbb{Z})^{m_{d,\lambda}} \times \mathcal{S}_{m_{d,\lambda}}$.

Let $(C_1, \dots, C_k)$ be a generic tuple of conjugacy classes of G^F of type $\boldsymbol{\mu}$, and let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be a generic tuple of irreducible characters of G^F of type $\boldsymbol{\mu}$.

THEOREM 4.3.1
We have
(1) $\sum_{\mathcal{X}} \prod_{i=1}^k \mathcal{X}(C_i) = \hat{\mathbf{H}}^{\boldsymbol{\mu}}_\omega(q)$ *where the sum is over the irreducible characters of G^F of type ω, and*
(2) $\sum_{\mathcal{O}} \prod_{i=1}^k \mathcal{X}_i(\mathcal{O}) = \mathbf{H}^{\boldsymbol{\mu}}_\omega(q)$ *where the sum is over the conjugacy classes of G^F of type ω.*

Remark 4.3.2
Let us denote by $\mathcal{X}_\omega(1)$ the degree of an irreducible character of G^F of type ω. (The degree depends only on the type.) Hence formula (3.1.2) in Proposition 3.1.4 applied to GL_n reads

$$\sum_{\mathcal{X}\in \mathrm{Irr}(G^F)} \frac{\mathcal{X}(1)^2}{|G^F|}\Big(\frac{|G^F|}{\mathcal{X}(1)}\Big)^{2g} \prod_{i=1}^k \frac{|C_i|\mathcal{X}(C_i)}{\mathcal{X}(1)}$$

$$= \sum_{\omega\in \mathbf{T}_n} \frac{\mathcal{X}_\omega(1)^2}{|G^F|}\Big(\frac{|G^F|}{\mathcal{X}_\omega(1)}\Big)^{2g} \Big(\prod_{i=1}^k \frac{|C_i|}{\mathcal{X}_\omega(1)}\Big) \sum_{\mathcal{X}} \prod_{i=1}^k \mathcal{X}(C_i) \qquad (4.3.1)$$

where the second sum in the right-hand side is over the irreducible characters of type ω. Theorem 4.3.1(1) will be used in Section 5.2 to obtain an expression of this formula in terms of symmetric functions.

Theorem 4.3.1(2) will be used to prove Theorem 6.1.1.

Proof of Theorem 4.3.1
Let $\mathcal{X}$ be an irreducible character of type $\alpha \in \mathbf{T}_n$, and let $\mathcal{O}$ be a conjugacy class of G^F of type $\beta \in \mathbf{T}_n$. We have (see formula (2.5.2))

$$\mathcal{X} = \epsilon_G \epsilon_M |W_M|^{-1} \sum_{w\in W_M} \tilde{\varphi}(wF) R^G_{T_w}(\theta^{T_w}).$$

The $\mathbb{F}_q$-rank of M is $f(\alpha)$, so $\epsilon_G\epsilon_M = (-1)^{n+f(\alpha)}$. Let $\sigma \in \mathcal{O}$, and put $L = C_G(\sigma_s)$. Then for $w \in W_M$,

$$R^G_{T_w}(\theta^{T_w})(\sigma) = |L^F|^{-1} \sum_{\{h\in G^F | \sigma\in h^{-1}T_w h\}} Q^L_{h^{-1}T_w h}(\sigma_u)\theta^{T_w}(h\sigma_s h^{-1}).$$

We have $\{h \in G^F \mid \sigma \in h^{-1}T_w h\} = \{h \in G^F \mid h^{-1}T_w h \subset L\}$. Put $A_w := \{h \in G \mid h^{-1}T_w h \subset L\}$. Note that the sum over A_w^F depends only on the F-conjugacy class of w in W_M. The F-conjugacy classes of W_M, and so the M^F-conjugacy classes of the F-stable maximal tori of M, are parameterized by the set of types $\{\tau \mid \tau \sim \alpha\}$ as in Section 4.1. From its definition, the value $\tilde{\varphi}(wF)$ depends also only on the

F-conjugacy class of w in W_M. For $\tau \in \mathbf{T}_n$, we write T_τ, A_τ, $\tilde{\varphi}(\tau)$ instead of T_w, A_w, $\tilde{\varphi}(wF)$ if the F-conjugacy class of w is of type τ. Let $c(\tau)$ be the cardinality of the corresponding F-conjugacy class in W_M. Then

$$\mathcal{X}(\sigma) = (-1)^{n+f(\alpha)}|L^F|^{-1} \sum_{\tau\sim\alpha} \sum_{h\in A_\tau^F} \frac{c(\tau)}{|W_M|} \tilde{\varphi}(\tau) Q^L_{h^{-1}T_\tau h}(\sigma_u)\theta^{T_\tau}(h\sigma_s h^{-1}).$$

We have $c(\tau)/|W_M| = z_\tau^{-1}$ and $\tilde{\varphi}(\tau) = \chi_\tau^\alpha$. Hence

$$\mathcal{X}(\sigma) = (-1)^{n+f(\alpha)}|L^F|^{-1} \sum_\tau z_\tau^{-1}\chi_\tau^\alpha \sum_{h\in A_\tau^F} Q^L_{h^{-1}T_\tau h}(\sigma_u)\theta^{T_\tau}(h\sigma_s h^{-1}).$$

Since by convention $\chi_\tau^\alpha = 0$ if $\tau \nsim \alpha$, we omit the condition $\tau \sim \alpha$ in the above sum. The map $h \mapsto h^{-1}T_\tau h$ is a surjective map from the set A_τ^F onto the set of F-stable maximal tori of L that are in the G^F-conjugacy class (of F-stable maximal tori of G) of type $[\tau] \in \mathcal{P}_n$. Therefore it induces a surjective map $A_\tau^F/L^F \to \{\nu \mid \nu \sim \beta, [\nu] = [\tau]\}$. Hence

$$\mathcal{X}(\sigma) = (-1)^{n+f(\alpha)} \sum_\tau z_\tau^{-1}\chi_\tau^\alpha \sum_{\{\nu|[\nu]=[\tau]\}} Q_\nu^\beta(q) \sum_{l\in\overline{A}_\nu} \theta^{T_\tau}(l\sigma_s l^{-1}) \qquad (4.3.2)$$

where $\overline{A}_\nu$ is the set of elements lL^F of A_τ^F/L^F such that the L^F-conjugacy class of $l^{-1}T_\tau l$ is of type ν.

Let us determine the set $\overline{A}_\nu$. The L^F-conjugacy classes of the F-stable maximal tori of L^F are parameterized by the set $\{\nu \mid \nu \sim \beta\}$. Let T_ν denote an F-stable maximal torus of L whose L^F-conjugacy class is of type $\nu \in \{\gamma \mid \gamma \sim \beta, [\gamma] = [\tau]\}$. Then the G^F-conjugacy class of T_ν is of type $[\nu] = [\tau]$, and so T_ν is G^F-conjugate to T_τ, say, $T_\tau = gT_\nu g^{-1}$ with $g \in G^F$. We put $B_\nu = \{h \in G \mid h^{-1}T_\nu h \subset L\}$. Then the map $h \mapsto g^{-1}h$ induces a bijection $(A_\tau^F/L^F) \simeq (B_\nu^F/L^F)$. Since the maximal tori of L are all L-conjugate, the map $N_G(T_\nu) \to (B_\nu/L)$, $n \mapsto nL$ is surjective and commutes with the Frobenius F. This map induces a bijection $(N_G(T_\nu)/N_L(T_\nu)) \xrightarrow{\sim} (B_\nu/L)$ which commutes with F. We thus have a bijection:

$$\big(W_G(T_\nu)/W_L(T_\nu)\big)^F \xrightarrow{\sim} (B_\nu/L)^F.$$

Since L is connected we get bijections:

$$\big(W_G(T_\nu)/W_L(T_\nu)\big)^F \xrightarrow{\sim} (B_\nu^F/L^F) \simeq (A_\tau^F/L^F).$$

Under this bijection, the elements of $\overline{A}_\nu$ correspond to the elements $u \in (W_G(T_\nu)/W_L(T_\nu))^F$ such that $(T_\nu)_{\dot{u}^{-1}F(\dot{u})}$, $\dot{u} \in W_G(T_\nu)$ being a representative of u, and T_ν are L^F-conjugate. Now saying that $(T_\nu)_{\dot{u}^{-1}F(\dot{u})}$ and T_ν are L^F-conjugate is equivalent to saying that $\dot{u}^{-1}F(\dot{u})$ is in the F-conjugacy class of 1 in $W_L(T_\nu)$; that is,

$\dot{u}^{-1}F(\dot{u}) = w^{-1}F(w)$ for some $w \in W_L(T_\nu)$. We know that $W_G(T_\nu)/W_L(T_\nu) \simeq \mathcal{S}_n/\prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$. Under this bijection, the automorphism F on $W_G(T_\nu)$ induces an automorphism on $\mathcal{S}_n$ which stabilizes $\prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$. Let us determine the automorphism obtained. Let v_β be an element of $\mathcal{S}_n$ such that the automorphism $z \mapsto v_\beta z v_\beta^{-1}$ induces an action on each component of $\prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$ by circular permutation of the d_i blocks of length $|\beta^i|$. Then $(W_L, F) \simeq \big(\prod_i(\mathcal{S}_{|\beta^i|})^{d_i}, v_\beta\big)$. Now let $w_\nu \in \prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$ be in the v_β-conjugacy class of $\prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$ corresponding to ν; then $(W_G(T_\nu), F) \simeq (\mathcal{S}_n, w_\nu v_\beta)$, where $w_\nu v_\beta : \mathcal{S}_n \to \mathcal{S}_n, z \mapsto w_\nu v_\beta z (w_\nu v_\beta)^{-1}$. We deduce that $\overline{A}_\nu$ is in bijection with the set $\overline{\mathcal{W}}_\nu$ of elements $x(\prod_i(\mathcal{S}_{|\beta^i|})^{d_i})$ with $x \in \mathcal{S}_n$ such that $x^{-1}(w_\nu v_\beta)x = t(w_\nu v_\beta)t^{-1}$ for some $t \in \prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$.

Let us determine the cardinality of $\overline{A}_\nu$ as we will need it later. Put $H = \prod_i(\mathcal{S}_{|\beta^i|})^{d_i}$. We have a bijective map $C_{\mathcal{S}_n}(w_\nu v_\beta)/C_H(w_\nu v_\beta) \to \overline{\mathcal{W}}_\nu$, $xC_H(w_\nu v_\beta) \mapsto xH$. But $|C_{\mathcal{S}_n}(w_\nu v_\beta)| = z_{[\tau]}$ and $|C_H(w_\nu v_\beta)| = z_\nu$, and therefore,

$$|\overline{A}_\nu| = |\overline{\mathcal{W}}_\nu| = z_{[\tau]} z_\nu^{-1}. \tag{4.3.3}$$

Now let us compute $\sum_{\mathcal{X}} \prod_i \mathcal{X}(\mathcal{C}_i)$ and $\sum_{\mathcal{O}} \prod_i \mathcal{X}_i(\mathcal{O})$. We first compute the second sum. Let (L, C) be a pair of type ω where L is an F-stable Levi subgroup and C and an F-stable unipotent conjugacy class of L. Let $u \in C$. We have a surjective map $(Z_L)^F_{\mathrm{reg}} \to \{G^F - \text{orbits of type } \omega\}$ that sends z to $\mathcal{O}^{G^F}_{zu}$. If $s, s' \in (Z_L)^F_{\mathrm{reg}}$, then s and s' have the same image if there exists $g \in G^F$ such that $g(sC)g^{-1} = s'C$, that is, $gsg^{-1} = s'$ and $gCg^{-1} = C$. The identity $gsg^{-1} = s'$ implies that $g \in N_G(L)$. Therefore the fibers of our map can be identified with $W_G(L, C) := \{g \in G^F | g \in N_G(L) \cap N_G(C)\}/L^F$, which is of cardinality $|W(\omega)|$. We thus have

$$\sum_{\mathcal{O}} \prod_{i=1}^k \mathcal{X}_i(\mathcal{O}) = \frac{1}{|W(\omega)|} \sum_{z \in (Z_L)^F_{\mathrm{reg}}} \prod_{i=1}^k \mathcal{X}_i(zu).$$

Applying the formula (4.3.2) with $(\alpha, \beta) = (\mu_i, \omega)$, we get

$$\sum_{\mathcal{O}} \prod_{i=1}^k \mathcal{X}_i(\mathcal{O}) = \frac{1}{|W(\omega)|} \sum_{\tau_1,\ldots,\tau_k} \sum_{\{(\nu_1,\ldots,\nu_k) | [\nu_i] = [\tau_i]\}} \prod_{i=1}^k \chi^{\mu_i}_{\tau_i} z_{\tau_i}^{-1} Q^{\omega}_{\nu_i}(q)$$
$$\times \sum_{(l_1,\ldots,l_k) \in \overline{A}_{\nu_1} \times \cdots \times \overline{A}_{\nu_k}} \Big(\sum_{z \in (Z_L)^F_{\mathrm{reg}}} \prod_{i=1}^k \theta_i^{T_{\tau_i}}(l_i z l_i^{-1}) \Big).$$

Put $\theta_i^{l_i^{-1} T_{\tau_i} l_i}(z) := \theta_i^{T_{\tau_i}}(l_i z l_i^{-1})$ for all $z \in Z_L^F$. Then $\prod_i \theta_i^{l_i^{-1} T_{\tau_i} l_i}$ is a linear character of Z_L^F. By assumption, it is generic and so by Proposition 4.2.1, we have

$\sum_{z\in(Z_L)^F_{\rm reg}}\prod_i \theta^{l_i^{-1}T_{\tau_i}l_i}(z)=(q-1)K^o_\omega$, from which we deduce that

$$\sum_{\mathcal{O}}\prod_{i=1}^k \mathcal{X}_i(\mathcal{O})$$
$$=\frac{(q-1)K^o_\omega}{|W(\omega)|}\sum_{\tau_1,\dots,\tau_k}\sum_{\{(\nu_1,\dots,\nu_k)|[\nu_i]=[\tau_i]\}}\prod_{i=1}^k \chi^{\mu_i}_{\tau_i} z^{-1}_{\tau_i} Q^{\omega}_{\nu_i}(q)|\overline{W}_{\nu_1}|\cdots|\overline{W}_{\nu_k}|.$$

The assertion (2) of the theorem follows then from formula (4.3.3).

Let us now compute $\sum_{\mathcal{X}}\prod_i \mathcal{X}(C_i)$. Let (L,χ) be of type ω with L an F-stable Levi subgroup of G and χ an F-stable irreducible character of W_L. Let $\mathcal{X}^L_\chi$ be the unipotent character of L^F associated to χ. The map $\mathrm{Irr}_{\rm reg}(L^F)\to\{\mathcal{X}\in\mathrm{Irr}(G^F)\mid \mathcal{X}\text{ of type }\omega\}$ that sends θ to $\epsilon_G\epsilon_L R^G_L(\theta\cdot\mathcal{X}^L_\chi)$ is surjective, and its fibers are of cardinality $|W(\omega)|$. We thus have

$$\sum_{\mathcal{X}}\prod_{i=1}^k \mathcal{X}(C_i)=\frac{1}{|W(\omega)|}\sum_{\theta\in\mathrm{Irr}_{\rm reg}(L^F)}\prod_{i=1}^k \epsilon_G\epsilon_L R^G_L(\theta\cdot\mathcal{X}^L_\chi)(C_i).$$

The value $\epsilon_G\epsilon_L R^G_L(\theta\cdot\mathcal{X}^L_\chi)(C_i)$ is of the form $\mathcal{X}(\sigma)$ (see formula (4.3.2)), with $(\alpha,\beta)=(\omega,\mu_i)$. Hence

$$\sum_{\mathcal{X}}\prod_{i=1}^k \mathcal{X}(C_i)=\frac{1}{|W(\omega)|}\sum_{\tau_1,\dots,\tau_k}\sum_{\{(\nu_1,\dots,\nu_k)|[\nu_i]=[\tau_i]\}}\prod_{i=1}^k \chi^{\omega}_{\tau} z^{-1}_{\tau_i} Q^{\mu_i}_{\nu_i}(q)$$
$$\times\sum_{(l_1,\dots,l_k)\in\overline{A}_{\nu_1}\times\cdots\times\overline{A}_{\nu_k}}\Big(\sum_{\theta\in\mathrm{Irr}_{\rm reg}(L^F)}\prod_{i=1}^k \theta^{T_{\tau_i}}(l_i\sigma_{i,s}l_i^{-1})\Big)$$

where $\sigma_{i,s}$ is the semisimple part of some fixed element $\sigma_i\in C_i$. Recall that for $\theta\in\mathrm{Irr}_{\rm reg}(L^F)$, $\theta^{T_{\tau_i}}$ is the restriction of θ to $T^F_{\tau_i}$. Assertion (1) of the theorem follows from Proposition 4.2.3 and formula (4.3.3). □

5. Character varieties

Fix a nonnegative integer g, and choose a generic tuple $(\mathcal{C}_1,\mathcal{C}_2,\dots,\mathcal{C}_k)$ of semisimple conjugacy classes of $\mathrm{GL}_n(\mathbb{C})$ of type $\boldsymbol{\mu}=(\mu^1,\dots,\mu^k)$ where $\mu^i=(\mu^i_1,\dots,\mu^i_{r_i})$ is a partition of n. Recall that the nonnegative integers $\mu^i_1,\dots,\mu^i_{r_i}$ are the multiplicities of the distinct eigenvalues of $\mathcal{C}_i$. Let $\mathcal{M}_{\boldsymbol{\mu}}$ be the corresponding complex character variety as defined in Section 2.1.

5.1. *Independence of the generic eigenvalues*

Though the variety $\mathcal{M}_{\boldsymbol{\mu}}$ depends on the choice of generic eigenvalues, our main Conjecture 1.2.1 predicts that the mixed Hodge polynomial $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t)$ should not. In general, in GIT problems depending on parameters, it is normal to see change in cohomology as one crosses a wall of a certain chamber structure in the space of parameters. In hyper-Kähler situations, however, it has been observed that no change takes place (see, e.g., [1], [13]).

Generalizing the argument of [22, Corollary 2.2.4], here we prove that, for a dense subset (in the analytic topology) of generic eigenvalues of multiplicities $\boldsymbol{\mu}$, the mixed Hodge polynomial of $\mathcal{M}_{\boldsymbol{\mu}}$ is constant. In particular, at least on this dense subset, there is no change of behavior across walls. We prove in Corollary 5.2.2 below that the E-polynomial of $\mathcal{M}_{\boldsymbol{\mu}}$ is completely independent of the choice of generic eigenvalues of multiplicities $\boldsymbol{\mu}$.

PROPOSITION 5.1.1

There is a dense subset (in the analytic topology) of generic eigenvalues of multiplicities $\boldsymbol{\mu}$ for which the mixed Hodge polynomial $H_c(\mathcal{M}_{\boldsymbol{\mu}}; x, y, t)$ is constant.

Proof

Let $r = r_1 + \cdots + r_k$ be the number of distinct eigenvalues of the conjugacy classes $\mathcal{C}_i$. With the notation of the proof of Lemma 2.1.2, pick $a' \in A' \cong \mathbb{G}_m^{r-1}$ corresponding to $r-1$ algebraically independent transcendental complex numbers. By a general fact on automorphisms of $\mathbb{C}/\mathbb{Q}$ any two such choices can be conjugated by an element of $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$. By functoriality, the two corresponding varieties have isomorphic mixed Hodge structures. This proves our claim. □

5.2. *E-polynomial*

In this section we prove that $\mathcal{M}_{\boldsymbol{\mu}}$ is polynomial-count, and we give a closed formula for $E(\mathcal{M}_{\boldsymbol{\mu}}; q)$. This formula will be used to compute the Euler characteristic in Section 5.3 and later to prove the connectedness of $\mathcal{M}_{\boldsymbol{\mu}}$ (see [20], [21]).

THEOREM 5.2.1

The variety $\mathcal{M}_{\boldsymbol{\mu}}$ is polynomial-count, and its E-polynomial is given by

$$E(\mathcal{M}_{\boldsymbol{\mu}}; q) = q^{(1/2)d_\mu}\, \mathbb{H}_{\boldsymbol{\mu}}\Big(\sqrt{q}, \frac{1}{\sqrt{q}}\Big)$$

where $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$ is defined in (2.3.25) and $d_{\boldsymbol{\mu}} = \dim(\mathcal{M}_{\boldsymbol{\mu}})$ (see (1.2.1)).

Proof

It is clear that $\mathbb{H}_{\boldsymbol{\mu}}(z, w) \in \mathbb{Q}(z, w)$. Hence Theorem 5.2.3 below implies that there exists $Q(x) \in \mathbb{Q}(x)$ such that for all r we have $\#\mathcal{M}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_{q^r}) = Q(q^r)$. In particular, $Q(x)$ is an integer for infinitely many integer values of x; hence $Q(x) \in \mathbb{Q}[x]$. Therefore $\mathcal{M}_{\boldsymbol{\mu}}$ is polynomial-count, and so our claim follows from Theorem 2.4.1 and Theorem 5.2.3 below. □

The theorem has the following straightforward consequence.

COROLLARY 5.2.2
The E-polynomial of $\mathcal{M}_{\boldsymbol{\mu}}$ does not depend on the choice of the generic semisimple conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ of a given type $\boldsymbol{\mu}$.

Let $\mathcal{U}_{\boldsymbol{\mu}} = \operatorname{Spec}(\mathcal{A})$ be the R-scheme defined in Appendix A. Put $\mathcal{X}_{\boldsymbol{\mu}} = \operatorname{Spec}(\mathcal{A}^{\mathrm{PGL}_n(R)})$. Then the R-scheme $\mathcal{X}_{\boldsymbol{\mu}}$ is a *spreading out* of $\mathcal{M}_{\boldsymbol{\mu}}$; that is, $\mathcal{X}_{\boldsymbol{\mu}}$ gives back $\mathcal{M}_{\boldsymbol{\mu}}$ after extension of scalars from R to $\mathbb{C}$. If $\phi : R \to k$ is a ring homomorphism into a field k, we denote by $\mathcal{M}_{\boldsymbol{\mu}}^{\phi}$ the k-scheme obtained from $\mathcal{X}_{\boldsymbol{\mu}}$ by extension of scalars.

THEOREM 5.2.3
For any ring homomorphism $\phi : R \to \mathbb{F}_q$,

$$\#\mathcal{M}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q) = q^{(1/2)d_{\boldsymbol{\mu}}}\, \mathbb{H}_{\boldsymbol{\mu}}\Big(\sqrt{q}, \frac{1}{\sqrt{q}}\Big).$$

Proof
Let k be an algebraic closure of $\mathbb{F}_q$. Since $\mathrm{PGL}_n(k)$ is connected any F-stable $\mathrm{PGL}_n(k)$-orbit of $\mathcal{U}_{\boldsymbol{\mu}}^{\phi}(k)$ contains an F-stable point, that is, an $\mathbb{F}_q$-rational point. Hence the natural map

$$\mathcal{U}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q)/\mathrm{PGL}_n(\mathbb{F}_q) \to \big(\mathcal{U}_{\boldsymbol{\mu}}^{\phi}(k)/\mathrm{PGL}_n(k)\big)^F = \mathcal{M}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q)$$

is surjective. The k-tuple of conjugacy classes $(\mathcal{C}_1^{\phi}, \ldots, \mathcal{C}_k^{\phi})$ being generic, the group $\mathrm{PGL}_n(\mathbb{F}_q)$ acts freely on $\mathcal{U}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q)$, and so the above map is injective. Hence

$$\#\mathcal{M}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q) = \frac{\#\mathcal{U}_{\boldsymbol{\mu}}^{\phi}(\mathbb{F}_q)}{|\mathrm{PGL}_n(\mathbb{F}_q)|}.$$

Let $\mathrm{Irr}(G^F)_\omega$ denote the set of irreducible characters of type ω. We denote by $\mathcal{X}_\omega(1)$ the degree of the irreducible characters in $\mathrm{Irr}(G^F)_\omega$. For $i \in \{1, \ldots, k\}$, let C_i be the conjugacy class $\mathcal{C}_i^{\phi}(\mathbb{F}_q)$ of $G^F = \mathrm{GL}_n(\mathbb{F}_q)$. From Proposition 3.1.4 and Theorem 4.3.1(1) (see Remark 4.3.2), we have

$$\begin{aligned}\#\mathcal{U}^{\phi}_{\boldsymbol{\mu}}(\mathbb{F}_q) &= |G^F|^{2g-1} \sum_{\mathcal{X}\in \mathrm{Irr}(G^F)} \frac{1}{\mathcal{X}(1)^{2g-2+k}} \prod_{i=1}^{k} |C_i|\mathcal{X}(C_i)\\
&= \sum_{\omega\in \mathbf{T}_n} \frac{|G^F|^{2g-1}\prod_{i=1}^k |C_i|}{\mathcal{X}_\omega(1)^{2g-2+k}} \sum_{\mathcal{X}\in \mathrm{Irr}(G^F)_\omega} \prod_{i=1}^{k} \mathcal{X}(C_i)\\
&= \sum_{\omega\in \mathbf{T}_n} \frac{|G^F|^{2g-1}\prod_{i=1}^k |C_i|}{\mathcal{X}_\omega(1)^{2g-2+k}} \hat{\mathbf{H}}^{\boldsymbol{\mu}_*}_{\omega}(q)\\
&= \sum_{\omega\in \mathbf{T}_n} \frac{|G^F|^{2g-1}(q-1)K^o_\omega\prod_{i=1}^k |C_i|}{|W(\omega)|\mathcal{X}_\omega(1)^{2g-2+k}} (-1)^{kn+kf(\omega)} \prod_{i=1}^{k} A(\omega,\mu^i)\end{aligned}$$

with $A(\omega,\mu^i_*)$ as in Lemma 2.3.5, where μ^i_* is the type in $\mathbf{T}_n$ corresponding to the partition μ^i (see beginning of this section). For a type $\omega=(d_1,\omega^1)\cdots(d_r,\omega^r)$, recall (see [40, Chapter IV, (6.7)])

$$\frac{|G^F|}{\mathcal{X}_\omega(1)} = (-1)^{f(\omega)} H_\omega(q) q^{\frac{1}{2}n(n-1)-n(\omega)}.$$

By formula (2.3.22) we have $\mathcal{H}^0_{\mu^i_*}(0,\sqrt{q}) = |C_i|/|G^F|$, and note that $C^0_\omega = K^o_\omega/|W(\omega)|$ (see formula (2.3.10) and Proposition 4.2.1). Using also Lemma 2.3.5, we thus deduce that

$$\begin{aligned}&\#\mathcal{U}^{\phi}_{\boldsymbol{\mu}}(\mathbb{F}_q)\\
&\quad= |G^F|(q-1)\sum_{\omega\in\mathbf{T}_n} \big((-1)^{f(\omega)} H_\omega(q) q^{(1/2)n(n-1)-n(\omega)}\big)^{2g+k-2}\\
&\qquad\qquad \times C^0_\omega (-1)^{kn+kf(\omega)}\\
&\qquad \times \prod_{i=1}^{k} \langle s_\omega(\mathbf{x}_i), \mathcal{H}^0_{\mu^i_*}(0,\sqrt{q})\tilde{H}_{\mu^i_*}(\mathbf{x}_i;q)\rangle\\
&\quad= |G^F|(q-1)(-1)^{kn} q^{(1/2)n(n-1)(2g+k-2)} \sum_{\omega\in\mathbf{T}_n} C^0_\omega (H_\omega(q) q^{-n(\omega)})^{2g+k-2}\\
&\qquad \times \prod_{i=1}^{k} \langle s_\omega(\mathbf{x}_i), \mathcal{H}^0_{\mu^i_*}(0,\sqrt{q})\tilde{H}_{\mu^i_*}(\mathbf{x}_i;q)\rangle\\
&\quad= |G^F|(q-1)(-1)^{kn} q^{(1/2)(n^2(k+2g-2)-kn)}\end{aligned}$$

$$\times\Big\langle\sum_{\omega\in\mathbf{T}} C^0_\omega q^{(1-g)|\omega|}(H_\omega(q)q^{-n(\omega)})^{2g+k-2}\prod_{i=1}^k s_\omega(\mathbf{x}_i),$$

$$\prod_{i=1}^k \mathcal{H}^0_{\mu^i_*}(0,\sqrt{q})\tilde{H}_{\mu^i_*}(\mathbf{x}_i;q)\Big\rangle$$

$$= |G^F|(q-1)(-1)^{kn}q^{(1/2)(n^2(k+2g-2)-kn)}$$

$$\times\Big\langle\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}} q^{(1-g)|\lambda|}(H_\lambda(q)q^{-n(\lambda)})^{2g+k-2}\prod_{i=1}^k s_\lambda(\mathbf{x}_i)\Big),$$

$$\prod_{i=1}^k \mathcal{H}^0_{\mu^i_*}(0,\sqrt{q})\tilde{H}_{\mu^i_*}(\mathbf{x}_i;q)\Big\rangle$$

$$= |G^F|(q-1)q^{(1/2)(n^2(k+2g-2)-kn)-\sum_i n(\mu^i_*)}$$

$$\times\Big\langle\mathrm{Log}\Big(\sum_{\lambda\in\mathcal{P}} q^{(1-g)|\lambda|}(H_\lambda(q)q^{-n(\lambda)})^{2g+k-2}\prod_{i=1}^k s_\lambda(\mathbf{x}_i)\Big),\prod_{i=1}^k h_{\mu^i}(\mathbf{x}_i\mathbf{y})\Big\rangle.$$

In the third equality $|\omega|$ is defined as the size of ω; that is, $|\omega| = n$ if $\omega\in\mathbf{T}_n$. The last equality follows from Lemma 2.3.6. For any symmetric functions u and v, $\langle u(\mathbf{xy}), v(\mathbf{x})\rangle = \langle u(\mathbf{x}), v(\mathbf{xy})\rangle$. This can be checked on the basis of power symmetric functions. We deduce from Lemma 2.3.8 that

$$\#\mathcal{U}^\phi_{\boldsymbol{\mu}}(\mathbb{F}_q) = |G^F|(q-1)q^{(1/2)(n^2(k+2g-2)-kn)-\sum_i n(\mu^i_*)}$$

$$\times\Big\langle\mathrm{Log}(\Omega(\sqrt{q},1/\sqrt{q})),\prod_{i=1}^k h_{\mu^i}(\mathbf{x}_i)\Big\rangle.$$

We thus have

$$\#\mathcal{M}^\phi_{\boldsymbol{\mu}}(\mathbb{F}_q)$$

$$= (q-1)^2 q^{(1/2)(n^2(k+2g-2)-kn)-\sum_i n(\mu^i_*)}\Big\langle\mathrm{Log}(\Omega(\sqrt{q},1/\sqrt{q})),\prod_{i=1}^k h_{\mu^i}(\mathbf{x}_i)\Big\rangle.$$

We have $\mathbb{H}_{\boldsymbol{\mu}}(\sqrt{q},1/\sqrt{q}) = ((q-1)^2/q)\langle\mathrm{Log}(\Omega(\sqrt{q},1/\sqrt{q})),\prod_{i=1}^k h_{\mu^i}(\mathbf{x}_i)\rangle$. It remains to check that the remaining power of q is $d_{\boldsymbol{\mu}}/2$, but this follows from the observation that $2n(\mu^i_*)+n=\sum_j(\mu^i_j)^2$. □

Here we can prove a consequence of the curious Poincaré duality Conjecture 1.2.2.

COROLLARY 5.2.4
The E-polynomial is palindromic; that is, it satisfies the "curious" Poincaré duality

$$E(\mathcal{M}_{\boldsymbol{\mu}};q) = q^{d_\mu} E(\mathcal{M}_{\boldsymbol{\mu}};q^{-1}) = \sum_i \Big(\sum_k (-1)^k h^{i,i;k}(\mathcal{M}_{\boldsymbol{\mu}})\Big) q^i.$$

Proof
By Theorem 2.1.5 the variety $\mathcal{M}_{\boldsymbol{\mu}}$ is nonsingular of pure dimension $d_{\boldsymbol{\mu}}$. Hence the second equality is a consequence of formula (2.4.1). From Theorem 5.2.1 we have

$$E(\mathcal{M}_{\boldsymbol{\mu}};q^{-1}) = q^{-d_\mu/2}\mathbb{H}_{\boldsymbol{\mu}}(1/\sqrt{q},\sqrt{q}) = q^{-d_\mu/2}\frac{(q-1)^2}{q}\Big\langle \prod_i h_{\mu^i}(\mathbf{x}_i), \mathrm{Log}(\Omega(1/\sqrt{q},\sqrt{q}))\Big\rangle.$$

From (2.3.24) we conclude that

$$E(\mathcal{M}_{\boldsymbol{\mu}};q^{-1}) = q^{-d_\mu/2}\frac{(q-1)^2}{q}\Big\langle \prod_i h_{\mu^i}(\mathbf{x}_i), \mathrm{Log}(\Omega(\sqrt{q},1/\sqrt{q}))\Big\rangle = q^{-d_\mu} E(\mathcal{M}_{\boldsymbol{\mu}};q). \qquad \square$$

5.3. *Euler characteristic*

The $2g$-dimensional torus $(\mathbb{C}^\times)^{2g}$ acts on the character variety $\mathcal{M}_{\boldsymbol{\mu}}$ by scalar multiplication on the first $2g$ coordinates. Let $\tilde{\mathcal{M}}_{\boldsymbol{\mu}}$ be the affine GIT quotient $\mathcal{M}_{\boldsymbol{\mu}}//(\mathbb{C}^\times)^{2g}$. Exactly as in [22, Theorem 2.2.12] we can argue that

$$H^*(\mathcal{M}_{\boldsymbol{\mu}}) \cong H^*\big((\mathbb{C}^\times)^{2g}\big) \otimes H^*(\tilde{\mathcal{M}}_{\boldsymbol{\mu}})$$

as mixed Hodge structures, which implies that

$$H_c(\mathcal{M}_{\boldsymbol{\mu}};x,y,t) = H_c(\tilde{\mathcal{M}}_{\boldsymbol{\mu}};x,y,t)\cdot(1+xyt)^{2g}$$

and hence also that

$$E(\mathcal{M}_{\boldsymbol{\mu}};q) = E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}};q)\cdot(1-q)^{2g}.$$

It follows that $E(\mathcal{M}_{\boldsymbol{\mu}}) = 0$ if $g > 0$. Here we compute $E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}})$ for $g > 0$.

Remark 5.3.1
Note, in particular, that Conjecture 1.2.1(iii) implies that $(z-w)^{2g}$ should divide $\mathbb{H}_{\boldsymbol{\mu}}(z,w)$. This is not readily visible from its definition (see (1.1.3)).

THEOREM 5.3.2
Assume that $g > 1$; then

$$E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = \begin{cases} \mu(n)n^{2g-3} & \text{if } \boldsymbol{\mu} = ((n), \ldots, (n)), \\ 0 & \text{otherwise,} \end{cases}$$

where μ is the ordinary Möbius function.

Proof
First note that

$$E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = \frac{\langle h_{\boldsymbol{\mu}}, \mathrm{Log}(\Omega(\sqrt{q}, 1/\sqrt{q}))\rangle}{(q-1)^{2g-2}}\Big|_{q=1}, \tag{5.3.1}$$

where, as before, $h_{\boldsymbol{\mu}} := \prod_{i=1}^{k} h_{\mu^i}(\mathbf{x}_i)$. We have by Section 2.3.6,

$$\Omega(\sqrt{q}, 1/\sqrt{q}) = \sum_{\lambda \in \mathcal{P}} A_\lambda, \quad A_\lambda := \left(q^{-\frac{1}{2}\langle \lambda, \lambda\rangle} H_\lambda(q)\right)^{2g-2} \prod_{i=1}^{k} \tilde{H}_\lambda(\mathbf{x}_i; q, q^{-1}). \tag{5.3.2}$$

Let $U_n = U_n(\mathbf{x}_1, \ldots, \mathbf{x}_k; q)$ be defined by

$$\log\left(\Omega(\sqrt{q}, 1/\sqrt{q})\right) = \sum_{n \geq 1} \frac{1}{n} U_n(\mathbf{x}_1, \ldots, \mathbf{x}_k; q);$$

then, as in (2.3.6),

$$\frac{U_n}{n} = \sum_{m_\lambda} (-1)^{m-1}(m-1)! \prod_\lambda \frac{A_\lambda^{m_\lambda}}{m_\lambda!} \tag{5.3.3}$$

where $m := \sum_\lambda m_\lambda$ and the sum is over all sequences $\{m_\lambda\}$ of nonnegative integers such that $\sum_\lambda m_\lambda |\lambda| = n$. Since $(q-1)^{|\lambda|}$ divides $H_\lambda(q)$, $(q-1)^{(2g-2)n}$ divides U_n as it divides each term in the sum (5.3.3). Let $V_n = V_n(\mathbf{x}_1, \ldots, \mathbf{x}_k; q)$ be defined by

$$\mathrm{Log}\left(\Omega(\sqrt{q}, 1/\sqrt{q})\right) = \sum_{n \geq 1} V_n(\mathbf{x}_1, \ldots, \mathbf{x}_k; q),$$

and then by (2.3.5),

$$\left\langle h_{\boldsymbol{\mu}}, \mathrm{Log}(\Omega(\sqrt{q}, 1/\sqrt{q}))\right\rangle = \langle h_{\boldsymbol{\mu}}, V_n\rangle = \frac{1}{n} \sum_{d|n} \mu(d) \langle h_{\boldsymbol{\mu}}, U_{n/d}(\mathbf{x}_1^d, \ldots, \mathbf{x}_k^d; q^d)\rangle.$$

Since $(q-1)^{(2g-2)(n/d)}$ divides $U_{n/d}(\mathbf{x}_1^d, \ldots, \mathbf{x}_k^d; q^d)$ for all d, we have

$$\frac{\langle h_{\boldsymbol{\mu}}, V_n\rangle}{(q-1)^{2g-2}}\Big|_{q=1} = \frac{1}{n}\mu(n)\Big\langle h_{\boldsymbol{\mu}}, \frac{U_1(\mathbf{x}_1^n, \ldots, \mathbf{x}_k^n; q^n)}{(q-1)^{2g-2}}\Big|_{q=1}\Big\rangle.$$

But

$$U_1(\mathbf{x}_1^n, \ldots, \mathbf{x}_k^n; q^n) = q^{n(g-1)}(q^n - 1)^{2g-2} \prod_{i=1}^{k} \tilde{H}_{(1)}(\mathbf{x}_i^n; q^n, q^{-n}),$$

and $\tilde{H}_{(1)}(\mathbf{x}^n) = p_{(1)}(\mathbf{x}^n) = p_{(n)}(\mathbf{x})$. Hence

$$\begin{aligned}\frac{\langle h_{\boldsymbol{\mu}}, V_n\rangle}{(q-1)^{2g-2}}\Big|_{q=1} &= \frac{1}{n}\mu(n)n^{2g-3}\prod_{i=1}^{k}\langle h_{\mu^i}(\mathbf{x}_i), p_{(n)}(\mathbf{x}_i)\rangle \\ &= \begin{cases} \frac{1}{n}\mu(n)n^{2g-3} & \text{if } \boldsymbol{\mu} = ((n), \ldots, (n)), \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The last equality follows from Lemma 2.3.1. □

THEOREM 5.3.3
For $g = 1$,

$$E(\tilde{\mathcal{M}}_{\boldsymbol{\mu}}) = \frac{1}{n} \sum_{d \mid \gcd(\mu_j^i)} \sigma(n/d)\mu(d) \frac{((n/d)!)^k}{\prod_{i,j}(\mu_j^i/d)!}$$

where $\sigma(m) = \sum_{d \mid m} d$.

Proof
By [40, Chapter VI, (8.16)], we have $K_{\lambda\mu}(1,1) = \chi^{\lambda}_{(1^n)} = n!/h(\lambda)$ where $h(\lambda)$ is the hook length of λ, and so for a partition μ of size n, we have (see [40, Chapter I, p. 66])

$$\tilde{H}_{\mu}(\mathbf{x}; 1, 1) = \sum_{\lambda} \frac{n!}{h(\lambda)} s_{\lambda}(\mathbf{x}) = e_1(\mathbf{x})^n = h_1(\mathbf{x})^n.$$

Hence

$$\Omega(1,1) = \sum_{\lambda} h^{|\lambda|}_{(1,1,\ldots,1)} = \prod_{m \geq 1}(1 - h_1^m)^{-1} \tag{5.3.4}$$

by Euler's formula. As before, let $U_n = U_n(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ and $V_n = V_n(\mathbf{x}_1, \ldots, \mathbf{x}_k)$ be the coefficients of $\log(\Omega^1(1,1))$ and $\operatorname{Log}(\Omega^1(1,1))$, respectively. Then $U_n = \sigma(n)h_1$ and

$$\begin{aligned}\big\langle h_{\boldsymbol{\mu}}, \operatorname{Log}(\Omega(1,1))\big\rangle &= \langle h_{\boldsymbol{\mu}}, V_n\rangle \\ &= \frac{1}{n}\sum_{d \mid n}\sigma(n/d)\mu(d)\prod_{i=1}^{k}\langle h_1(\mathbf{x}_i^d)^{n/d}, h_{\mu^i}(\mathbf{x}_i)\rangle\end{aligned}$$

$$= \frac{1}{n}\sum_{d|n}\sigma(n/d)\mu(d)\prod_{i=1}^{k}\langle p_{(d^{n/d})}(\mathbf{x}_i), h_{\mu^i}(\mathbf{x}_i)\rangle$$

$$= \frac{1}{n}\sum_{d|\gcd(\mu^i_j)}\sigma(n/d)\mu(d)\frac{((n/d)!)^k}{\prod_{i,j}(\mu^i_j/d)!}.$$

The last equality follows from Lemma 2.3.1. □

Remark 5.3.4

The task to evaluate the Euler characteristic when $g = 0$ is more complicated, due to the presence of high-order poles in $\mathcal{H}^0_\lambda(\sqrt{q}, 1/\sqrt{q})$ at $q = 1$ (see (1.5.8) for a computation in a specific example).

6. The pure part of $\mathbb{H}_\mu(z, w)$

In this section we fix once and for all a multipartition $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k) \in (\mathcal{P}_n)^k$ where $\mu^i = (\mu^i_1, \dots, \mu^i_{l_i})$. We give both a representation theoretical and a cohomological interpretation of the pure part $\mathbb{H}_{\boldsymbol{\mu}}(0, w)$ of $\mathbb{H}_{\boldsymbol{\mu}}(z, w)$.

6.1. Multiplicities in tensor products

In this section $G = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$. For a partition $\mu = (n_1, \dots, n_r)$ we define $\mu_\dagger$ to be the type $(1, (n_1)^1)\cdots(1, (n_r)^1) \in \mathbf{T}$. Let $(\mathcal{X}_1, \dots, \mathcal{X}_k)$ be a generic tuple of k-irreducible characters of type $\boldsymbol{\mu}_\dagger := (\mu^1_\dagger, \dots, \mu^k_\dagger) \in \mathbf{T}_n$. The irreducible characters $\mathcal{X}_1, \dots, \mathcal{X}_k$ are then semisimple. Put

$$R_{\boldsymbol{\mu}} := \bigotimes_{i=1}^{k}\mathcal{X}_i.$$

Let $\Lambda : G^F \to \overline{\mathbb{Q}}_\ell$ be defined by $x \mapsto q^{g \dim C_G(x)}$. Note that the map $x \mapsto q^{\dim C_G(x)}$ is the character of the representation of G^F in the group algebra $\overline{\mathbb{Q}}_\ell[\mathfrak{g}^F]$ where G^F acts on $\mathfrak{g}^F$ by the adjoint action.

Let $\langle\cdot,\cdot\rangle_{G^F}$ be the nondegenerate bilinear form on $C(G^F)$ defined by

$$\langle f, g\rangle_{G^F} = |G^F|^{-1}\sum_{x\in G^F} f(x)\overline{g(x)}.$$

THEOREM 6.1.1

We have

$$\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle_{G^F} = \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}).$$

Proof
Recall Lemma 4.1.1, which says that if C is a conjugacy class of G^F of type $\omega \in \mathbf{T}_n$, then $\mathcal{H}_\omega(0, \sqrt{q}) = q^{g \dim C_G(x)}|C|/|G^F|$ where $x \in C$. Hence by Theorem 4.3.1(2),

$$
\begin{aligned}
&\Big\langle \Lambda \otimes \bigotimes_{i=1}^{k} \mathcal{X}_i, \mathrm{Id} \Big\rangle_{G^F} \\
&\qquad = \sum_C \frac{|C|}{|G^F|} \Lambda(C) \prod_{i=1}^{k} \mathcal{X}_i(C) \\
&\qquad = \sum_{\omega \in \mathbf{T}_n} \mathcal{H}_\omega(0, \sqrt{q}) \mathbf{H}_\omega^{\boldsymbol{\mu}^\dagger}(q) \\
&\qquad = \sum_{\omega \in \mathbf{T}_n} \frac{(q-1)K_\omega^o}{|W(\omega)|} \mathcal{H}_\omega(0, \sqrt{q}) \prod_{i=1}^{k} (-1)^{n+f(\mu_\dagger^i)} \langle s_{\mu_\dagger^i}(\mathbf{x}_i), \tilde{H}_\omega(\mathbf{x}_i; q) \rangle \\
&\qquad = (-1)^{kn+\sum_i f(\mu_\dagger^i)} \sum_{\omega \in \mathbf{T}_n} (q-1) C_\omega^0 \mathcal{H}_\omega(0, \sqrt{q}) \Big\langle \prod_i s_{\mu_\dagger^i}(\mathbf{x}_i), \prod_i \tilde{H}_\omega(\mathbf{x}_i; q) \Big\rangle \\
&\qquad = (-1)^{kn+\sum_i f(\mu_\dagger^i)} (q-1) \Big\langle \prod_i s_{\mu_\dagger^i}(\mathbf{x}_i), \sum_{\omega \in \mathbf{T}_n} C_\omega^0 \mathcal{H}_\omega(0, \sqrt{q}) \prod_i \tilde{H}_\omega(\mathbf{x}_i; q) \Big\rangle \\
&\qquad = (q-1) \Big\langle \prod_i h_{\mu^i}(\mathbf{x}_i), \mathrm{Log}(\Omega(0, \sqrt{q})) \Big\rangle.
\end{aligned}
$$

The last equality follows from the fact that $f(\mu_\dagger^i) = n$ and $s_{\mu_\dagger^i}(\mathbf{x}) = s_{(\mu_1^i)^1}(\mathbf{x}) \cdots s_{(\mu_{l_i}^i)^1}(\mathbf{x}) = h_{\mu^i}(\mathbf{x})$. $\square$

6.2. *Poincaré polynomial of quiver varieties*

Here we assume that $\boldsymbol{\mu}$ is indivisible so that we can choose a generic tuple $(\mathcal{O}_1, \ldots, \mathcal{O}_k)$ of semisimple adjoint orbits of $\mathfrak{gl}_n(\mathbb{C})$ of type $\boldsymbol{\mu}$. Let $\mathcal{Q}_{\boldsymbol{\mu}}$ be the corresponding complex quiver variety as in Section 2.2.

The aim of this section is to prove the following theorem.

THEOREM 6.2.1
The compactly supported Poincaré polynomial of $\mathcal{Q}_{\boldsymbol{\mu}}$ is given by

$$P_c(\mathcal{Q}_{\boldsymbol{\mu}}; t) = t^{d_{\boldsymbol{\mu}}} \mathbb{H}_{\boldsymbol{\mu}}(0, t).$$

As we did for the character variety in Appendix A, we define a spreading out $\mathcal{Y}_{\boldsymbol{\mu}}/\mathcal{R}$ of $\mathcal{Q}_{\boldsymbol{\mu}}$ such that for any ring homomorphism $\phi : \mathcal{R} \to \mathbb{K}$ into an algebraically closed field $\mathbb{K}$, the adjoint orbits $\mathcal{O}_1^\phi, \ldots, \mathcal{O}_k^\phi$ of $\mathfrak{gl}_n(\mathbb{K})$ are generic and of the same type as $\mathcal{O}_1, \ldots, \mathcal{O}_k$. Let $\mathcal{Q}_{\boldsymbol{\mu}}^\phi$ denote the corresponding quiver variety over $\mathbb{K}$.

THEOREM 6.2.2
For any ring homomorphism $\phi : \mathcal{R} \to \mathbb{F}_q$ we have

$$\#\mathcal{Q}_{\boldsymbol{\mu}}^\phi(\mathbb{F}_q) = q^{(1/2)d_{\boldsymbol{\mu}}} \mathbb{H}_{\boldsymbol{\mu}}(0, \sqrt{q}). \tag{6.2.1}$$

Theorem 6.2.1 follows from Propositions 2.4.2 and 2.2.6 and Theorem 6.2.2. Indeed, Theorem 6.2.2 implies that $\mathcal{Q}_{\boldsymbol{\mu}}^g/\mathbb{C}$ is polynomial-count.

We now prove Theorem 6.2.2.

For $i \in \{1, \ldots, k\}$, let O_i be the adjoint orbit $\mathcal{O}_i^\phi(\mathbb{F}_q)$ of $\mathfrak{g}^F = \mathfrak{gl}_n(\mathbb{F}_q)$. As in the character variety case we show that

$$\#\mathcal{Q}_{\boldsymbol{\mu}}^\phi(\mathbb{F}_q) = \frac{\#\mathcal{V}_{\boldsymbol{\mu}}^\phi(\mathbb{F}_q)}{|\mathrm{PGL}_n(\mathbb{F}_q)|}.$$

Let $\Lambda : \mathfrak{g}^F \to \overline{\mathbb{Q}}_\ell$, $x \mapsto q^{g \dim C_G(x)}$. By Proposition 3.2.2 and Remark 3.2.3, we have

$$\begin{aligned}\#\mathcal{Q}_{\boldsymbol{\mu}}^\phi(\mathbb{F}_q) &= q^{n^2(g-1)}(q-1)\sum_O \frac{|O|}{|G^F|}\Lambda(O)\prod_{i=1}^k \mathcal{F}^{\mathfrak{g}}(1_{O_i})(O)\\ &= q^{n^2(g-1)}(q-1)\sum_{\omega\in\mathbf{T}_n}\mathcal{H}_\omega(0,\sqrt{q})\sum_O\prod_{i=1}^k \mathcal{F}^{\mathfrak{g}}(1_{O_i})(O)\end{aligned}$$

where the second sum is over the adjoint orbits O of $\mathfrak{g}^F$ of type ω. The type of adjoint orbits is defined exactly as for conjugacy classes (see Section 4.1). We need the following lemma.

LEMMA 6.2.3
Given $\omega \in \mathbf{T}_n$, we have

$$\sum_O \prod_{i=1}^k \mathcal{F}^{\mathfrak{g}}(1_{O_i})(O) = \frac{q^{1+\sum_i d_i/2}}{q-1}\mathbf{H}_\omega^{\boldsymbol{\mu}^\dagger}(q)$$

where the sum is over the adjoint orbits of type ω, where $\boldsymbol{\mu}_\dagger$ is as in Section 6.1, and where $d_i = n^2 - \sum_j (\mu_j^i)^2$.

Proof
We first remark that if C is a semisimple adjoint orbit of $\mathfrak{g}^F$ of type $(1, 1^{n_1})(1, 1^{n_2})\cdots$

$(1, 1^{n_r})$, then by formula (2.5.5),

$$\mathcal{F}^{\mathfrak{g}}(1_C) = \epsilon_G \epsilon_L |W_L|^{-1} \sum_{w \in W_L} q^{d_L/2} \mathcal{R}^{\mathfrak{g}}_{\mathfrak{t}_w}\big(\mathcal{F}^{\mathfrak{t}_w}(1_\sigma^{T_w})\big)$$

where $L = \prod_{i=1}^r \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)$ and where $\sigma \in C \cap L$.

If $\mathcal{X}$ is an irreducible character of type $(1,(n_1)^1)(1,(n_2)^1)\cdots(1,(n_r)^1)$, then by formula (2.5.2) we have

$$\mathcal{X} = \epsilon_G \epsilon_L |W_L|^{-1} \sum_{w \in W_L} R^G_{T_w}(\theta^{T_w})$$

where $L = \prod_{i=1}^r \mathrm{GL}_{n_i}(\overline{\mathbb{F}}_q)$. Hence from formulas (2.5.1) and (2.5.4) we see that the calculation of the values of $\mathcal{X}$ and $\mathcal{F}^{\mathfrak{g}}(1_C)$ is completely similar. We thus may follow the proof of Theorem 4.3.1(2) to compute $\sum_{\mathcal{O}} \prod_{i=1}^k \mathcal{F}^{\mathfrak{g}}(1_C)(O)$. To do that we need to use the Lie algebra analogue of Proposition 4.2.1, which is as follows. Let M be an F-stable Levi subgroup of G of type $\omega \in \hat{\mathbf{T}}_n$ with Lie algebra $\mathfrak{m}$. We say that a linear character $\Theta : z(\mathfrak{m})^F \to \overline{\mathbb{Q}}_\ell$ is *generic* if its restriction to $z(\mathfrak{g})^F$ is trivial and if for any proper F-stable Levi subgroup H containing M, its restriction to $z(\mathfrak{h})^F$ is nontrivial. Put

$$z(\mathfrak{m})_{\mathrm{reg}} := \big\{ x \in z(\mathfrak{m}) \,\big|\, C_G(x) = M \big\}.$$

Then

$$\sum_{z \in z(\mathfrak{m})^F_{\mathrm{reg}}} \Theta(z) = q K^o_\omega$$

where K^o_ω is as in Proposition 4.2.1. The proof of this identity is completely similar to that of Proposition 4.2.1 except that here we are working with additive characters of $\mathbb{F}_q$ instead of multiplicative characters of $\mathbb{F}_q^\times$. This explains the coefficient q instead of $q-1$. □

We thus have

$$\begin{aligned}\#\mathcal{Q}^\phi_{\boldsymbol{\mu}}(\mathbb{F}_q) &= q^{n^2(g-1)}(q-1) \sum_{\omega \in \mathbf{T}_n} \mathcal{H}_\omega(0, \sqrt{q}) \frac{q^{1+\sum_i d_i/2}}{q-1} \mathbf{H}^{\boldsymbol{\mu}^\dagger}_\omega(q) \\ &= q^{d_{\boldsymbol{\mu}}/2} \sum_{\omega \in \mathbf{T}_n} \mathcal{H}_\omega(0, \sqrt{q}) \mathbf{H}^{\boldsymbol{\mu}^\dagger}_\omega(q).\end{aligned}$$

We may now proceed as in the proof of Theorem 6.1.1 to complete the proof of Theorem 6.2.2.

6.3. Quiver representations, Kac-Moody algebras, and the character ring of $\mathrm{GL}_n(\mathbb{F}_q)$

Let Γ be the comet-shaped quiver associated to g and $\boldsymbol{\mu}$ as in Section 2.2, and let $\mathbf{v}$ be the dimension vector with dimension $\sum_{j=1}^{l} \mu_j^i$ at the lth vertex on the ith leg. Then we have the following.

THEOREM 6.3.1

For $\boldsymbol{\mu}$ indivisible, the following are equivalent:

(a) $\langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle \neq 0$*;*

(b) *the quiver variety $\mathcal{Q}_{\boldsymbol{\mu}}$ is nonempty.*

For $g = 0$, (a) or (b) hold if and only if $\mathbf{v}$ *is a root of the Kac-Moody algebra associated to* Γ.

Proof

The equivalence between (a) and (b) follows from Theorems 6.2.1 and 6.1.1. If $g = 0$, then it is proved by Crawley-Boevey [3, Section 6] that $\mathcal{Q}_{\boldsymbol{\mu}}$ is nonempty if and only if $\mathbf{v}$ is a root. □

As mentioned in the introduction, the problem of the nonemptiness of $\mathcal{Q}_{\boldsymbol{\mu}}$ in the genus $g = 0$ case, which is part of the Deligne-Simpson problem, was first solved by Kostov (see [29], [30]). The equivalence of (a) and (b) in Theorem 6.3.1 is formally similar to the connection between Horn's problem (which asks for which partitions λ, μ, ν does $H_\lambda + H_\mu + H_\nu = 0$ have solutions in Hermitian matrices) and the problem of the nontrivial appearance of the trivial representation in the tensor product $V_\lambda \otimes V_\mu \otimes V_\nu$ of the irreducible representations V_λ, V_μ, V_ν of $\mathrm{GL}_n(\mathbb{C})$ (see [28]).

We conclude with a naturally arising question: Can the identity $A_{\boldsymbol{\mu}}(q) = \langle \Lambda \otimes R_{\boldsymbol{\mu}}, 1\rangle$ in Section 1.4 be strengthened by establishing an explicit bijection between the set of isomorphic classes of absolutely indecomposable representations of Γ and a basis of $(V_\Lambda \otimes V_1 \otimes \cdots \otimes V_k)^{\mathrm{GL}_n(\mathbb{F}_q)}$ where $V_\Lambda := (\overline{\mathbb{Q}}_\ell[\mathfrak{gl}_n(\mathbb{F}_q)])^{\otimes g}$ and V_i is a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ which affords the character $\mathcal{X}_i$?

Appendices

Appendix A

Fix integers $g \geq 0$, $k, n > 0$. We now construct a scheme whose points parameterize representations of the fundamental group of a k-punctured Riemann surface of genus g into GL_n with prescribed images in conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ at the punctures. We give the construction of this scheme in stages to alleviate the notation.

Fix $\boldsymbol{\mu} = (\mu^1, \mu^2, \ldots, \mu^k) \in \mathcal{P}_n{}^k$, and let a^i_j, for $i = 1, \ldots, k; j = 1, \ldots, r_i := l(\mu^i)$, be indeterminates. We should think of $a^i_1, \ldots, a^i_{r_i}$ as the distinct eigenvalues of $\mathcal{C}_i$ each with multiplicity μ^i_j; it will be in fact convenient to work with the multiset $\mathbf{A}_i := \{a^i_1, \ldots, a^i_1, a^i_2, \ldots, a^i_2, \ldots, a^i_{r_i}, \ldots, a^i_{r_i}\}$. To simplify we write $[\mathbf{A}] := \prod_{a \in \mathbf{A}} a$ for any multiset $\mathbf{A} \subseteq \mathbf{A}_i$.

Let

$$R_0 := \mathbb{Z}[a^i_j]/(1 - [\mathbf{A}_1] \cdots [\mathbf{A}_k]),$$

and consider the multiplicative set $S \subseteq R_0$ generated by (the classes of) $a^i_{j_1} - a^i_{j_2}$ for $j_1 \neq j_2$ and $1 - [\mathbf{A}'_1] \cdots [\mathbf{A}'_k]$ for $\mathbf{A}'_i \subseteq \mathbf{A}_i$ of the same cardinality n' with $0 < n' < n$.

Since R_0 is reduced and S does not contain 0, the localization

$$R := S^{-1} R_0$$

is not trivial (R is a ring with 1). We refer to it as the *ring of generic eigenvalues of type* $\boldsymbol{\mu}$.

In the special case where $k = 1$ and $\mu = (n)$ we have

$$R_0 = \mathbb{Z}[a]/(1 - a^n),$$

and $S \subseteq R_0$ is the multiplicative set generated by $1 - a^{n'}$ for $1 \leq n' < n$.

LEMMA A.1

For $k = 1$ and $\mu = (n)$ the ring $R = S^{-1} R_0$ is isomorphic to $\mathbb{Z}[1/n, \zeta_n]$, where ζ_n is a primitive nth root of unity.

Proof

The natural map $\psi : R_0 \to R = S^{-1} R_0$ has kernel the ideal generated by $(1 - a^n)/(1 - a^{n'})$ for $1 \leq n' < n$. This means that ψ factors through $\mathbb{Z}[\zeta_n] \hookrightarrow R$ with $\psi(a) = \zeta_n$. Since

$$\prod_{i=1}^{n-1} (1 - \zeta_n^i) = n$$

and each factor is in the image of S, it follows that $1/n \in R$. Hence $\mathbb{Z}[1/n, \zeta_n] \hookrightarrow R$.

By the same token, the map $\phi : R_0 \to \mathbb{Z}[1/n, \zeta_n]$ sending a to ζ_n takes $1 - a^{n'}$ to a unit. Hence by the universal property of R there is a unique extension $\phi : R \to \mathbb{Z}[1/n, \zeta_n]$. This completes the proof. □

In general, we have a map $\mathbb{Z}[a]/(1 - a^d) \hookrightarrow R_0$, where $d := \gcd(\mu^i_j)$, defined by sending a to $\prod_{i,j} (a^i_j)^{\mu^i_j/d}$. By the lemma we get $\mathbb{Z}[1/d, \zeta_d] \hookrightarrow R$.

Recall the definitions from Section 2.1. Note that, up to a possible reordering of eigenvalues of equal multiplicity, a map $\phi : R \to \mathbb{K}$ uniquely determines a k-tuple of semisimple generic conjugacy classes $(\mathcal{C}_1^\phi, \mathcal{C}_2^\phi, \ldots, \mathcal{C}_k^\phi)$ of type $\boldsymbol{\mu}$ in $\mathrm{GL}_n(\mathbb{K})$ satisfying (2.1.2); $\mathcal{C}_i^\phi$ has eigenvalues $\phi(a_j^i)$ of multiplicities μ_j^i.

Consider the algebra $\mathcal{A}_0$ over R of polynomials in $n^2(2g+k)$ variables, corresponding to the entries of $(n \times n)$-matrices $A_1, \ldots, A_g; B_1, \ldots, B_g; X_1, \ldots, X_k$, with

$$\det A_1, \ldots, \det A_k, \qquad \det B_1, \ldots, \det B_k, \qquad \det X_1, \ldots, \det X_k$$

inverted. Let I_n be the identity matrix, and for elements A, B of a group put $(A, B) := ABA^{-1}B^{-1}$.

Define $\mathcal{I}_0 \subset \mathcal{A}_0$ to be the radical of the ideal generated by the entries of

$$(A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k - I_n, \qquad (X_i - a_1^i I_n) \cdots (X_i - a_{r_i}^i I_n), \quad i = 1, \ldots, k,$$

and the coefficients of the polynomial

$$\det(tI_n - X_i) - \prod_{j=1}^{r_i} (t - a_j^i)^{\mu_j^i}$$

in an auxiliary variable t. Finally, let $\mathcal{A} := \mathcal{A}_0 / \mathcal{I}_0$ and $\mathcal{U}_{\boldsymbol{\mu}} := \mathrm{Spec}(\mathcal{A})$.

Let $\phi : R \to K$ be a map to a field K, and let $\mathcal{U}_{\boldsymbol{\mu}}^\phi$ be the corresponding base change of $\mathcal{U}_{\boldsymbol{\mu}}$ to K. A K-point of $\mathcal{U}_{\boldsymbol{\mu}}^\phi$ is a solution in $\mathrm{GL}_n(K)$ to

$$(A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n, \quad X_i \in \mathcal{C}_i^\phi,$$

where, as before, $\mathcal{C}_i^\phi$ is the semisimple conjugacy class in $\mathrm{GL}_n(K)$ with eigenvalues $\phi(a_1^i), \ldots, \phi(a_{r_i}^i)$ of multiplicities $\mu_1^i, \ldots, \mu_{r_i}^i$.

Hence, if Σ_g is a compact Riemann surface of genus g with punctures $S = \{s_1, \ldots, s_k\} \subseteq \Sigma_g$, then $\mathcal{U}_{\boldsymbol{\mu}}^\phi(K)$ can be identified with the set

$$\left\{\rho \in \mathrm{Hom}(\pi_1(\Sigma_g \setminus S), \mathrm{GL}_n(K)) \;\middle|\; \rho(\gamma_i) \in \mathcal{C}_i^\phi\right\}$$

(for some choice of base point, which we omit from the notation). Here we use the standard presentation

$$\pi_1(\Sigma_g \setminus S) = \langle \alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g; \gamma_1, \ldots, \gamma_g \mid (\alpha_1, \beta_1) \cdots (\alpha_g, \beta_g) \gamma_1 \cdots \gamma_k = 1 \rangle$$

(γ_i is the class of a simple loop around s_i with orientation compatible with that of Σ_g).

Remark A.2

A completely analogous construction works for the quiver case in the case when $\boldsymbol{\mu}$ is indivisible yielding an affine scheme $\mathcal{V}_{\boldsymbol{\mu}}$ with similar properties. For example, in the definition of R_0 and R we replace the product of elements in a multiset by their sum to guarantee genericity (see Section 2.2.1). The primes $p \in \mathbb{Z}$ that become invertible in R are those that are smaller than $\min_i \max_j \mu_i^j$ (cf. Section 2.2).

Appendix B

Here we prove a version of the smooth-proper base change theorem. A closely related result was obtained by Nakajima [5, Appendix].

THEOREM B.1

Let X be a nonsingular complex algebraic variety, and let $f : X \to \mathbb{C}$ be a smooth morphism, that is, a surjective submersion. Let $\mathbb{C}^\times$ act on X covering a positive power of the standard action on $\mathbb{C}$ such that the fixed point set $X^{\mathbb{C}^\times}$ is complete and for all $x \in X$ the $\lim_{\lambda\to 0} \lambda x$ exists. Then the fibers have isomorphic cohomology supporting pure mixed Hodge structures.

Proof

The proof is similar to that of [23, Lemma 6.1]; we give the details to be self-contained. By base change, if necessary, we can assume that the $\mathbb{C}^\times$-action on X covers the standard action on $\mathbb{C}$. Let $\mathbb{C}^\times$ act on $\mathbb{C}^2$ by $\lambda(z, w) = (\lambda z, w)$. Then $\mathbb{C}^2 \to \mathbb{C}$ given by $(z, w) \mapsto zw$ is $\mathbb{C}^\times$-equivariant with the standard action on $\mathbb{C}$. Let now X' denote the base change of X via this map; in other words, $X' = \{(x, z, w) \in X \times \mathbb{C}^2 | f(x) = zw\}$. Then X' inherits the $\mathbb{C}^\times$ action given by $\lambda(x, z, w) = (\lambda x, \lambda z, w)$, and f induces the map $f' : X' \to \mathbb{C}$ by $f(x, z, w) = w$ which is equivariant with respect to the trivial action on the base. By [44, Theorem 11.2], the set $U \subset X'$ of points $u \in X'$ such that $\lim_{\lambda\to\infty} \lambda u$ does not exist is open, and there exists a geometric quotient $\overline{X} := U//\mathbb{C}^\times$ which is proper over $\mathbb{C}$ via the induced map $\overline{f} : \overline{X} \to \mathbb{C}$. Indeed it is a completion of X over $\mathbb{C}$ as $X \subset \overline{X}$ naturally by the embedding $x \mapsto \mathbb{C}^\times(x, 1, f(x))$.

We now show that $\overline{f}$ is topologically trivial. It is not entirely straightforward, as $\overline{X}$ is only an orbifold, because the action of $\mathbb{C}^\times$ on U may not be free—there could be points with finite stablizers. However, the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers acts on U as a subgroup of $\mathbb{C}^\times$. Therefore the action of $\mathbb{R}^\times$ on U is free. It is properly discontinuous because the action of $\mathbb{C}^\times$ on U is properly discontinuous as $U \to \overline{X}$ is a geometric quotient. The quotient space $U/\mathbb{R}_+^\times$ is therefore a smooth manifold and the total space of a principal $\mathbb{T} := \mathrm{U}(1)$ orbi-bundle over the orbifold $\overline{X}$, which is proper over $\mathbb{C}$. Hence the induced map $f_+ : U/\mathbb{R}_+^\times \to \mathbb{C}$ is a proper submersion. Thus by choosing a $\mathbb{T}$-invariant Riemannian metric on $U/\mathbb{R}_+^\times$ and flowing perpendicular to the projection, we find a $\mathbb{T}$-equivariant trivialization of f_+ in the analytic topology. Dividing out by the $\mathbb{T}$-action yields a trivialization of $\overline{f}$ in the analytic topology. Consequently, the restriction $H^*(\overline{X}) \to H^*(\overline{X}_w)$ to the cohomology of any fiber of $\overline{f}$ is an isomorphism.

Note that $Z := \overline{X} \setminus X = \{\mathbb{C}^\times(x, 0, w) \mid \lim_{\lambda\to\infty} \lambda x \text{ exists}\}$ is trivial over $\mathbb{C}$; therefore $H^*(Z) \to H^*(Z_w)$ is an isomorphism. Applying the five lemma to the long exact sequences of the pairs $(\overline{X}, Z)$ and $(\overline{X}_w, Z_w)$, we get that $H^*(\overline{X}, Z) \cong H^*(\overline{X}_w, Z_w) \cong H^*_{cpt}(X_w)$. Thus any two fibers of f have isomorphic cohomology; in particular, $H^*_{cpt}(X_w) \cong H^*_{cpt}(X_0)$ for all $w \in \mathbb{C}$. As $\overline{X}_0$ is a proper orbifold (in particular a rational homology manifold), [7, Theorem 8.2.4] implies that its cohomology has pure mixed Hodge structure. Finally, by standard Morse theory arguments, $H^*(\overline{X}_0) \to H^*(X_0)$ is surjective, and thus $H^*(X_0)$ also has pure mixed Hodge structure. The proof is complete. □

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