Topology of Hitchin systems and Hodge theory of character varieties: the case A_1

By Mark Andrea A. de Cataldo, Tamás Hausel, and Luca Migliorini

Abstract

For G = GL_2, PGL_2, SL_2 we prove that the perverse filtration associated with the Hitchin map on the rational cohomology of the moduli space of twisted G-Higgs bundles on a compact Riemann surface C agrees with the weight filtration on the rational cohomology of the twisted G character variety of C when the cohomologies are identified via non-Abelian Hodge theory. The proof is accomplished by means of a study of the topology of the Hitchin map over the locus of integral spectral curves.

Contents

1. **Introduction**

Starting with the paper of Weil [52], and its commentary by Grothendieck [26], the moduli space of holomorphic vector bundles on a projective curve has become the focus of much important work in mathematics, and there is now extensive literature concerning its construction and properties. As is well known, the construction of this moduli space via geometric invariant theory is naturally paired with the notions of *stable* and *semistable* vector bundle.

A central result is the theorem of Narasimhan and Seshadri [42], which asserts that, roughly speaking, the semistable vector bundles of degree zero on a complex nonsingular projective curve C (that we assume to be of genus $g \geq 2$) are precisely the ones associated with unitary representations of the fundamental group of C or, if we consider bundles with nonzero degree on C, of the punctured curve $C \setminus p$, with a prescribed scalar monodromy around the puncture. Let us spell out the theorem in the case of rank 2 bundles: the fundamental group $\pi_1(C \setminus p)$ has free generators $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ such that $a_1^{-1} b_1^{-1} a_1 b_1 \cdots a_g^{-1} b_g^{-1} a_g b_g$ is the homotopy class of a loop around p. Unitary local systems of rank 2 on $C \setminus p$ with local monodromy $-\mathrm{I}$ around p are automatically irreducible, and the set of their isomorphism classes is

$$\mathcal{N}_B := \{A_1, B_1, \ldots, A_g, B_g \in \mathrm{U}(2) \mid A_1^{-1} B_1^{-1} A_1 B_1 \cdots$$
$$A_g^{-1} B_g^{-1} A_g B_g = -\mathrm{I}\}/\mathrm{U}(2),$$

where the unitary group $\mathrm{U}(2)$ acts by conjugation on the matrices A_i, B_i. This action factors through a free action of $\mathrm{PU}(2)$; hence the quotient $\mathcal{N}_B$ is a real analytic variety. The theorem of Narasimhan and Seshadri states that there is a canonical diffeomorphism $\mathcal{N}_B \simeq \mathcal{N}$, where $\mathcal{N}$ is the moduli space of stable rank 2 vector bundles of degree 1 on C.

A "complexified" version of this set-up, taking into consideration the analogue of the variety $\mathcal{N}_B$ obtained by replacing the unitary group $\mathrm{U}(2)$ with its complexification $\mathrm{GL}_2(\mathbb{C})$ (or, more generally, with any complex reductive group G, in which case the matrix $-\mathrm{I}$ is replaced by a suitable element in the center of G) arose in the work of Hitchin [35], [36]. Even though this paper considers the variants of this construction for the complex algebraic groups groups $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$, in this introduction we focus on the group $\mathrm{GL}_2(\mathbb{C})$; more details can be found in Section 1.2.

The representations of $\pi_1(C \setminus p)$ into $\mathrm{GL}_2(\mathbb{C})$ with monodromy $-\mathrm{I}$ around p are automatically irreducible, and their isomorphism classes are parametrized by the *twisted character variety*

$$\mathcal{M}_{\mathrm{B}} := \big\{ A_1, B_1, \dots, A_g, B_g \in \mathrm{GL}_2(\mathbb{C}) \mid A_1^{-1} B_1^{-1} A_1 B_1 \cdots$$
$$A_g^{-1} B_g^{-1} A_g B_g = -\mathrm{I} \big\} // \mathrm{GL}_2(\mathbb{C}),$$

where the quotient is taken in the sense of geometric invariant theory. As in the unitary picture, the action of $\mathrm{GL}_2(\mathbb{C})$ factors through a free action of $\mathrm{PGL}_2(\mathbb{C})$, and $\mathcal{M}_{\mathrm{B}}$ is a nonsingular irreducible complex affine variety of dimension $8g-6$.

The *non-Abelian Hodge theorem* states that, just as in the Narasimhan-Seshadri correspondence, $\mathcal{M}_{\mathrm{B}}$ is naturally diffeomorphic to another quasi-projective variety, i.e., the *moduli space of semistable Higgs bundles* $\mathcal{M}_{\mathrm{Dol}}$ parametrizing stable pairs (E, ϕ) consisting of a degree 1 rank 2 vector bundle E on C together with a Higgs field $\phi \in H^0(C, \mathrm{End}\,(E) \otimes K_C)$, subject to a natural condition of stability. If E itself is a stable vector bundle, then ϕ is in a natural way a cotangent vector at the point $[\mathrm{E}] \in \mathcal{N}$. It follows that $\mathcal{M}_{\mathrm{Dol}}$ contains the cotangent bundle of $\mathcal{N}$ as a Zariski open subset, which turns out to be dense.

The variety $\mathcal{M}_{\mathrm{Dol}}$ has a rich geometry; it has a natural hyperkähler metric, an S^1-action by isometries and, importantly, it carries a projective map $\chi : \mathcal{M}_{\mathrm{Dol}} \longrightarrow \mathcal{A}$, the *Hitchin fibration*, where the target $\mathcal{A}$ is (noncanonically) isomorphic to $\mathbb{C}^{4g-3}$, and the fibre of χ over a general point $s \in \mathcal{A}$ is isomorphic to the Jacobian of a branched double covering of C associated with s, the *spectral curve C_s*. This description of $\mathcal{M}_{\mathrm{Dol}}$ is usually referred to as *abelianization* since it reduces, to some extent, the study of Higgs bundles on C to that of line bundles on the spectral curves.

While the algebraic varieties $\mathcal{M}_{\mathrm{B}}$ and $\mathcal{M}_{\mathrm{Dol}}$ are diffeomorphic, they are not biholomorphic. The former is affine, and the latter is foliated by the fibers of the Hitchin map that are compact $(4g-3)$-dimensional algebraic subvarieties, Lagrangian with respect to the natural holomorphic symplectic structure associated with the hyperkähler metric on $\mathcal{M}_{\mathrm{Dol}}$. Furthermore, just as in the case of $\mathcal{N}_B$ and $\mathcal{N}$, the variety $\mathcal{M}_{\mathrm{B}}$ does not depend on the complex structure of C, whereas $\mathcal{M}_{\mathrm{Dol}}$ does.

It is natural to investigate the relation between some of the invariants of $\mathcal{M}_{\mathrm{B}}$ and $\mathcal{M}_{\mathrm{Dol}}$. This paper takes a step in this direction.

The paper [32] investigates in depth one of the important algebro-geometric invariants of $\mathcal{M}_{\mathrm{B}}$, namely the mixed Hodge structure on its cohomology groups. In view of [32, Cor. 4.1.11], the mixed Hodge structure of $H^*(\mathcal{M}_{\mathrm{B}})$ is of Hodge-Tate type. The quotient pure Hodge structures Gr_i^W satisfy

(1.1.1) $\mathrm{Gr}_{2i+1}^W H^*(\mathcal{M}_{\mathrm{B}}) = 0$ for all i, and $\mathrm{Gr}_{2i}^W H^*(\mathcal{M}_{\mathrm{B}})$ is of type (i, i).

The weight filtration $W_\bullet$ has a natural splitting, and it is nontrivial in certain cohomological degrees. For instance, $H^4(\mathcal{M}_B)$ contains classes of type $(2,2)$ and $(4,4)$. A remarkable property of $W_\bullet$ is the "curious hard Lefschetz theorem": there is a cohomology class $\tilde\alpha \in H^2(\mathcal{M}_B)$, of type $(2,2)$, such that the map given by iterated cup products with $\tilde\alpha$ defines isomorphisms

$$(1.1.2) \qquad \cup\, \tilde\alpha^l : \mathrm{Gr}^W_{8g-6-2l} H^*(\mathcal{M}_B) \xrightarrow{\ \cong\ } \mathrm{Gr}^W_{8g-6+2l} H^{*+2l}(\mathcal{M}_B).$$

Note that the class $\tilde\alpha$ raises the cohomological degree by two and the weight type by four, and that $\mathcal{M}_B$ is affine; both facts are in contrast with the hypotheses of the classical hard Lefschetz theorem, hence the "curiousity" of (1.1.2).

On the other hand, the Hodge structure on $H^*(\mathcal{M}_{\mathrm{Dol}})$ is pure; i.e., its weight filtration $W_\bullet$ is trivial in every cohomological degree. The class $\tilde\alpha \in H^2(\mathcal{M}_{\mathrm{Dol}})$ has pure type $(1,1)$. This raises the following question: what is the meaning of the weight filtration $W_\bullet$ of $H^*(\mathcal{M}_B)$ when viewed in $H^*(\mathcal{M}_{\mathrm{Dol}})$ via the diffeomorphism $\mathcal{M}_B \simeq \mathcal{M}_{\mathrm{Dol}}$ coming from the non-Abelian Hodge theorem? The answer we give in this paper brings into the picture the perverse Leray filtration P of $H^*(\mathcal{M}_{\mathrm{Dol}})$ that is naturally associated with the Hitchin map $\chi : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{A}$.

The perverse Leray filtration has been implicitly introduced in [7], and it has been studied and employed in [12], [14], [15], [8], [9]. This filtration is the abutment of the *perverse Leray spectral sequence* which, in turn, is a variant of the classical Leray spectral sequence. In the case of proper, but not necessarily smooth maps, e.g., our Hitchin map χ, this variant is better behaved than the classical Leray one. In fact, it always degenerates at E_2, and the graded pieces of the abutted perverse Leray filtration satisfy a version, called the relative hard Lefschetz, of the hard Lefschetz theorem, involving the operation of cupping with the first Chern class of a line bundle which is relatively ample with respect to the proper map. Both the Leray and the perverse Leray filtration originate from filtrations of the derived direct image complex of sheaves $\chi_*\mathbb{Q}$ on $\mathcal{A}$.

Since the target $\mathcal{A}$ of the map χ is affine, the perverse filtration has the following simple geometric characterization (see [15], where a different numbering convention is used). Let $s \geq 0$, and let $\Lambda^s \subseteq \mathcal{A}$ be a general s-dimensional linear section of $\mathcal{A}$ relative to a chosen identification of $\mathcal{A}$ with $\mathbb{C}^{4g-3}$; then

$$(1.1.3) \qquad P_p H^d(\mathcal{M}_{\mathrm{Dol}}) = \mathrm{Ker}\,\{H^d(\mathcal{M}_{\mathrm{Dol}}) \longrightarrow H^d(\chi^{-1}(\Lambda^{d-p-1}))\}.$$

The main result of this paper is that, up to a trivial renumbering of the filtrations, *the weight filtration $W_\bullet$ on $H^*(\mathcal{M}_B)$ coincides with the perverse Leray filtration $P_\bullet$ on $H^*(\mathcal{M}_{\mathrm{Dol}})$.*

THEOREM 1.1.1 ("P=W"). *In terms of the isomorphism $H^*(\mathcal{M}_B) \xrightarrow{\simeq} H^*(\mathcal{M}_{Dol})$ induced by the diffeomorphism $\mathcal{M}_B \xrightarrow{\simeq} \mathcal{M}_{Dol}$ stemming from the non-Abelian Hodge theorem, we have*

$$W_{2k}H^*(\mathcal{M}_B) = W_{2k+1}H^*(\mathcal{M}_B) = P_k H^*(\mathcal{M}_{Dol}).$$

Since the class $\tilde{\alpha} \in H^*(\mathcal{M}_{Dol})$ is relatively ample with respect to the Hitchin map, the curious hard Lefschetz theorem for $\tilde{\alpha}$ on $(H^*(\mathcal{M}_B), W)$ can thus be explained in terms of the relative hard Lefschetz theorem for $\tilde{\alpha}$ on $(H^*(\mathcal{M}_{Dol}), P)$.

One may say informally that the weight filtration on $H^*(\mathcal{M}_B)$ keeps track of certain topological properties of the Hitchin map on $\mathcal{M}_{Dol}$. This is even more remarkable in view of the fact that the structure of algebraic variety on $\mathcal{M}_B$, and thus the shape of $W_\bullet$, depend only on the topology of the curve C, while the complex/algebraic structure of the Higgs moduli space $\mathcal{M}_{Dol}$ and thus the Hitchin map depend on the complex structure of C.

In fact, as far as $P = W$ goes, we prove a more precise result. There are natural splittings (constructed by Deligne in [20]) of the perverse Leray filtration of $H^*(\mathcal{M}_{Dol})$. The splittings induced on $H^*(\mathcal{M}_{Dol})$ are equal, and they coincide with the splitting mentioned above of the weight filtration of $H^*(\mathcal{M}_B)$. We also prove that these results hold for the varieties associated with $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{PGL}_2(\mathbb{C})$. In Section 4.2 we also prove a version of the main theorem "$P = W$" for the moduli spaces of Higgs bundles with poles on C, namely when the canonical bundle is replaced by a different line bundle of high enough degree. In this case, there is no character variety $\mathcal{M}_B$ to be compared with $\mathcal{M}_{Dol}$. However, the cohomology ring $H^*(\mathcal{M}_{Dol})$ admits yet another filtration which is quite visible in terms of generators and relations. We prove that this third filtration coincides with the perverse Leray filtration associated with the Hitchin map (which is also defined in the context of poles). In the case where there are no poles, this third filtration coincides, after a simple renumbering, with the weight filtration.

Finally, as we need it in the course of our proof of $P = W$ in the case when $\mathrm{G} = \mathrm{SL}_2$, in Remark 4.4.9 we give a description of the cohomology ring for $\mathrm{G} = \mathrm{SL}_2$ that does not appear in the literature. This ring had been earlier determined by M. Thaddeus in unpublished work.

Since the proof of our main result is lengthy, we sketch below the main steps leading to it. Of course, for the sake of clarity, we do so by overlooking many technical issues.

The ring structure of $H^*(\mathcal{M}_{Dol})$ is known in terms of generators and relations; see [34], [33]. By using a result of M. Thaddeus, we prove that the place of the multiplicative generators in the perverse Leray filtration of $H^*(\mathcal{M}_{Dol})$ is the same as in the weight filtration of $H^*(\mathcal{M}_B)$ (Theorem 3.1.1).

If the perverse Leray filtration were compatible with cup products, then we could infer the same conclusion for the other cohomology classes.

However, only the weaker compatibility

$$P_i H(\mathcal{M}_{\mathrm{Dol}}) \cup P_j H(\mathcal{M}_{\mathrm{Dol}}) \longrightarrow P_{i+j+d} H(\mathcal{M}_{\mathrm{Dol}})$$

holds *a priori* for the perverse Leray filtration (see Proposition 1.4.7, and [11, Th. 6.1]), where d is the relative dimension of the map χ. In contrast, the compatibility in the strong form holds for the ordinary Leray filtration.

The Leray filtration is contained in the perverse Leray filtration. At the level of the direct image complex, the two filtrations coincide on the open subset of regular values on the target of the map.

One key to our approach is that we prove that, for the Hitchin map, there is a significantly larger open set of $\mathcal{A}$ where the Leray and the perverse Leray filtration coincide on the direct image complex. We define the "elliptic" locus $\mathcal{A}_{\mathrm{ell}} \subseteq \mathcal{A}$ to be the subset of points $s \in \mathcal{A}$ for which the corresponding spectral curve C_s is integral. Let $\mathcal{A}_{\mathrm{reg}} \subseteq \mathcal{A}_{\mathrm{ell}}$ be the set of regular values for the Hitchin map χ; the corresponding spectral curves are irreducible nonsingular. The key result is then the following, which we believe to be of independent interest.

THEOREM 1.1.2. *Let* $j : \mathcal{A}_{\mathrm{reg}} \longrightarrow \mathcal{A}_{\mathrm{ell}}$ *be the inclusion and, for* $l \geq 0$, *let* R^l *denote the local system* $s \mapsto H^l(\chi^{-1}(s))$ *on* $\mathcal{A}_{\mathrm{reg}}$. *Then, there is an isomorphism in the derived category of sheaves on* $\mathcal{A}_{\mathrm{ell}}$:

$$(\chi_* \mathbb{Q})_{|\chi^{-1}(\mathcal{A}_{\mathrm{ell}})} \simeq \bigoplus_l R^0 j_* \, R^l \, [-l].$$

This theorem contains two distinct statements:

(1) The perverse sheaves on $\mathcal{A}_{\mathrm{ell}}$ appearing in the decomposition theorem ([7]) for the Hitchin map restricted over the open set $\mathcal{A}_{\mathrm{ell}}$ are supported on the whole $\mathcal{A}_{\mathrm{ell}}$; This is a special case of Ngô's support theorem 7.1.13 in [45].

(2) These perverse sheaves, which are the intersection cohomology complexes on $\mathcal{A}_{\mathrm{ell}}$ of the local systems R^l on the smooth locus $\mathcal{A}_{\mathrm{reg}}$, are ordinary sheaves, as opposed to complexes. Up to a shift, they agree with the higher direct images appearing in the Leray spectral sequence.

The theorem implies that the classical and the perverse Leray filtrations coincide on $\mathcal{A}_{\mathrm{ell}}$. This puts us in a position to compute the "perversity" of most monomials generators of $H^*(\mathcal{M}_{\mathrm{Dol}})$; see Lemma 4.3.1. As explained in (1.1.3), the perversity of a class is tested by restricting it to the inverse image of generic linear sections of $\mathcal{A}$. The algebraic subset $\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}$ is of high codimension in $\mathcal{A}$. It follows that, in a certain range of dimensions, the general linear section can be chosen to lie entirely in $\mathcal{A}_{\mathrm{ell}}$, where we know, by Theorem 1.1.2, that the perverse Leray filtration is compatible with the cup product since it coincides with the Leray filtration.

At this point, we can conclude in the case of Higgs bundles for poles; see Section 4.2. In the geometrically more significant case where there are no poles, some monomials are not covered by the above line of reasoning, for the corresponding linear sections must meet $\mathcal{A} \setminus \mathcal{A}_{\text{ell}}$ by simple reasons of dimension. We treat these remaining classes using an ad hoc argument based on the properties of the Deligne splitting mentioned above; see Section 4.3.

In order to prove Theorem 1.1.2 we first determine an upper bound (see Theorem 2.2.7) for the Betti numbers of the fibres of the Hitchin map over $\mathcal{A}_{\text{ell}}$. In the case when $\mathrm{G} = \mathrm{GL}_2$, these fibres are the compactified Jacobians of the spectral curves, which, being double coverings of a nonsingular curve, have singularities analytically isomorphic to $y^2 - x^k = 0$, a fact we use in an essential way in our computations. Next, we give a lower bound (see Theorem 2.3.1) for the dimension of the stalks of the intersection cohomology complexes. This bound is based on the computation of the local monodromy of the family of nonsingular spectral curves around a singular integral spectral curve. It is achieved by a repeated use of the Picard-Lefschetz formula. Since the upper and lower bounds coincide, the decomposition theorem ([7]) gives the wanted result.

We see at least two difficulties to extend the results in this paper for complex reductive groups of higher rank:

- the monodromy computation of Theorem 2.3.12 which leads to the proof of Theorem 2.3.1 would be more complicated, and
- we do not know enough about compactified Jacobians of curves with singularities which are not double points.

Already for the group GL_3, Theorem 1.1.2(2) fails, and the intersection cohomology complexes are not shifted sheaves.

On the other hand, a curious hard Lefschetz theorem is conjectured in [32, Conj. 4.2.7] to hold for the character variety for PGL_n, which would of course follow, if $P = W$, from the relative hard Lefschetz theorem. Additionally, in a recent work of physicists Chuang-Diaconescu-Pan [16], a certain refined Gopakumar-Vafa conjecture for local curves in a Calabi-Yau 3-fold leads to a conjecture on the dimension of the graded pieces of the perverse filtration on the cohomology of the moduli space of twisted GL_n-Higgs bundles on C. Their conjecture agrees with the conjectured [32, Conj. 4.2.1] dimension of the graded pieces of the weight filtration on the cohomology of the twisted GL_n-character variety. The compatibility of these two conjectures may be considered the strongest indication so far that $P = W$ should hold for higher rank Higgs bundles as well.

In the paper [10] we prove that a result analogous to our main theorem $P = W$ holds in a situation that is expected to be closely related to the moduli space of certain parabolic Higgs bundles of rank n on a genus one

curve. Interestingly, in this case, the coincidence of the two filtrations holds, whereas the result (1) above, concerning the supports of the perverse sheaves being maximal on a large open set, fails, due to the fact that every new stratum contributes a new direct summand sheaf.

While property (2) above seems to hold only for Hitchin fibrations associated with groups of type A_1, the case studied in the present paper, and property (1) may not hold for parabolic Higgs bundles, we expect that the $P = W$ phenomenon should be a general feature of non-Abelian Hodge theory for curves. More generally, in [10, §4.4] we also conjecture that this exchange of filtration phenomenon should hold for holomorphic symplectic varieties with a $\mathbb{C}^*$-action, that, roughly speaking, behave like an algebraically completely integrable system.

1.2. *Cohomology of moduli spaces.*

1.2.1. *Character variety.* In this section, we recall some definitions and results from [32]. Throughout the paper, the singular homology and cohomology groups are taken with rational coefficients.

Let Σ be a closed Riemann surface of genus $g \geq 2$, and let G be a complex reductive group. In this paper, we consider only the cases $G = GL_2, PGL_2$, and SL_2. We are interested in the variety parametrizing certain twisted representations of the fundamental group $\pi_1(\Sigma)$ into G modulo isomorphism. Specifically, we consider the GL_2-*character variety*

$$\mathcal{M}_B := \{A_1, B_1, \ldots, A_g, B_g \in GL_2 \mid A_1^{-1}B_1^{-1}A_1B_1$$
$$\cdots A_g^{-1}B_g^{-1}A_gB_g = -I\}//PGL_2,$$

i.e., the affine GIT quotient by the diagonal adjoint action of PGL_2 on the matrices A_i, B_i. We also define the SL_2-*character variety*:

$$\check{\mathcal{M}}_B := \{A_1, B_1, \ldots, A_g, B_g \in SL_2 \mid A_1^{-1}B_1^{-1}A_1B_1$$
$$\cdots A_g^{-1}B_g^{-1}A_gB_g = -I\}//PGL_2.$$

The torus GL_1^{2g} acts on GL_2^{2g} by coordinate-wise multiplication, and this yields an action of GL_1^{2g} on $\mathcal{M}_\mathrm{B}$. Similarly, the finite subgroup of order 2 elements $\mu_2^{2g} \subset \mathrm{GL}_1^{2g}$, with $\mu_2 := \{\pm 1\} \subset \mathrm{GL}_1$, acts on SL_2^{2g} by coordinate-wise multiplication and we define the PGL_2-*character variety* as

$$(1.2.1) \qquad \hat{\mathcal{M}}_\mathrm{B} := \mathcal{M}_\mathrm{B}//\mathrm{GL}_1^{2g} = \check{\mathcal{M}}_\mathrm{B}/\mu_2^{2g}.$$

The surjective group homomorphism $\mathrm{SL}_2 \times \mathrm{GL}_1 \to \mathrm{GL}_2$ with finite kernel μ_2 induces a covering

$$(1.2.2) \qquad \check{\mathcal{M}}_\mathrm{B} \times \mathrm{GL}_1^{2g} \to \mathcal{M}_\mathrm{B}$$

with covering group μ_2^{2g}.

The varieties $\mathcal{M}_\mathrm{B}$, $\check{\mathcal{M}}_\mathrm{B}$ are nonsingular and affine (cf. [32, §2.2]), whereas $\hat{\mathcal{M}}_\mathrm{B}$ is affine with finite quotient singularities and parametrizes the representations of $\pi_1(\Sigma)$ to PGL_2 that do not admit a lift to representations of SL_2. We have $\dim \mathcal{M}_\mathrm{B} = 8g - 6$ and $\dim \check{\mathcal{M}}_\mathrm{B} = \dim \hat{\mathcal{M}}_\mathrm{B} = 6g - 6$. In view of (1.2.1), we have that

$$H^*(\hat{\mathcal{M}}_\mathrm{B}) = H^*(\check{\mathcal{M}}_\mathrm{B})^{\mu_2^{2g}},$$

the subring of μ_2^{2g} invariants, while (1.2.2) implies that

$$(1.2.3) \qquad H^*(\mathcal{M}_\mathrm{B}) = H^*(\mathrm{GL}_1^{2g}) \otimes H^*(\check{\mathcal{M}}_\mathrm{B})^{\mu_2^{2g}} = H^*(\mathrm{GL}_1^{2g}) \otimes H^*(\hat{\mathcal{M}}_\mathrm{B}).$$

The cohomology ring $H^*(\mathcal{M}_\mathrm{B})$ is generated by certain universal classes $\varepsilon_i \in H^1(\mathrm{GL}_1^{2g}) \subset H^1(\mathcal{M}_\mathrm{B})$ for $i = 1, \ldots, 2g$, $\alpha \in H^2(\check{\mathcal{M}}_\mathrm{B})^{\mu_2^{2g}} \subset H^2(\mathcal{M}_\mathrm{B})$, $\psi_i \in H^3(\check{\mathcal{M}}_\mathrm{B})^{\mu_2^{2g}} \subset H^3(\mathcal{M}_\mathrm{B})$ for $i = 1, \ldots, 2g$, and $\beta \in H^4(\check{\mathcal{M}}_\mathrm{B})^{\mu_2^{2g}} \subset H^4(\mathcal{M}_\mathrm{B})$. The proof can be found in [34] (generators) and in its sequel [33] (relations). The construction of these universal classes is explained in [32, §4.1]. The paper [32] uses this information to determine the mixed Hodge structure on $H^*(\mathcal{M}_\mathrm{B})$. For use in this paper, we summarize these results as follows. Let $(H, W_\bullet, F^\bullet)$ be a mixed Hodge structure. (See the textbook [47] for a comprehensive treatment of mixed Hodge theory.)

A class $\sigma \in H$ is said to be of *homogeneous weight* k ([32, Def. 4.1.6]) if its image in $H_\mathbb{C}$, still denoted by σ, satisfies

$$(1.2.4) \qquad \sigma \in W_{2k}H_\mathbb{C} \cap F^k H_\mathbb{C}.$$

Note that if σ has *homogeneous weight* k, then its image in $\mathrm{Gr}_{2k}^W H_\mathbb{C}$ is of type (k, k).

The natural mixed Hodge structure on $H^i(\mathcal{M}_\mathrm{B})$ satisfies $W_k H^i(\mathcal{M}_\mathrm{B}) = H^i(\mathcal{M}_\mathrm{B})$ for $k \geq 2i$, and, as $\mathcal{M}_\mathrm{B}$ is nonsingular, $W_k H^i(\mathcal{M}_\mathrm{B}) = 0$ for $k \leq i - 1$. The following is proved in [32, Th. 4.1.8].

THEOREM 1.2.1. *The cohomology classes* $\varepsilon_i \in H^1(\mathcal{M}_\mathrm{B})$ *have homogenous weight* 1, *while the classes* $\alpha \in H^2(\mathcal{M}_\mathrm{B})$, $\psi_i \in H^3(\mathcal{M}_\mathrm{B})$ *for* $i = 1, \ldots, 2g$, *and* $\beta \in H^4(\mathcal{M}_\mathrm{B})$ *have homogenous weight* 2.

It follows that homogenous elements generate $H^*(\mathcal{M}_\mathrm{B})$. The following is Corollary 4.1.11. in [32].

THEOREM 1.2.2. *The weight filtration* $W_\bullet H^*(\mathcal{M}_\mathrm{B})$ *satisfies*

(1) $W_{2k}H^*(\mathcal{M}_\mathrm{B}) = W_{2k+1}H^*(\mathcal{M}_\mathrm{B})$ *for all* k;

(2) $\left(\mathrm{Gr}_{2k}^W H^*(\mathcal{M}_\mathrm{B})_\mathbb{C}\right)^{pq} = 0$ *if* $(p,q) \neq (k,k)$.

Denoting by $W_k^d(\mathcal{M}_\mathrm{B}) \subset H^d(\mathcal{M}_\mathrm{B})$ the subspace of degree d homogenous weight k cohomology classes, we have the following splittings:

$$(1.2.5) \qquad H^d(\mathcal{M}_\mathrm{B}) = \bigoplus_k W_k^d(\mathcal{M}_\mathrm{B}), \qquad W_{2k}H^d(\mathcal{M}_\mathrm{B}) = \bigoplus_{i \leq k} W_i^d(\mathcal{M}_\mathrm{B}).$$

Theorem 1.1.3 of [32] gives a formula for the mixed Hodge polynomials of $\hat{\mathcal{M}}_\mathrm{B}$ and $\mathcal{M}_\mathrm{B}$ which implies the curious symmetries

$$\dim \mathrm{Gr}^W_{\dim \mathcal{M}_\mathrm{B}-2k} H^*(\mathcal{M}_\mathrm{B}) = \dim \mathrm{Gr}^W_{\dim \mathcal{M}_\mathrm{B}+2k} H^{*+2k}(\mathcal{M}_\mathrm{B})$$

and

$$\dim \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}_\mathrm{B}-2k} H^*(\hat{\mathcal{M}}_\mathrm{B}) = \dim \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}_\mathrm{B}+2k} H^{*+2k}(\hat{\mathcal{M}}_\mathrm{B}).$$

These equalities, called *curious Poincaré duality* in [32], are made more precise and significant by the *curious hard Lefschetz* theorems. Consider the class $\alpha \in H^2(\hat{\mathcal{M}}_\mathrm{B})$ introduced above and the class $\tilde{\alpha} \in H^2(\mathcal{M}_\mathrm{B})$ defined in terms of the isomorphism (1.2.3) by

$$(1.2.6) \qquad \tilde{\alpha} := 1 \otimes \alpha + \left(\sum_{i=1}^g \varepsilon_i \varepsilon_{i+g}\right) \otimes 1.$$

We then have ([32, Th. 1.1.5])

THEOREM 1.2.3 (Curious hard Lefschetz). *The map given by iterated cup product with* $\tilde{\alpha}$ *induces isomorphisms*

$$(1.2.7) \quad \cup \tilde{\alpha}^k : \mathrm{Gr}^W_{\dim \mathcal{M}_\mathrm{B}-2k} H^*(\mathcal{M}_\mathrm{B}) \xrightarrow{\cong} \mathrm{Gr}^W_{\dim \mathcal{M}_\mathrm{B}+2k} H^{*+2k}(\mathcal{M}_\mathrm{B}), \quad \forall k \geq 0.$$

Similarly, cupping with α *defines isomorphisms*

$$(1.2.8) \quad \cup \alpha^k : \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}_\mathrm{B}-2k} H^*(\hat{\mathcal{M}}_\mathrm{B}) \xrightarrow{\cong} \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}_\mathrm{B}+2k} H^{*+2k}(\hat{\mathcal{M}}_\mathrm{B}), \quad \forall k \geq 0.$$

The present paper was partly motivated by the desire to give a more conceptual explanation for these curious hard Lefschetz theorems.

1.2.2. *Moduli of Higgs bundles and their cohomology ring.* Let C be a smooth complex projective curve of genus $g \geq 2$. A Higgs bundle is a pair (E, ϕ) of a vector bundle E on C and a Higgs field $\phi \in H^0(C, \mathrm{End}\, E \otimes K_C)$. Let $\mathcal{M}_\mathrm{Dol}$ denote the GL_2-*Higgs moduli space*, i.e., the moduli space of stable Higgs bundles of rank 2 and degree 1. It is a nonsingular quasi-projective variety with $\dim \mathcal{M}_\mathrm{Dol} = 8g - 6$.

Let us fix a degree 1 line bundle Λ on C. Let $\check{\mathcal{M}}_{\mathrm{Dol}}$ be the SL_2-*Higgs moduli space* of stable Higgs bundles (E, ϕ) or rank 2, with determinant $\det(E) \simeq \Lambda$ and trace-free $\mathrm{tr}(\phi) = 0$ Higgs field. The moduli space $\check{\mathcal{M}}_{\mathrm{Dol}}$ is a nonsingular quasi-projective variety with $\dim \check{\mathcal{M}}_{\mathrm{Dol}} = 6g - 6$. Defining the map

$$\lambda_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}} \mapsto \mathrm{Pic}^1_C \times H^0(C, K_C), \qquad \lambda_{\mathrm{Dol}}(E, \phi) := (\det(E), \mathrm{tr}(\phi)),$$

we have

$$\check{\mathcal{M}}_{\mathrm{Dol}} = \lambda_{\mathrm{Dol}}^{-1}((\Lambda, 0)).$$

Let $\mathcal{M}^0_{\mathrm{Dol}} \subseteq \mathcal{M}_{\mathrm{Dol}}$ be the subset of stable Higgs bundles with traceless Higgs field:

$$\mathcal{M}^0_{\mathrm{Dol}} = \{(E, \phi) \text{ with } \mathrm{tr}(\phi) = 0\}.$$

The group Pic^0_C of degree 0 holomorphic line bundles on C acts on $\mathcal{M}^0_{\mathrm{Dol}}$ as follows: $L \in \mathrm{Pic}^0_C$ sends (E, ϕ) to $(E \otimes L, \phi \otimes \mathrm{Id}_L)$. The group $\Gamma := \mathrm{Pic}^0_C[2] \cong \mathbb{Z}_2^{2g}$ of order 2 line bundles on C acts naturally on $\check{\mathcal{M}}_{\mathrm{Dol}}$ in the same way. The two resulting quotients are easily seen to be isomorphic. We call the resulting variety the PGL_2-*Higgs moduli space* and denote it by

$$\hat{\mathcal{M}}_{\mathrm{Dol}} = \mathcal{M}^0_{\mathrm{Dol}} / \mathrm{Pic}^0_C = \check{\mathcal{M}}_{\mathrm{Dol}} / \Gamma.$$

It is a quasi-projective $(6g - 6)$-dimensional algebraic variety with finite quotient singularities.

The fundamental theorem of non-Abelian Hodge theory on the curve C for the groups $\mathrm{G} = \mathrm{GL}_2$, SL_2, and PGL_2 under consideration can be stated as follows ([35], [50], [22], [18]).

THEOREM 1.2.4 (Non-Abelian Hodge theorem). *There are canonical diffeomorphisms*

$$\mathcal{M}_{\mathrm{B}} \cong \mathcal{M}_{\mathrm{Dol}}, \;\; \check{\mathcal{M}}_{\mathrm{B}} \cong \check{\mathcal{M}}_{\mathrm{Dol}}, \;\; \hat{\mathcal{M}}_{\mathrm{B}} \cong \hat{\mathcal{M}}_{\mathrm{Dol}}.$$

At the level of cohomology, the non-Abelian Hodge theorem yields canonical isomorphisms
(1.2.9)
$$H^*(\mathcal{M}_{\mathrm{B}}) \cong H^*(\mathcal{M}_{\mathrm{Dol}}), \;\; H^*(\check{\mathcal{M}}_{\mathrm{B}}) \cong H^*(\check{\mathcal{M}}_{\mathrm{Dol}}), \;\; H^*(\hat{\mathcal{M}}_{\mathrm{B}}) \cong H^*(\hat{\mathcal{M}}_{\mathrm{Dol}}).$$

Remark 1.2.5. The Hodge structure on the cohomology of these Higgs moduli spaces is pure, and its Hodge polynomial is known; see Conjecture 5.6 in [30], which also proposes a conjectural formula for any rank.

Given a line bundle D on C, we can consider, more generally, the moduli space of stable pairs (E, ϕ), where E is a rank 2 degree 1 bundle on C and $\phi \in H^0(C, \mathrm{End}\, E \otimes D)$. The corresponding moduli space is connected by Theorem 7.5 in [46], and if $\deg D > 2g - 2$ or if $D = K_C$, it is nonsingular ([46, Prop. 7.4]).

Notation 1.2.6. For the sake of notational simplicity, this moduli space is denoted by $\mathcal{M}$ in the sequel of this paper, without mentioning its dependence on the line bundle D, always meant to satisfy $\deg D > 2g - 2$, or $D = K_C$. Whenever we talk specifically of the case $D = K_C$, we denote the corresponding moduli space by $\mathcal{M}_{\mathrm{Dol}}$.

We still have the map $\lambda_D : \mathcal{M} \longrightarrow \mathrm{Pic}^1_C \times H^0(C, D)$ defined by $\lambda_D(E, \phi) = (\det(E), \mathrm{tr}(\phi))$. We set $\check{\mathcal{M}} := \lambda_D^{-1}((\Lambda, 0))$, $\hat{\mathcal{M}} := \mathcal{M}^0/\mathrm{Pic}^0_C = \check{\mathcal{M}}/\Gamma$, where, as above,
$$\mathcal{M}^0 = \{(E, \phi) \in \mathcal{M} \text{ with } \mathrm{tr}(\phi) = 0\},$$
and $\Gamma = \mathrm{Pic}^0_C[2] \cong \mathbb{Z}_2^{2g}$ is the group of order 2 line bundles on C; see Section 2.4.

It is proved in [34, (4.4)] that there is a Higgs bundle $(\mathbb{E}, \boldsymbol{\Phi})$ on $\mathcal{M} \times C$ with the property that, for every family of Higgs bundles $(\mathbb{E}_S, \boldsymbol{\Phi}_S)$ parametrized by an algebraic variety S, there is a unique map $a : S \longrightarrow \mathcal{M}$ and an isomorphism
$$(\mathbb{E}_S, \boldsymbol{\Phi}_S) \simeq \mathcal{L} \otimes (a \times \mathrm{Id})^*(\mathbb{E}, \boldsymbol{\Phi})$$
for a uniquely determined line bundle $\mathcal{L}$ on S.

Remark 1.2.7. The vector bundle $\mathbb{E}$ with the universal property stated above is determined up to twisting with a line bundle pulled back from $\mathcal{M}$; hence, given two different choices $\mathbb{E}, \mathbb{E}'$, we have a canonical isomorphism of the associated endomorphisms bundles $\mathrm{End}\,\mathbb{E} \simeq \mathrm{End}\,\mathbb{E}'$. The vector bundle $\mathrm{End}\,\mathbb{E}$ on $\mathcal{M} \times C$ is thus unambiguously defined.

Let $e_1, \ldots, e_{2g}$ be a symplectic basis of $H^1(C)$ and $\omega \in H^2(C)$ be the Poincaré dual of the class of a point. The Künneth decomposition of the second Chern class of $\mathrm{End}\,\mathbb{E}$,

$$(1.2.10) \qquad -c_2(\mathrm{End}\mathbb{E}) = \alpha \otimes \omega + \sum_{i=1}^{2g} \psi_i \otimes e_i + \beta \otimes 1 \in H^*(\mathcal{M}) \otimes H^*(C),$$

defines the classes $\alpha \in H^2(\mathcal{M})$, $\psi_i \in H^3(\mathcal{M})$, and $\beta \in H^4(\mathcal{M})$. These classes define also classes in $H^*(\check{\mathcal{M}})$ by restriction and in $H^*(\hat{\mathcal{M}})$ by restriction and $\mathrm{Pic}^0_C[2]$-invariance. They will be denoted with the same letters. In the case $D = K_C$, these classes coincide, via the isomorphisms (1.2.9), with the classes in $H^*(\mathcal{M}_{\mathrm{B}})$, denoted by the same symbols, defined in Section 1.2.1.

The generators of $H^*(\mathrm{Pic}^1_C)$ pull back to the classes $\varepsilon_1, \ldots, \varepsilon_{2g} \in H^1(\mathcal{M})$ via the morphism $\mathcal{M} \to \mathrm{Pic}^1_C$ given by $(E, \phi) \mapsto \det(E)$.

The paper [34] shows that the universal classes
$$\{\varepsilon_1, \ldots, \varepsilon_{2g}, \alpha, \psi_1, \ldots, \psi_{2g}, \beta\}$$
are a set of multiplicative generators for $H^*(\mathcal{M})$; the relations among these universal classes were determined in [33]. Due to the role these relations play in this paper we summarize the main result of [33].

Because of the isomorphism $H^*(\mathcal{M}) \simeq H^*(\hat{\mathcal{M}}) \otimes H^*(\operatorname{Pic}_C^0)$, it is enough to describe the ring $H^*(\hat{\mathcal{M}})$. We introduce the element

$$\gamma := -2 \sum_i \psi_i \psi_{i+g},$$

we set $\Psi := \operatorname{Span}(\psi_i) \subseteq H^3(\hat{\mathcal{M}})$, and we define

$$\Lambda_0^k := \operatorname{Ker}\left\{ \gamma^{g+1-k} : \bigwedge^k \Psi \longrightarrow \bigwedge^{2g+2-k} \Psi \right\} \text{ for } 0 \le k \le g \text{ and } \Lambda_0^k = 0 \text{ for } k > g.$$

By the standard representation theory of the symplectic group, there is a direct sum decomposition

$$\bigwedge^k \Psi = \bigoplus_i \gamma^i \Lambda_0^{k-2i}.$$

Definition 1.2.8. Given two integers $a, b \ge 0$, we define I_b^a to be the ideal of $\mathbb{Q}[\alpha, \beta, \gamma]$ generated by γ^{a+1} and

$$(1.2.11) \qquad \rho_{r,s,t}^c := \sum_{i=0}^{\min(c,r,s)} \frac{\alpha^{r-i}}{(r-i)!} \frac{\beta^{s-i}}{(s-i)!} \frac{(2\gamma)^{t+i}}{i!},$$

where $c := r + 3s + 2t - 2a + 2 - b$ for all the $r, s, t \ge 0$ such that

$$(1.2.12) \qquad r + 3s + 3t > 3a - 3 + b \text{ and } r + 2s + 2t \ge 2a - 2 + b.$$

Remark 1.2.9. If $r = 0$ and $b > 0$, then the second inequality in (1.2.12) is strictly stronger than the first.

The main result of [34] is then

THEOREM 1.2.10. *The cohomology ring of $\hat{\mathcal{M}}$ has the presentation*

$$H^*(\hat{\mathcal{M}}) = \sum_{k=0}^{g} \Lambda_0^k(\psi) \otimes \left(\mathbb{Q}[\alpha, \beta, \gamma] / I_{\deg D + 2 - 2g + k}^{g-k} \right).$$

The form of the relations (1.2.11) affords the following

Definition 1.2.11. We define the grading w on $H^*(\hat{\mathcal{M}})$ by setting

$$w(\alpha) = w(\beta) = w(\psi_i) = 2$$

and extending it by multiplicativity. This grading is well defined since the relations of Theorem 1.2.10 are homogenous with respect to this grading. We denote by $W_\bullet'$ the increasing filtration associated to this grading.

If $D = K_C$, thanks to the results of [32] described in Section 1.2.1, $W_\bullet'$ on $H^*(\hat{\mathcal{M}})$ coincides, up to a simple renumbering and via (1.2.9), with the weight filtration associated with the mixed Hodge structure on $H^*(\hat{\mathcal{M}}_B)$. As

mentioned in [32, Rem. 5.2.3], for a general D, even though there is no associated Betti moduli space, the filtration $W'_\bullet$ on $H^*(\hat{\mathcal{M}})$ turns out to have the same formal properties of the weight filtration on $H^*(\hat{\mathcal{M}}_\mathrm{B})$ described in Section 1.2.1. In particular, [32, Lemma 5.3.3] implies that it satisfies the following curious hard Lefschetz property completely analogous to (1.2.8) of Theorem 1.2.3.

THEOREM 1.2.12. *For $\alpha \in H^2(\hat{\mathcal{M}})$, we have the isomorphisms*

$$(1.2.13) \quad \cup \alpha^k : \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}-2k} H^*(\hat{\mathcal{M}}) \xrightarrow{\cong} \mathrm{Gr}^W_{\dim \hat{\mathcal{M}}+2k} H^{*+2k}(\hat{\mathcal{M}}), \quad \forall k \geq 0.$$

Finally, setting $w(\varepsilon_i) = 1$, we get a grading and an associated filtration on $H^*(\mathcal{M})$, and all the discussion above goes through without any change.

1.3. *The Hitchin fibration and spectral curves.*

1.3.1. *The Hitchin fibration* $(\mathrm{G} = \mathrm{GL}_2)$. Given a Higgs field

$$\phi \in H^0(C, \mathrm{End}\, E \otimes D),$$

we have $\mathrm{tr}(\phi) \in H^0(C, D)$ and $\det(\phi) \in H^0(C, 2D)$. The Hitchin map, $\chi : \mathcal{M} \longrightarrow \mathcal{A}$ assigns

$$(1.3.1) \quad \mathcal{M} \ni (E, \phi) \longmapsto (\mathrm{tr}(\phi), \det(\phi)) \in \mathcal{A} := H^0(C, D) \times H^0(C, 2D).$$

Note that we do not indicate the dependence on the line bundle D in the notation for the target $\mathcal{A}$ of the Hitchin map (cf. the conventions introduced in Notation 1.2.6). It follows from Theorem 6.1 in [46] that the map χ is proper.

In the case of $\check{\mathcal{M}} \subseteq \mathcal{M}$, the corresponding Hitchin fibration $\check{\chi}$ is just the restriction of χ to $\check{\mathcal{M}}$. Since, by definition, if $(E, \phi) \in \check{\mathcal{M}}$, then $\mathrm{tr}(\phi) = 0$, we have

$$(1.3.2) \qquad\qquad \check{\chi} : \check{\mathcal{M}} \longrightarrow \mathcal{A}^0 := H^0(C, 2D) \subseteq \mathcal{A}.$$

The map descends to the quotient $\hat{\mathcal{M}} = \check{\mathcal{M}}/\Gamma$, and we have

$$(1.3.3) \qquad\qquad\qquad \hat{\chi} : \hat{\mathcal{M}} \longrightarrow \mathcal{A}^0.$$

In the rest of this section, we concentrate on the map χ. The necessary changes for dealing with the cases of $\hat{\chi}$ and $\check{\chi}$ are discussed in Section 2.4.

1.3.2. *The spectral curve construction.* Let $\pi_D : \mathbb{V}(D) \longrightarrow C$ be the total space of the line bundle D. For $s := (s_1, s_2) \in \mathcal{A}$ as in (1.3.1), the *spectral curve* C_s is the curve on $\mathbb{V}(D)$ defined by the equation

$$(1.3.4) \qquad\qquad \left\{ y \in \mathbb{V}(D) : y^2 - \pi_D^*(s_1)y + \pi_D^*(s_2) = 0 \right\}.$$

Spectral curves can be singular, reducible, even nonreduced; they are locally planar, and, in force of our assumptions on the genus of C and the degree of D, connected. The restriction $\pi_s : C_s \longrightarrow C$ of the projection $\pi_D : \mathbb{V}(D) \longrightarrow C$

exhibits C_s as a double cover of C. Equation (1.3.4) in $\mathbb{V}(D) \times \mathcal{A}$ defines the flat family u of spectral curves

(1.3.5)
$$
\begin{array}{ccc}
\mathscr{C}_{\mathcal{A}} & \xrightarrow{\pi} & C \times \mathcal{A} \\
& \searrow_{u} \quad \swarrow_{p_2} & \\
& \mathcal{A}, &
\end{array}
$$

where $u^{-1}(s) = C_s$ for all $s \in \mathcal{A}$. The family is equipped with the involution $\iota : \mathscr{C}_{\mathcal{A}} \longrightarrow \mathscr{C}_{\mathcal{A}}$ over $C \times \mathcal{A}$ exchanging the two sheets of the covering.

The restriction of the relative Picard scheme of the family u to the smooth locus

$$\mathcal{A}_{\mathrm{reg}} := \{s \in \mathcal{A} \text{ such that } C_s \text{ is smooth}\}$$

is the disjoint union over $l \in \mathbb{Z}$ of the proper families $p^l : \mathscr{P}^l_{\mathrm{reg}} \longrightarrow \mathcal{A}_{\mathrm{reg}}$ such that $(p^l)^{-1}(s) = \mathrm{Pic}^l_{C_s}$ is the component of the Picard variety of C_s parametrizing degree l line bundles on C_s.

Remark 1.3.1. Fix a degree 1 line bundle $\mathcal{L}$ on C. The operation of tensoring line bundles of fixed degree with $\pi^* \mathcal{L}$ defines isomorphisms $\mathscr{P}^l_{\mathrm{reg}} \longrightarrow \mathscr{P}^{l+2}_{\mathrm{reg}}$ of schemes over $\mathcal{A}_{\mathrm{reg}}$. It follows that, up to isomorphisms, there are only two such families, the abelian scheme $\mathscr{P}^0_{\mathrm{reg}}$ and the $\mathscr{P}^0_{\mathrm{reg}}$-torsor $\mathscr{P}^1_{\mathrm{reg}}$. Sending a point $\hat{c} \in C_s$ to the line bundle $\mathcal{O}_{C_s}(\hat{c})$ defines an Abel-Jacobi-type $\mathcal{A}_{\mathrm{reg}}$-map $\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}} \longrightarrow \mathscr{P}^1_{\mathrm{reg}}$.

The Riemann-Hurwitz formula and (1.3.4) imply at once the following

PROPOSITION 1.3.2. *Let* $s = (s_1, s_2) \in \mathcal{A}$. *Assume* $s_1^2 - 4s_2 \neq 0 \in H^0(C, 2D)$.

(1) *The spectral curve C_s is reduced, and the covering $\pi_s : C_s \longrightarrow C$ is branched at the zeros of $s_1^2 - 4s_2$. The point $s = (s_1, s_2) \in \mathcal{A}_{\mathrm{reg}}$ if and only if $s_1^2 - 4s_2$ has simple zeros, in which case $g(C_s) = 2g - 1 + \deg D$.*

(2) *If $s_1^2 - 4s_2$ vanishes to finite order $k \geq 2$ at a point $c \in C$, then the spectral curve C_s has a planar singularity at the point $\pi_s^{-1}(c)$ that is locally analytically isomorphic to $\{y^2 - x^k = 0\} \subseteq \mathbb{C}^2$.*

Remark 1.3.3. Associating with $s = (s_1, s_2) \in \mathcal{A}$ its discriminant divisor $(s_1^2 - 4s_2) \in C^{(2r)}$, where $r := \deg D$ and $C^{(2r)}$ is the $2r$-th symmetric product of C, gives a map $\Theta : \mathcal{A} \longrightarrow C^{(2r)}$.

We recall that if $\mathcal{F}$ is a torsion-free sheaf on an integral curve $\mathcal{C}$, then the rank of $\mathcal{F}$ is the dimension of its stalk at the generic point of $\mathcal{C}$ and the degree $\deg \mathcal{F}$ is defined as $\deg \mathcal{F} := \chi(\mathcal{C}, \mathcal{F}) - \mathrm{rank}(\mathcal{F}) \chi(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. For $l \in \mathbb{Z}$, then the *compactified Jacobian* $\overline{\mathrm{Pic}}_{\mathcal{C}}^l$ *of degree l* parametrizes torsion-free sheaves of rank 1 and degree l on $\mathcal{C}$ (see [23], [2]). Tensoring with a line bundle of

degree l gives an isomorphism $\overline{\mathrm{Pic}_\mathcal{C}}^0 \simeq \overline{\mathrm{Pic}_\mathcal{C}}^l$. If $\mathcal{C}$ is smooth, then every rank 1 torsion free sheaf is locally free and $\overline{\mathrm{Pic}_\mathcal{C}}^l = \mathrm{Pic}_\mathcal{C}^l$.

The following theorem ([6, Prop. 3.6]) describes the fibres of the Hitchin map over a rather large open subset of the base $\mathcal{A}$. Recall Remark 1.3.1 and that we are considering Higgs bundles of *odd* degree.

THEOREM 1.3.4. *Let $s \in \mathcal{A}$ be such that the spectral curve C_s is integral. There is an isomorphism of varieties $\chi^{-1}(s) \simeq \overline{\mathrm{Pic}_{C_s}}^a$ with $a = 0$ if $\deg D$ is odd and $a = 1$ if $\deg D$ is even.*

The isomorphisms assigns to a torsion free sheaf $\mathcal{F}$ of degree a on C_s its direct image $\pi_{s*}\mathcal{F}$, which is a torsion free $\mathcal{O}_C$-module on C. Since C is smooth, the sheaf $\pi_{s*}\mathcal{F}$ is locally free of rank 2, in view of the fact that π_s has degree 2. Since the map π_s is finite, there are no higher cohomology sheaves, and $\chi(C_s, \mathcal{F}) = \chi(C, \pi_{s*}\mathcal{F})$. The Riemann-Roch theorem

$$\deg \pi_{s*}\mathcal{F} + 2(1-g) = \chi(C, \pi_*\mathcal{F}) = \chi(C_s, \mathcal{F})$$
$$= \deg \mathcal{F} + \chi(C_s, \mathcal{O}_{C_s}) = \deg \mathcal{F} + 2(1-g) - \deg D$$

implies that $\pi_{s*}\mathcal{F}$ has odd degree if $\deg \mathcal{F} - \deg D$ is odd. The Higgs field arises as multiplication by y (see (1.3.4)) in view of the natural structure of $\pi_{s*}\mathcal{O}_{C_s}$-module on $\pi_{s*}\mathcal{F}$ (see [6, §3] for details).

In particular, for every $s \in \mathcal{A}_{\mathrm{reg}}$, the fiber $\chi^{-1}(s)$ can be identified, non-canonically, with the Jacobian variety of the smooth spectral curve C_s. In fact, the Abelian scheme $\mathscr{P}^0_{\mathrm{reg}}$ acts on $\mathcal{M}_{\mathrm{reg}} := \chi^{-1}(\mathcal{A}_{\mathrm{reg}})$ making it into a torsor (see [45, §4.3]).

The following is well known.

LEMMA 1.3.5. *Let $\alpha : A \longrightarrow S$ be an Abelian scheme, let $\tau : P \longrightarrow S$ be an A-torsor, and let $j \geq 0$. Then there are natural isomorphisms of local systems*

$$R^j\tau_*\mathbb{Q}_P \simeq R^j\alpha_*\mathbb{Q}_A \simeq \bigwedge^j R^1\alpha_*\mathbb{Q}_A.$$

Proof. Since the fibers of A are connected, the action by translation on the cohomology of the fibers of P is trivial. Hence, the isomorphism of local systems $(R^j\tau_*\mathbb{Q}_P)_{|U} \simeq (R^j\alpha_*\mathbb{Q}_A)_{|U}$ associated with a local trivialization of P does not depend on the chosen trivialization. Consequently, the isomorphisms associated to a trivializing cover $\{U_i\}$ of S glue to a global isomorphism of local systems. The second isomorphism follows from the Künneth isomorphism $H^l((S^1)^a) \simeq \bigwedge^l H^1((S^1)^a)$. $\square$

COROLLARY 1.3.6. *There are canonical isomorphisms of local systems on $\mathcal{A}_{\mathrm{reg}}$:*

$$R^l\chi_{\mathrm{reg}*}\mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}} \simeq R^l p_*\mathbb{Q}_{\mathscr{P}^1_{\mathrm{reg}}} \simeq \bigwedge^l R^1 p_*\mathbb{Q}_{\mathscr{P}^1_{\mathrm{reg}}} \simeq \bigwedge^l R^1 u_{\mathrm{reg}*}\mathbb{Q}_{\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}}}.$$

Proof. The first and second isomorphisms follow by applying Lemma 1.3.5 to the $\mathscr{P}^0_{\mathrm{reg}}$-torsor $\mathcal{M}_{\mathrm{reg}}$. The Abel-Jacobi $\mathcal{A}_{\mathrm{reg}}$-map $\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}} \longrightarrow \mathscr{P}^1_{\mathrm{reg}}$ of Remark 1.3.1 induces via pullback a map of local systems $R^1 p^1_* \mathbb{Q} \longrightarrow R^1 u_{\mathrm{reg}\,*} \mathbb{Q}$ that is an isomorphism on each stalk, and this proves the third isomorphism. $\square$

1.4. *Perverse filtration.* Let

$$h : M^{a+f} \longrightarrow A^a$$

be a proper map of relative dimension f between irreducible varieties of the indicated dimensions. We assume that M is nonsingular, or with at worst finite quotient singularities, and that the fibres have constant dimension f. Let $\eta \in H^2(M)$ be the first Chern class of a relatively ample (or h-ample) line bundle on M, i.e., a line bundle which is ample when restricted to every fiber of h.

The goal of this section is to define the perverse Leray filtration P on the cohomology groups $H^*(M)$ and to list and discuss some of its relevant properties.

1.4.1. *Definition of the perverse filtration P on $H^*(M)$.* We employ freely the language of derived categories and perverse sheaves (see the seminal paper [7], the survey [13], or for example the paper [12]). Standard textbooks on the subject are [21], [37], [38].

We denote by D_A the "constructible derived category." Its objects are bounded complexes of sheaves of $\mathbb{Q}$-vector spaces with constructible cohomology sheaves; it is a full subcategory of the bounded derived category of sheaves of $\mathbb{Q}$-vector spaces. We denote the derived direct image Rh_* simply by h_* and, for $i \in \mathbb{Z}$, the i-th hypercohomology group of A with coefficients in $K \in D_A$ by $H^i(A, K)$. If the index i is unimportant (but fixed), we simply write $H^*(A, K)$. We set $H^\bullet(A, K) := \oplus_i H^i(A, K)$. We work with the middle perversity t-structure. The corresponding category of perverse sheaves is denoted by P_A. Given $K \in D_A$, we have the sequence of maps of "truncated" complexes

$$\cdots \longrightarrow {}^{\mathfrak{p}}\tau_{\leq p-1} K \longrightarrow {}^{\mathfrak{p}}\tau_{\leq p} K \longrightarrow {}^{\mathfrak{p}}\tau_{\leq p+1} K \longrightarrow \cdots \longrightarrow K, \qquad p \in \mathbb{Z},$$

where ${}^{\mathfrak{p}}\tau_{\leq p} K = 0$ for every $p \ll 0$ and ${}^{\mathfrak{p}}\tau_{\leq p} K = K$ for every $p \gg 0$. The (increasing) perverse filtration P on the cohomology groups $H^*(A, K)$ is defined by taking the images of the truncation maps in cohomology:

$$(1.4.1) \qquad P_p H^*(A, K) := \mathrm{Im}\,\{H^*(A, {}^{\mathfrak{p}}\tau_{\leq p} K) \longrightarrow H^*(A, K)\}.$$

Clearly, the perverse filtration on $H^*(M) = H^*(M, \mathbb{Q}_M)$ becomes trivial after a dimensional shift. On the other hand, we also have the perverse filtration on $H^*(A, h_* \mathbb{Q}) = H^*(M)$ which, as it is the case for its variant given by the

Leray filtration, is highly nontrivial. This is what may be called the perverse Leray filtration on $H^*(M)$ associated with h.

For the needs of this paper, we want the perverse Leray filtration P on $H^*(M)$ to be of type $[0, 2f]$, i.e., $P_{-1} = \{0\}$ and $P_{2f} = H^*(M)$, and to satisfy $1 \in P_0 H^0(M)$. In order to achieve this, we define (with slight abuse of notation) the perverse Leray filtration on $H^*(M)$ (with respect to h) by setting

$$P_p H^*(M) := P_p H^{*-a}(A, h_* \mathbb{Q}_M[a]).$$

Note that in [12], the perverse Leray filtration is defined so that it is of type $[-f, f]$.

In order to simplify the notation, we set

$$(1.4.2) \qquad H^*_{\leq p}(M) := P_p H^*(M), \qquad H^*_p(M) := \mathrm{Gr}^P_p H^*(M) := P_p/P_{p-1}.$$

In this paper, we also use the graded spaces for the weight filtration $\mathrm{Gr}^W_w H^*(M)$ and we employ the same notation $H^*_w(M)$. In those cases, we make it clear which meaning should be given to the symbols.

1.4.2. *Decomposition and relative hard Lefschetz theorems, primitive decomposition.* Define

$$(1.4.3) \qquad \mathcal{P}^p := {}^{\mathfrak{p}}\mathcal{H}^p(h_* \mathbb{Q}_M[a]) \in P_A, \quad p \in \mathbb{Z},$$

where ${}^{\mathfrak{p}}\mathcal{H}^p(-)$ denotes the p-th perverse cohomology functor. We have that $\mathcal{P}^p = 0$ for $p \notin [0, 2f]$. The decomposition theorem for the proper map $h : M \to A$ then gives the existence of isomorphisms in D_A

$$(1.4.4) \qquad \varphi : \bigoplus_{p=0}^{2f} \mathcal{P}^p[-p] \xrightarrow{\simeq} h_* \mathbb{Q}_M[a].$$

We have identifications

$$(1.4.5) \qquad H^*_{\leq p}(M) = \bigoplus_{p'=0}^{p} \varphi\left(H^*_{p'}(M)\right), \qquad H^*_p(M) = H^{*-a-p}(A, \mathcal{P}^p).$$

Remark 1.4.1. The images $\varphi\left(H^*_p(M)\right) \subseteq H^*(M)$ depend on φ. If $H^*_{\leq p-1}(M) = \{0\}$, then the image $\varphi\left(H^*_p(M)\right) = H^*_{\leq p}(M)$ is independent of φ. In particular, the image $\varphi\left(H^*_0(M)\right) = H^*_{\leq 0}(M)$ is independent of φ.

One of the deep assertions of the decomposition theorem is that each perverse sheaf $\mathcal{P}^p$ is semisimple and splits canonically into a direct sum

$$(1.4.6) \qquad \mathcal{P}^p = \bigoplus_Z \mathrm{IC}_Z(L_{Z,p})$$

of intersection complexes over a finite collection of distinct irreducible closed subvarieties Z in A with coefficients given by semisimple local systems $L_{Z,p}$ defined on a dense open subset $Z^o \subseteq Z_{\mathrm{reg}} \subseteq Z$ of the regular part of Z.

There are the following three basic symmetries:

(1) (PVD) Poincaré-Verdier duality: If we denote the Verdier dual of K by $K^\vee$, then we have

$$(1.4.7) \qquad \mathcal{P}^{f-i} \simeq (\mathcal{P}^{f+i})^\vee, \qquad \forall i \in \mathbb{Z}.$$

(2) (RHL) Relative hard Lefschetz: For every $i \geq 0$, the i-th iteration of the operation of cupping with the h-ample line bundle η yields isomorphisms of perverse sheaves

$$(1.4.8) \qquad \eta^i : \mathcal{P}^{f-i} \xrightarrow{\simeq} \mathcal{P}^{f+i};$$

in particular, we have the hard Lefschetz isomorphisms at the level of graded groups (still called RHL):

$$(1.4.9) \qquad \eta^i : H^*_{f-i}(M) \xrightarrow{\simeq} H^{*+2i}_{f+i}(M), \qquad \forall\, i \geq 0.$$

(3) (Self-duality): The isomorphisms PVD and RHL are compatible with the direct sum decomposition (1.4.6) and, in particular, the local systems $L_{Z,p}$ are self-dual.

Recall that P_A is an Abelian category. By a standard abuse of notation, which greatly simplifies the notation, we view kernels and images as subobjects.

For $0 \leq i \leq f$ and $0 \leq j \leq f - i$, define

$$(1.4.10)$$
$$\mathcal{Q}^{i,0} := \mathrm{Ker}\left\{\eta^{f-i+1} : \mathcal{P}^i \longrightarrow \mathcal{P}^{2f-i+2}\right\}, \qquad \mathcal{Q}^{i,j} := \mathrm{Im}\left\{\eta^j : \mathcal{Q}^{i,0} \longrightarrow \mathcal{P}^{i+2j}\right\},$$

and set $\mathcal{Q}^{i,j} = 0$ for all the other values of (i,j). The RHL (1.4.8) then yields the natural primitive decompositions in P_A:

$$(1.4.11) \qquad \mathcal{P}^k = \bigoplus_{j \geq 0} \mathcal{Q}^{k-2j,j}, \qquad \forall k \in \mathbb{Z}^{\geq 0}.$$

1.4.3. *Deligne's Q-splitting associated with the relatively ample η.* The paper [20] defines three preferred decomposition isomorphisms (1.4.4) associated with the h-ample line bundle η. We consider the first of them (see also [14]), which we denote by ϕ_η, and name the Deligne isomorphism; notice, however, that the indexing scheme employed here differs from that of [20], [14]. The cohomological properties of the Deligne isomorphism needed in this paper are the following.

Fact 1.4.2. The map

$$\phi_\eta : \bigoplus_{p=0}^{2f} \mathcal{P}^p[-p] \xrightarrow{\simeq} h_* \mathbb{Q}_M[a]$$

is characterized by the following properties. Let $0 \leq i \leq f$. Then

(i) applying the functor ${}^{\mathfrak{p}}\mathcal{H}^i(-)$ to the map $\phi_{\eta_|} : \mathcal{Q}^{i,0}[-i] \to h_* \mathbb{Q}[a]$ gives the canonical inclusion $\mathcal{Q}^{i,0} \subseteq \mathcal{P}^i$;

(ii) for every $s > f - i$, the composition below is zero:

$$\mathcal{Q}^{i,0}[-i] \xrightarrow{\phi_{\eta_|}} h_*\mathbb{Q}[a] \xrightarrow{\eta^s} h_*\mathbb{Q}[a][2s] \longrightarrow ({}^{\mathfrak{p}}\tau_{\geq f+s}\mathbb{Q}[a])[2s]$$

or, equivalently, the composition $\eta^s \circ \phi_{\eta_|}$ factors through ${}^{\mathfrak{p}}D_A^{\leq f-s-1}$.

For $d \geq 0$, $0 \leq i \leq f$ and $0 \leq j \leq f - i$, define

$$(1.4.12) \qquad Q^{i,j;d} := \phi_\eta\left(H^{d-a-i-2j}\left(A, \mathcal{Q}^{i,j}\right)\right) \subseteq H_{\leq i+2j}^d(M),$$

$$(1.4.13) \qquad Q^{i,j} := \bigoplus_{d \geq 0} Q^{i,j;d} \subseteq \bigoplus_{d \geq 0} H_{\leq i+2j}^d(M),$$

and define $Q^{i,j} = Q^{i,j;d} = \{0\}$ for all the other values of $(i, j; d)$. We then have the following decompositions, which depend on ϕ_η:

$$(1.4.14) \qquad H^\bullet(M) = \bigoplus_{i,j} Q^{i,j}, \qquad H^d(M) = \bigoplus_{i,j} Q^{i,j;d},$$

$$(1.4.15) \qquad H_{\leq p}^\bullet(M) = \bigoplus_d H_{\leq p}^d(M) = \bigoplus_{i,j,d,\, i+2j \leq p} Q^{i,j;d} = \bigoplus_{i,j,\, i+2j \leq p} Q^{i,j}.$$

Every $u \in H^\bullet(M)$ admits the Q-*decomposition* associated with the splitting ϕ_η:

$$(1.4.16) \qquad\qquad u = \sum u^{i,j}, \qquad u^{i,j} \in Q^{i,j}.$$

By construction, we have

$$H_p^d(M) = \bigoplus_{i+2j=p} Q^{i,j;\,d}, \qquad H_p(M) = \bigoplus_{i+2j=p} Q^{i,j}.$$

The properties of the Deligne splitting that we need, and that follow from Fact 1.4.2, are

$$(1.4.17)$$
$$\eta\, Q^{i,j} = Q^{i,j+1},\ \forall\, 0 \leq j < f - i, \qquad \eta\, Q^{i,f-i} \subseteq \bigoplus_{0 \leq l \leq \min(f-i,f-k)} Q^{k,l}.$$

In particular, we have the simple relation

$$(1.4.18) \qquad\qquad Q^{i,j} = \eta^j Q^{i,0}, \qquad \forall\, 0 \leq j \leq f - i.$$

Here is some ad hoc notation and terminology. Let $p \in \mathbb{Z}$ and $u \in H_{\leq p}^\bullet(M)$. Denote by $[u]_p \in H_p^\bullet(M)$ the natural projection to the graded group. In what follows, we add over $0 \leq i \leq f$ and $0 \leq j \leq f - i$. We have

$$u = \sum_{i+2j \leq p} u^{i,j}, \qquad [u]_p = \sum_{i+2j=p} [u^{i,j}]_p.$$

We say that

- u *has perversity* $\leq p$.
- u *has perversity* p if $[u]_p \neq 0$.
- The class $0 \in H^\bullet(M)$ has perversity p, for every $p \in \mathbb{Z}$.

- u is *sharp* if $u^{i,j} = 0$ whenever $i + 2j < p$; note that the zero class is automatically sharp and that a class may have a given perversity without being sharp.
- $u = u^{p,0} \in Q^{p,0} \subseteq H^{\bullet}_{\leq p}(M)$ is RHL-*primitive*; note that such a class is automatically sharp.

One should not confuse RHL-primitivity with primitivity: if $u \in Q^{p,0}$, then

$$(1.4.19) \quad \eta^{f-p+1} u^{p,0} \in H^{\bullet}_{\leq 2f-p+1}(M), \qquad \text{i.e.,} \qquad [\eta^{f-p+1} u^{p,0}]_{2f-p+2} = 0,$$

whereas one could have $\eta^{f-p+1} u^{p,0} \neq 0$.

Recall that we have chosen the Deligne splitting ϕ_{η} associated with η. The following lemma does not hold for an arbitrary splitting φ in (1.4.4).

LEMMA 1.4.3 (Nonmixing lemma). *Let $u \in H^{\bullet}_{\leq p}(M)$. If $\eta^{f-p+1} u = 0$, then u is RHL-primitive, i.e., $u = u^{p,0} \in Q^{p,0}$.*

Proof. In view of the Q-decomposition, we can write

$$u = u^{p,0} + \sum_{j \geq 1} u^{p-2j,j} + \sum_{s+2t<p} u^{s,t},$$

where the first two summands are sharp and have perversity p and the third has perversity $\leq p - 1$. By (1.4.17) and (1.4.18) we deduce that

$$\eta^{f-p+1} u^{p,0} \in \bigoplus_{0 \leq l \leq f-p, f-k} Q^{k,l}, \quad \eta^{f-p+1} u^{p-2j,j} \in Q^{p-2j,f-p+1+j},$$

$$\eta^{f-p+1} u^{s,t} \in Q^{s,t+f-p+1}.$$

The three collections of Q-spaces above have no term in common. It follows that all three terms in $\eta^{f-p+1} u = 0$ are zero. By RHL (1.4.8), cupping with η^{f-p+1} is injective on the spaces $Q^{p-2j,j}$, $j \geq 1$, and $Q^{s,t}$ above. We deduce that $u^{p+2j,j} = u^{s,t} = 0$. □

1.4.4. *The perverse filtration and cup-product.* The following is a crude, completely general, estimate.

LEMMA 1.4.4. *Let $u \in H^d(M)$. Then the cup product map with u satisfies*

$$\cup u : H^*_{\leq p}(M) \longrightarrow H^{*+d}_{\leq p+d}(M).$$

Proof. We have $H^d(M) = \mathrm{Hom}_{D_M}(\mathbb{Q}_M, \mathbb{Q}_M[d])$ so that we may view the cohomology class u as a map $u : \mathbb{Q}_M[a] \to \mathbb{Q}_M[a+d]$. The cup product map $\cup u$ coincides with the map induced in cohomology by the pushed-forward map $h_* u : h_* \mathbb{Q}_M[a] \to h_* \mathbb{Q}_M[a+d]$. We apply truncation and obtain the map ${}^{\mathfrak{p}}\tau_{\leq p} h_* \mathbb{Q}_M[a] \to {}^{\mathfrak{p}}\tau_{\leq p} h_* \mathbb{Q}_M[a+d] = ({}^{\mathfrak{p}}\tau_{\leq p+d} h_* \mathbb{Q}_M[a])[d]$. The assertion follows after taking cohomology. □

A much better estimate, leading to the key Proposition 1.4.11, holds under the following.

Assumption 1.4.5. The intersection complexes $IC_Z(L_{Z,p})$ (1.4.6) appearing in the decomposition theorem for $h_*\mathbb{Q}_M[a]$ have strict support A (i.e., each $Z = A$) and $IC_A(L_{A,p}) = R^0 j_*^o L_{A,p}[a]$, where $j^o : A^o \to A$ is the immersion of an open dense subset.

Fact 1.4.6. Take A^o to be the open set over which h is smooth, and set $R^p := (R^p h_* \mathbb{Q}_M)_{|A^o}$. We may re-phrase Assumption 1.4.5 as follows:

$$h_*\mathbb{Q}_M[a] \simeq \bigoplus_{p=0}^{2f} IC_A(R^p)[-p] = \bigoplus_{p=0}^{2f} R^0 j_*^o R^p[a][-p] = \bigoplus_{p=0}^{2f} R^p h_* \mathbb{Q}_M[a][-p].$$

As a consequence, if Assumption 1.4.5 holds, then the perverse Leray filtration on $H^*(M) = H^{*-a}(A, h_*\mathbb{Q}_M[a])$ coincides with the standard Leray filtration on $H^*(M) = H^*(A, h_*\mathbb{Q}_M)$.

PROPOSITION 1.4.7. *If* Assumption 1.4.5 *holds, then we have*

(1.4.20) $$H_{\leq p}^*(M) \otimes H_{\leq q}^*(M) \longrightarrow H_{\leq p+q}^{*+*}(M).$$

Proof. It is a known fact that the multiplicativity property (1.4.20) holds with respect to the standard Leray filtration (see [11, Th. 6.1] for a proof). The statement then follows, since, as noticed in Fact 1.4.6, Assumption 1.4.5 implies that the perverse Leray filtration and the standard Leray filtration coincide. □

Let us assume that the target A of the map $h : M \to A$ is affine of dimension a, and let $A \subseteq \mathbb{C}^N$ be an arbitrary closed embedding. Let $s \geq 0$, $\Lambda^s \subseteq A$ be a general s-dimensional linear section and let $M_{\Lambda^s} := h^{-1}(\Lambda^s)$. For $s < 0$, we define $M_{\Lambda^s} := \emptyset$.

The following is the main result of [15] (Theorem 4.1.1).

THEOREM 1.4.8. *A class* $u \in H_{\leq p}^d(M)$ *if and only if* $u_{|M_{\Lambda^{d-p-1}}} = 0$.

Remark 1.4.9. Theorem 1.4.8 implies, in particular, that $H_{\leq p}^d(M) = 0$ if $p < d - a$ and that $H_{\leq p}^d(M) = H^d(M)$ if $p \geq d$.

Remark 1.4.10. For Λ^s general, transversality implies the following (see [12, Lemma 4.3.8]). If $u \in H_{\leq p}^d(M)$, then, $u_{|M_{\Lambda^s}} \in H_{\leq p}^d(M_{\Lambda^s})$; in other words, the change in perversity is compensated by the change in codimension.

Let $U \subseteq A$ be a Zariski dense open subset satisfying Assumption 1.4.5 (with U replacing A), and hence the conclusions of Fact 1.4.6. Let $Y := A \setminus U$ be the closed complement. Note that such an open set U always exists, e.g., $U = A^o$, the set over which the map is smooth. However, Y could be rather

large, i.e., have small codimension. The following proposition is key to our analysis of the perverse filtration in the cohomology ring of the moduli of Higgs bundles, where, as it turns out, the set Y is small just enough to let us go by.

PROPOSITION 1.4.11. *Let* $u_i \in H^{d_i}_{\leq p_i}(M)$ *for* $i = 1, \ldots, l$. *Let* $d := \sum d_i$ *and* $p := \sum p_i$. *If* $d - p - 1 < \operatorname{codim} Y$, *then* $v := u_1 \cup u_2 \cup \cdots \cup u_l \in H^d_{\leq p}(M)$.

Proof. By the assumption on $\operatorname{codim} Y$, a general Λ^{d-p-1} misses Y. By applying Remark 1.4.10 to the classes u_i, and then Proposition 1.4.7 to their restriction to $M_{\Lambda^{d-p-1}}$, we conclude that the resulting $v_{|M_{\Lambda^{d-p-1}}} \in H^d_{\leq p}(M_{\Lambda^{d-p-1}})$. As noticed in Remark 1.4.9, we have $v_{|M_{\Lambda^{d-p-1}}} = 0$. We conclude by Theorem 1.4.8. $\square$

1.4.5. *Extra vanishing when* $H^j(M) = 0$ *for all* $j > 2f$.

PROPOSITION 1.4.12. *Assume that* A *is affine and that* $H^j(M) = 0$ *for all* $j > 2f$. *Then the perversity of* $u \in H^d(M)$ *is in the interval* $\left[\lceil \frac{d}{2} \rceil, d\right]$.

Proof. We may assume that $u \neq 0$. Let p be the perversity of u. Assume that $p < \lceil \frac{d}{2} \rceil$. In particular, $p < f$ and $2p < d$. By RHL (1.4.9) and by the assumption on vanishing, we reach the contradiction $0 \neq \eta^{f-p}u \in H^{2f-2p+d}(M) = \{0\}$. The upper bound follows from Theorem 1.4.8, as noticed in Remark 1.4.9. $\square$

COROLLARY 1.4.13. *Under the hypothesis of Proposition 1.4.12, we have that*

$$H^d_{\leq \lceil \frac{d}{2} \rceil}(M) = Q^{\lceil \frac{d}{2} \rceil, 0; d}, \qquad H^d_{\leq \lceil \frac{d}{2} \rceil + 1}(M) = Q^{\lceil \frac{d}{2} \rceil, 0; d} \bigoplus Q^{\lceil \frac{d}{2} \rceil - 1, 1; d}.$$

Proof. By Proposition 1.4.12, we have $H^d_{\leq \lceil \frac{d}{2} \rceil - 1}(M) = \{0\}$. Then (1.4.5) implies that $H^d_{\leq \lceil \frac{d}{2} \rceil}(M) = \phi_\eta(H^{d-a-\lceil \frac{d}{2} \rceil}(A, \mathcal{P}^{\lceil \frac{d}{2} \rceil}))$. Equation (1.4.11) implies that

$$H^d_{\leq \lceil \frac{d}{2} \rceil}(M) = \bigoplus_{j \geq 0} Q^{\lceil \frac{d}{2} \rceil - 2j, j; d} = \bigoplus_{j \geq 0} \eta^j Q^{\lceil \frac{d}{2} \rceil - 2j, 0; d - 2j}.$$

By Proposition 1.4.12, since $\lceil \frac{d}{2} \rceil - 2j < \lceil (d-2j)/2 \rceil$, we have that $Q^{\lceil \frac{d}{2} \rceil - 2j, 0; d-2j} = \{0\}$ for $j > 0$. The assertion in perversity $\lceil \frac{d}{2} \rceil + 1$ is proved in the same way. $\square$

2. Cohomology over the elliptic locus

2.1. *Statement of Theorem 2.1.4.* We go back to the setup of Section 1.3.

Definition 2.1.1. The *elliptic locus* $\mathcal{A}_{\mathrm{ell}} \subseteq \mathcal{A}$ is the set of points $s = (s_1, s_2) \in \mathcal{A}$ for which the associated spectral curve C_s is integral. We set $\mathcal{M}_{\mathrm{ell}} := \chi^{-1}(\mathcal{A}_{\mathrm{ell}})$.

Remark 2.1.2. Let $s = (s_1, s_2) \in \mathcal{A}$. Since the covers $C_s \to C$ have degree 2, if the section $s_1^2 - 4s_2 \in H^0(C, 2D)$ vanishes with odd multiplicity at least at one point of C, then $s \in \mathcal{A}_{\mathrm{ell}}$. Since $2D$ has even degree, there is an even number of points on C where $s_1^2 - 4s_2$ has odd vanishing order.

LEMMA 2.1.3. *The set $\mathcal{A}_{\mathrm{ell}}$ is Zariski open and dense in $\mathcal{A}$ and contains $\mathcal{A}_{\mathrm{reg}}$. The complement $\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}$ is a closed algebraic subset of codimension $\deg D$ if $\deg D > 2g - 2$ and of codimension $2g - 3$ if $D = K_C$.*

Proof. Since $\deg D > 0$, given $s = (s_1, s_2) \in \mathcal{A}$, the zero locus of its discriminant divisor $s_1^2 - 4s_2$ is not empty, so that the spectral covering $\pi_s : C_s \longrightarrow C$ is never étale. A nonsingular spectral curve must therefore be irreducible, namely $\mathcal{A}_{\mathrm{reg}} \subseteq \mathcal{A}_{\mathrm{ell}}$. The spectral curve associated with $s = (s_1, s_2)$ is a divisor on the nonsingular surface $\mathbb{V}(D)$, and it is not integral precisely when s is in the image of the finite map $H^0(C, D) \times H^0(C, D) \longrightarrow \mathcal{A}$ sending (t_1, t_2) to $(t_1 + t_2, t_1 t_2)$; therefore, the image $\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}$ is a closed subset. By the Riemann-Roch theorem on C, we have that if $\deg D > 2g - 2$, then $\dim(\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}) = 2(\deg D + 1 - g)$ and $\dim \mathcal{A} = 3 \deg D + 2(1 - g)$, while, for $D = K_C$, we have that $\dim(\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}) = 2g$ and $\dim \mathcal{A} = 4g - 3$. $\qquad \square$

We denote by $j : \mathcal{A}_{\mathrm{reg}} \longrightarrow \mathcal{A}_{\mathrm{ell}}$ the open imbedding and by b_l the l-th Betti number.

Recall from Theorem 1.3.4 the noncanonical isomorphism $\overline{\mathrm{Pic}}_{C_s}^0 \simeq \chi^{-1}(s)$. Section 2 is devoted to the proof of the following

THEOREM 2.1.4. *For $s \in \mathcal{A}_{\mathrm{ell}}$ and for $l \geq 0$, we have*

$$(2.1.1) \qquad \dim\left((R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q})_s\right) = b_l(\overline{\mathrm{Pic}}_{C_s}) = b_l(\chi^{-1}(s)).$$

Theorem 2.1.4 readily implies the following.

COROLLARY 2.1.5. *The perverse sheaves $\mathcal{P}^l_{\mathcal{A}_{\mathrm{ell}}}$ appearing in the statement of the decomposition theorem (1.4.6) of Section 1.4.2 for the Hitchin map over the open set $\mathcal{A}_{\mathrm{ell}}$ satisfy*

$$\mathcal{P}^l_{\mathcal{A}_{\mathrm{ell}}} = \mathrm{IC}_{\mathcal{A}_{\mathrm{ell}}}(R^l \chi_{\mathrm{reg}*} \mathbb{Q}) = R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q}\,[\dim \mathcal{A}], \quad \forall\, l;$$

i.e., there is only one intersection complex, supported on the whole $\mathcal{A}_{\mathrm{ell}}$, given by a sheaf in the single cohomological degree $-\dim \mathcal{A}$. In particular, Assumption 1.4.5 of Section 1.4.4 is fulfilled.

Proof that Theorem 2.1.4 implies Corollary 2.1.5. Set $R^l := R^l \chi_{\mathrm{reg}*} \mathbb{Q}$ and $a := \dim \mathcal{A}$. On the smooth locus $\mathcal{A}_{\mathrm{reg}}$, the decomposition theorem takes the form $\chi_{\mathrm{reg}*} \mathbb{Q} \simeq \bigoplus R^l[-l]$. It follows that $(\chi_* \mathbb{Q}[a])_{|\mathcal{A}_{\mathrm{ell}}} \simeq \left(\bigoplus IC(R^l)[-l]\right) \oplus K$, where K is a direct sum of shifted semisimple perverse sheaves supported on

proper subsets of $\mathcal{A}_{\mathrm{ell}}$. Taking the stalk at $s \in \mathcal{A}_{\mathrm{ell}}$ of the cohomology sheaves,

$$H^{k+a}(\chi^{-1}(s)) \simeq \mathcal{H}^k(\chi_* \mathbb{Q}[a])_s \simeq \left(\bigoplus_l \mathcal{H}^{k-l}(IC(R^l))_s \right) \bigoplus \mathcal{H}^k(K)_s$$

and

$$b_{k+a}(\chi^{-1}(s)) = \dim \mathcal{H}^k(h_* \mathbb{Q}[a])_s = \sum_l \dim \mathcal{H}^{k-l}(IC(R^l))_s + \dim \mathcal{H}^k(K)_s.$$

By the very definition of intersection cohomology complex, $\mathcal{H}^{-a}(IC(R^l)) = R^0 j_*^0 R^l$, hence the equality (2.1.1) forces $\mathcal{H}^r(IC(R^l)) = 0$ for $r \neq -a$ and $\mathcal{H}^r(K) = 0$ for all r. $\qquad\square$

Let us briefly outline the structure of the proof of Theorem 2.1.4, which occupies the remainder of Section 2. In Section 2.2, we prove an upper bound, Theorem 2.2.7, on the Betti numbers of the compactified Jacobian of an integral curve with A_k-singularities. The partial normalizations of such a curve define a natural stratification of the compactified Jacobian; the cohomology groups of the strata are easy to determine, and the spectral sequence arising from the stratification gives the desired upper bound. In Section 2.3, we complement this upper bound estimate with a lower bound estimate, Theorem 2.3.1, for the dimension of the stalks $(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q})_s$. The proof of Theorem 2.3.1 consists of a monodromy computation which is completed in Section 2.3.7. In Section 2.3, we also prove that the decomposition theorem forces the equality of the two bounds. This completes the proof of Theorem 2.1.4.

Remark 2.1.6. The arguments used in the proof of Theorem 2.1.4 do not depend on the specific features of the Hitchin map and hold more generally in the following setting. Suppose S is a nonsingular complex variety and $\mathscr{C} \longrightarrow S$ is a proper family of integral curves, smooth over the open set $S_{\mathrm{reg}} \overset{j}{\hookrightarrow} S$, which are branched double coverings of a fixed curve C. Let $\mathscr{I} \overset{f}{\longrightarrow} S$ be the associated family of compactified Jacobians. If $\mathscr{I}$ is nonsingular, or has at worst finite quotient singularities, then Theorem 2.1.4 and its Corollary 2.1.5 hold. In particular, for $s \in S$, we have $b_l(f^{-1}(s)) = \dim \left((R^0 j_* R^l f_* \mathbb{Q})_s \right)$.

2.2. *The upper bound estimate.* In this section, $\mathcal{C}$ denotes an integral projective curve whose singularities are double points of type A_k, i.e., analytically isomorphic to $(y^2 - x^{k+1} = 0) \subseteq \mathbb{C}^2$ for some $k \geq 1$. In view of Proposition 1.3.2 and of the fact that we work exclusively with integral spectral curves, this is the generality we need. If $k \geq 3$, then blowing up a point of type A_k produces a point of type A_{k-2}, and if $k = 1, 2$, respectively corresponding to an ordinary node and a cusp, then blowing up a point resolves the singularity. The invariant $\delta_c := \dim_{\mathbb{C}} \widetilde{\mathcal{O}}_{\mathcal{C},c} / \mathcal{O}_{\mathcal{C},c}$ measures the drop of the arithmetic genus under normalization. If $c \in \mathcal{C}$ is a singular point of type A_k, then $\delta_c = \left\lceil \frac{k}{2} \right\rceil$.

The point c must be blown up δ_c times in order to be resolved, and, with the exception of the blowing up of A_1, each blowing up map is bijective.

The following theorem lists a few well-known facts concerning the Picard variety of a curve with singularities of type A_k.

THEOREM 2.2.1. *Let $\mathcal{C}$ be a reduced and connected projective curve with at worst A_k-singularities. Let $\mathcal{C}_{\mathrm{sing}} \subseteq \mathcal{C}$ be its singular locus, and let $\nu : \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$ be the normalization map. Let $\nu^* : \mathrm{Pic}^0_{\mathcal{C}} \longrightarrow \mathrm{Pic}^0_{\widetilde{\mathcal{C}}}$ be the map induced by pull-back, where $\mathrm{Pic}^0_{\mathcal{C}}$ (resp. $\mathrm{Pic}^0_{\widetilde{\mathcal{C}}}$) is the connected component of the identity of the Picard scheme of $\mathcal{C}$ (resp. $\widetilde{\mathcal{C}}$). If $c \in \mathcal{C}_{\mathrm{sing}}$ is a singular point of type A_k, set*

$$\mathcal{P}_c := \begin{cases} \mathbb{C}^{\delta_c} & \text{if } k \text{ is even,} \\ \mathbb{C}^\times \times \mathbb{C}^{\delta_c - 1} & \text{if } k \text{ is odd,} \end{cases}$$

and define the commutative algebraic group $\mathcal{P} := \prod_{c \in \mathcal{C}_{\mathrm{sing}}} \mathcal{P}_c$. Then

(1) *If $\mathcal{C}$ is irreducible, there is an exact sequence of commutative algebraic groups*

$$(2.2.1) \qquad 1 \longrightarrow \mathcal{P} \longrightarrow \mathrm{Pic}^0_{\mathcal{C}} \longrightarrow \mathrm{Pic}^0_{\widetilde{\mathcal{C}}} \longrightarrow 1.$$

(2) *If $\mathcal{C}$ is reducible and $\sharp$ is the number of irreducible components, we have an exact sequence*

$$(2.2.2) \qquad 1 \longrightarrow \mathcal{P}/(\mathbb{C}^\times)^{\#-1} \longrightarrow \mathrm{Pic}^0_{\mathcal{C}} \longrightarrow \mathrm{Pic}^0_{\widetilde{\mathcal{C}}} \longrightarrow 1.$$

Proof. These facts follow directly from the exact sequence of sheaves of groups on $\mathcal{C}$,

$$1 \longrightarrow \mathcal{O}^\times_{\mathcal{C}} \longrightarrow \nu_* \mathcal{O}^\times_{\widetilde{\mathcal{C}}} \longrightarrow \nu_* \mathcal{O}^\times_{\widetilde{\mathcal{C}}}/\mathcal{O}^\times_{\mathcal{C}} \longrightarrow 1,$$

and a local computation (see [40, §7.5, especially Th. 5.19]). $\qquad \square$

The connected group $\mathrm{Pic}^0_{\mathcal{C}}$ acts, via tensor product, on the compactification $\overline{\mathrm{Pic}}^0_{\mathcal{C}}$, which is obtained by adding degree zero rank 1 torsion free sheaves on $\mathcal{C}$ which are not locally free.

Let $\nu : \mathcal{C}' \longrightarrow \mathcal{C}$ be a finite birational map. There is the direct image map

$$\nu_* : \overline{\mathrm{Pic}}^0_{\mathcal{C}'} \longrightarrow \overline{\mathrm{Pic}}^0_{\mathcal{C}}, \qquad \mathcal{F}' \longmapsto \nu_* \mathcal{F}'.$$

The following theorem summarizes most of the properties of the compactified Jacobians of blow ups that we need in the sequel of the paper.

THEOREM 2.2.2. *Let $\mathcal{C}$ be an integral, projective curve with at worst A_k-singularities, let $\nu : \mathcal{C}' \longrightarrow \mathcal{C}$ be a finite birational map and let $\mathcal{F} \in \overline{\mathrm{Pic}}^0_{\mathcal{C}}$. Then we have*

(1) *The compactified Jacobian $\overline{\mathrm{Pic}}^0_{\mathcal{C}}$ is irreducible. The action of $\mathrm{Pic}^0_{\mathcal{C}}$ has finitely many orbits. The orbit corresponding to locally free sheaves is dense.*

(2) *The direct image map $\nu_* : \overline{\mathrm{Pic}}_{\mathcal{C}'}^0 \longrightarrow \overline{\mathrm{Pic}}_{\mathcal{C}}^0$ is a closed imbedding with image a closed $\mathrm{Pic}_{\mathcal{C}}^0$-invariant subset of $\overline{\mathrm{Pic}}_{\mathcal{C}}^0$. The image of $\mathrm{Pic}_{\mathcal{C}'}^0$ is a locally closed $\mathrm{Pic}_{\mathcal{C}}^0$-invariant subset of $\overline{\mathrm{Pic}}_{\mathcal{C}}^0$.*

(3) *There are a unique finite birational map $\mu : \mathcal{C}_{\mathcal{F}} \longrightarrow \mathcal{C}$, obtained as a composition of simple blow ups, and a line bundle $\mathcal{L}_{\mathcal{F}}$ on $\mathcal{C}_{\mathcal{F}}$, such that $\mathcal{F} = \mu_* \mathcal{L}_{\mathcal{F}}$.*

(4) *Let $\mathfrak{S}$ be the poset of blow ups $\mathcal{C}' \longrightarrow \mathcal{C}$. There is a decomposition into locally closed subsets*

$$(2.2.3) \qquad \overline{\mathrm{Pic}}_{\mathcal{C}}^0 = \coprod_{\{\mathcal{C}' \to \mathcal{C}\} \in \mathfrak{S}} \mathrm{Pic}_{\mathcal{C}'}^0.$$

Proof. The proof of (1) can be found in [48], [1]. The proof of (2) can be found in [5]. The proof of (3) and (4) can be found in [25, Prop. 3.4]. $\qquad\square$

The main goal of this section is to prove Theorem 2.2.7, which gives an upper bound for the Betti numbers of the compactified Jacobian $\overline{\mathrm{Pic}}_{\mathcal{C}}^0$. In order to achieve this upper bound, we study the decomposition (2.2.3) by describing the poset $\mathfrak{S}$ of all the blowing-ups of $\mathcal{C}$.

Definition 2.2.3. An integral projective curve $\mathcal{C}$ with A_k-singularities is said to be of *singular type*

$$\underline{k} := (k_1, \ldots, k_o; k_{o+1}, \ldots, k_{o+e})$$

if its singular locus consists of $o + e$ distinct points $\{c_1, \ldots, c_o, c_{o+1}, \ldots, c_{o+e}\}$, where c_a is singular of type A_{k_a}, with k_a odd for $1 \leq a \leq o$ and k_a even for $o + 1 \leq a \leq o + e$. We say that each singular point is of one of two possible types, odd or even. We set $O := \{c_1, \ldots, c_o\} \subseteq \mathcal{C}_{\mathrm{sing}}$, the set of odd singular points, and $E := \{c_{o+1}, \ldots, c_{o+e}\} \subseteq \mathcal{C}_{\mathrm{sing}}$, the set of even singular points.

Recall that for each entry k_a above, we have defined an integer $\delta_{c_a} := \lceil k_a/2 \rceil$.

LEMMA 2.2.4. *Let $\mathcal{C}$ be of singular type $\underline{k}$, let $\widetilde{\mathcal{C}}$ be its normalization and let $\widetilde{g} := g(\widetilde{\mathcal{C}})$. Then*

$$\sum_l b_l(\mathrm{Pic}_{\mathcal{C}}^0) = 2^{2\widetilde{g}+o}.$$

Proof. A connected commutative Lie group is isomorphic to $(S^1)^r \times \mathbb{R}^s$ for some r and s. The Betti numbers satisfy $\sum_l b_l((S^1)^r \times \mathbb{R}^s) = 2^r$. In our case, Theorem 2.2.1 implies that $r = 2\widetilde{g} + o$. $\qquad\square$

The poset of blowing ups of $\mathcal{C}$ can be described as the set

$$\mathfrak{S} = \{I = (i_1, \ldots, i_o; i_{o+1}, \ldots, i_{o+e}) \in \mathbb{N}^{o+e} \mid 0 \leq i_a \leq \delta_{c_a}, \ \forall a = 1, \ldots, o+e\},$$

where we say that $I \geq I'$ if $i_a \geq i'_a$ for all $a = 1, \ldots, o + e$. Let $\mathcal{C}_I$ be the curve obtained from $\mathcal{C}$ by blowing up, in any order, i_1 times the point c_1, i_2 times the point c_2, etc. Let $\nu_I : \mathcal{C}_I \longrightarrow \mathcal{C}$ be the corresponding finite birational map. The singular points of $\mathcal{C}_I$ are still of type A_k.

Theorems 2.2.2 and 2.2.1 can be applied to $\mathrm{Pic}^0_{\mathcal{C}_I}$ and to $\overline{\mathrm{Pic}}^0_{\mathcal{C}_I}$. Note that $\mathcal{C}_I \longrightarrow \mathcal{C}$ factors through $\mathcal{C}_{I'} \longrightarrow \mathcal{C}$ if and only if $I \geq I'$, and that if $I = (\delta_{c_1}, \ldots, \delta_{c_{o+e}})$, then $\widetilde{\mathcal{C}} = \mathcal{C}_I \longrightarrow \mathcal{C}$ is the normalization. Define $|I| := \sum i_a$. Theorem 2.2.1 implies that

$$(2.2.4) \qquad \dim \mathrm{Pic}^0_{\mathcal{C}_I} = \dim \mathrm{Pic}^0_{\mathcal{C}} - |I|.$$

For $I \in \mathfrak{S}$, the direct image $\nu_{I,*}$ defines a locally closed imbedding $\mathrm{Pic}^0_{\mathcal{C}_I} \longrightarrow \overline{\mathrm{Pic}}^0_{\mathcal{C}}$. By applying (2.2.3) to the natural maps $\mathcal{C}_{I'} \longrightarrow \mathcal{C}_I$ for $I' \geq I$, we see that

$$\overline{\mathrm{Pic}}^0_{\mathcal{C}_I} = \coprod_{I' \geq I} \mathrm{Pic}^0_{\mathcal{C}_{I'}}.$$

PROPOSITION 2.2.5. *We have the following inequality concerning Betti numbers:*

$$\sum_{l \geq 0} b_l(\overline{\mathrm{Pic}}^0_{\mathcal{C}}) \leq \sum_{I \in \mathfrak{S}} \sum_{l \geq 0} b_l(\mathrm{Pic}^0_{\mathcal{C}_I}).$$

Proof. Let r be a nonnegative integer. Define the subset of $\overline{\mathrm{Pic}}^0_{\mathcal{C}}$:

$$Z_r = \coprod_{|I| \geq r} \mathrm{Pic}^0_{\mathcal{C}_I}.$$

In view of the discussion above, we have

(1) Z_r is a closed subset of $\overline{\mathrm{Pic}}^0_{\mathcal{C}}$. In particular, it is compact;
(2) there are closed inclusions $\emptyset \subseteq Z_\delta \subseteq \ldots \subseteq Z_1 \subseteq Z_0 = \overline{\mathrm{Pic}}^0_{\mathcal{C}}$, where $\delta := \sum_{c \in \mathcal{C}_{\mathrm{sing}}} \delta_c$;
(3) $Z_r \setminus Z_{r+1} = \coprod_{|I| = r} \mathrm{Pic}^0_{\mathcal{C}_I}$, where the union is over the connected components, all of which have the same dimension by (2.2.4).

The nested inclusions (2) yield the classical spectral sequence

$$(2.2.5) \qquad E_1^{p,q} = H^{p+q}(Z_{-p}, Z_{-p+1}) \Longrightarrow H^{p+q}(\overline{\mathrm{Pic}}^0_{\mathcal{C}}).$$

In view of the compactness (1), the E_1-term reads

$$E_1^{p,q} = H_c^{p+q}(Z_{-p} \setminus Z_{-p+1}) = \bigoplus_{|I| = -p} H_c^{p+q}(\mathrm{Pic}^0_{\mathcal{C}_I}).$$

By Poincaré duality, we have that $\sum_{l \geq 0} b_l(\mathrm{Pic}^0_{\mathcal{C}_I}) = \sum_{l \geq 0} \dim H_c^l(\mathrm{Pic}^0_{\mathcal{C}_I})$. It follows that

$$\sum_{p,q} \dim E_1^{p,q} = \sum_{I \in \mathfrak{S}} \sum_{l \geq 0} b_l(\mathrm{Pic}^0_{\mathcal{C}_I}).$$

Clearly, $\sum_{p,q} \dim E_r^{p,q} \leq \sum_{p,q} \dim E_1^{p,q}$, for every $r \geq 1$, and the statement follows. $\qquad\square$

In what follows, we adopt the convention that a product over the empty set equals 1. For every subset $J \subseteq O$, let $\delta_J := \prod_{c \in J} \delta_c$. We have the following lemma.

LEMMA 2.2.6. *We have*

$$\sum_{I \in \mathfrak{S}} \sum_{l \geq 0} b_l(\operatorname{Pic}_{\mathcal{C}_I}^0) = 2^{2\widetilde{g}} \left(\prod_{c \in O} (2\delta_c + 1) \right) \left(\prod_{c \in E} (\delta_c + 1) \right).$$

Proof. Let o_I be the number of odd points on $\mathcal{C}_I$. For every $0 \leq r \leq o$, let $\#_r$ be the number of curves $\mathcal{C}_I$ with a given $o_I = r$. Since Lemma 2.2.4 holds for every $\mathcal{C}_I$, we have that

$$(2.2.6) \qquad \sum_{I \in \mathfrak{S}} \sum_{l \geq 0} b_l(\operatorname{Pic}_{\mathcal{C}_I}^0) = \sum_{I \in \mathfrak{S}} 2^{2\widetilde{g} + o_I} = \sum_{r=0}^{o} \#_r \, 2^{2\widetilde{g} + r}.$$

We have

$$\#_o = \left(\prod_{c \in O} \delta_c \right) \left(\prod_{c \in E} (\delta_c + 1) \right);$$

in fact, the following two operations leave the number of odd points unchanged. Blowing up t times, $0 \leq t \leq \delta_c$, an even point $c \in E$, and blowing up t times, $0 \leq t < \delta_c$ an odd point $c \in O$.

In order to have precisely $o - 1$ odd points, we need to first blow up δ_c times an odd point c. Once this is done, we repeat the count above and deduce that

$$\#_{o-1} = \left(\sum_{j=1}^{o} \prod_{\substack{c \in O \\ c \neq c_j}} \delta_c \right) \prod_{c \in E} (\delta_c + 1) = \left(\sum_{\substack{J \subseteq O \\ \sharp J = o-1}} \delta_J \right) \prod_{c \in E} (\delta_c + 1).$$

It is clear that we can repeat this argument and rewrite the last term in (2.2.6) as

$$2^{2\widetilde{g}} \left(\sum_{r=0}^{o} 2^r \sum_{\substack{J \subseteq O \\ \sharp J = r}} \delta_J \right) \prod_{c \in E} (\delta_c + 1) = \left(\prod_{c \in O} (2\delta_c + 1) \right) \left(\prod_{c \in E} (\delta_c + 1) \right),$$

by the elementary equality $\sum_{r=0}^{o} 2^r \sum_{\substack{J \subseteq O \\ \sharp J = r}} \delta_J = \prod_{c \in O} (2\delta_c + 1)$. $\qquad\square$

Finally, we combine Proposition 2.2.5 and Lemma 2.2.6 and obtain the desired upper bound.

THEOREM 2.2.7. *Let $\mathcal{C}$ be an integral curve all of whose singularities are of type A_k. Denote by $O := \{c_1, \ldots, c_o\}$ the set of its singular points of type A_k with k odd, and denote by $E := \{c_{o+1}, \ldots, c_{o+e}\}$ the set of its singular points of type A_k with k even. Denote by $\widetilde{g}$ the genus of the normalization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$. Then*

$$\sum_l b_l \left(\overline{\mathrm{Pic}_{\mathcal{C}}^0} \right) \leq 2^{2\widetilde{g}} \left(\prod_{c \in O} (2\delta_c + 1) \right) \left(\prod_{c \in E} (\delta_c + 1) \right).$$

In fact, Theorem 2.1.4 below implies that the inequality above is in fact an equality. In particular, see Corollary 2.3.22; the spectral sequence (2.2.5) degenerates at E_1.

2.3. *The lower bound estimate.* The aim of this section is to prove Theorem 2.3.1, which, as we show below, readily implies Theorem 2.1.4.

THEOREM 2.3.1. *Let $s \in \mathcal{A}_{\mathrm{ell}}$, and let C_s be the corresponding spectral curve with its singular locus $\{c_1, \ldots, c_o, c_{o+1}, \ldots, c_{o+e}\}$. Let $O := \{c_1, \ldots, c_o\}$ be the set of points of type A_k with k odd, and let $E := \{c_o, \ldots, c_{o+e}\}$ be the set of points of type A_k with k even. Denote by $j : \mathcal{A}_{\mathrm{reg}} \longrightarrow \mathcal{A}_{\mathrm{ell}}$ the open imbedding. Then*

$$2^{2\widetilde{g}} \left(\prod_{c \in O} (2\delta_c + 1) \right) \left(\prod_{c \in E} (\delta_c + 1) \right) \leq \sum_l \dim \left(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q} \right)_s,$$

where $\widetilde{g}$ denotes the genus of the normalization $\widetilde{C_s}$ of C_s.

Proof that Theorem 2.3.1 implies Theorem 2.1.4. We have the following inequalities:

$$\sum_l \dim \left(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q} \right)_s \leq \sum_l b_l \left(\overline{\mathrm{Pic}_{C_s}^0} \right) \leq \sum_l \dim \left(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q} \right)_s,$$

where the first one follows from the general equality

$$\mathcal{H}^{-\dim \mathcal{A}}(\mathrm{IC}_{\mathcal{A}}(R^l \chi_{\mathrm{reg}*} \mathbb{Q}))_s = \left(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q} \right)_s$$

combined with the decomposition theorem (1.4.4) and (1.4.6) in Section 1.4.2 for the Hitchin map over $\mathcal{A}_{\mathrm{ell}}$ where we add up only the summands supported on $\mathcal{A}_{\mathrm{ell}}$, and the second inequality follows immediately by combining Theorems 2.2.7 and 2.3.1. $\qquad \square$

Outline of the strategy for the proof of Theorem 2.3.1. Let $s \in \mathcal{A}_{\mathrm{ell}}$. In view of Corollary 1.3.6, we have the natural isomorphism

$$\left(R^0 j_* R^l \chi_{\mathrm{reg}*} \mathbb{Q} \right)_s \simeq \left(R^0 j_* \bigwedge^l R^1 u_{\mathrm{reg}*} \mathbb{Q} \right)_s = \varinjlim \Gamma \left(N \cap \mathcal{A}_{\mathrm{reg}}, \bigwedge^l R^1 u_{\mathrm{reg}*} \mathbb{Q} \right),$$

where the direct limit is taken over the set of connected neighborhoods N of s in $\mathcal{A}$.

Fix a base-point $n_0 \in N \cap \mathcal{A}_{\mathrm{reg}}$. We have the monodromy representation

$$\pi_1(N \cap \mathcal{A}_{\mathrm{reg}}, n_0) \longrightarrow \mathrm{Aut}(H_1(C_{n_0}))$$

and its exterior powers

$$\pi_1(N \cap \mathcal{A}_{\mathrm{reg}}, n_0) \longrightarrow \mathrm{Aut}(\overset{l}{\bigwedge} H_1(C_{n_0})).$$

The evaluation map $\Gamma(N \cap \mathcal{A}_{\mathrm{reg}}, \bigwedge^l R^1 u_{\mathrm{reg}*} \mathbb{Q}) \longrightarrow \bigwedge^l H_1(C_{n_0})$ at the point n_0 identifies the vector space of sections $\Gamma(N \cap \mathcal{A}_{\mathrm{reg}}, \bigwedge^l R^1 u_{\mathrm{reg}*} \mathbb{Q})$ with the subspace of monodromy invariants of $\bigwedge^l H_1(C_{n_0})$. Thus, in order to prove Theorem 2.3.1 we need to investigate the monodromy of the restriction of the spectral curve family $u_{\mathrm{reg}} : C_{\mathcal{A}_{\mathrm{reg}}} \to \mathcal{A}_{\mathrm{reg}}$ to $N \cap \mathcal{A}_{\mathrm{reg}}$, where N is a small enough connected neighborhood of s in $\mathcal{A}$.

We consider the local family $\mathscr{C}_{\mathcal{U}} \longrightarrow \mathcal{U}$ of double coverings of C whose branch locus is "close" to that of C_s; i.e., it is contained in a neighborhood $\mathcal{U}$ of the divisor $(s_1^2 - 4s_2)$ in the symmetric product of C. The family has the property that every other family of double coverings whose branch locus is contained in $\mathcal{U}$ is the pull-back of $\mathscr{C}_{\mathcal{U}} \longrightarrow \mathcal{U}$ via a uniquely determined map; see Proposition 2.3.11 for a precise statement. We investigate the monodromy of the smooth part of this family, and we determine the dimension of the subspace of monodromy invariants in the exterior powers of the associated local system. Since the spectral curve family, restricted to a small enough neighborhood of s in $\mathcal{A}$, is isomorphic to the pullback of this local family via the map Θ of Remark 1.3.3, the dimension of the subspace of monodromy invariants of the local family gives a lower bound for the dimension of the monodromy invariants of the spectral curve family, thus proving Theorem 2.3.1.

Remark 2.3.2. While our analysis of the monodromy is purely local, a detailed study of the global monodromy of the family $\mathcal{M}_{\mathrm{reg}} \longrightarrow \mathcal{A}_{\mathrm{reg}}$ has been carried out, for C a hyperelliptic curve, by Copeland in [17].

Notation 2.3.3. In the remaining of Section 2.3, for notational simplicity, we denote with the same symbol a cycle (resp. relative cycle) and the homology (resp. relative homology) class it defines. In particular, an equality of cycles (resp. relative cycles) will always mean equality of their homology (resp. relative homology) classes.

2.3.1. *The double covering of a disc.* We review some basic facts (see [4, Part 1]) concerning the topology of a holomorphic branched double covering $\rho : \mathbb{S} \longrightarrow \overline{\mathbb{D}}$ of the closed unit disc $\overline{\mathbb{D}} \subseteq \mathbb{C}$, with boundary $\partial\mathbb{D}$ and interior $\mathbb{D}$, under the following

Assumption 2.3.4. The map ρ is the restriction of a holomorphic mapping from thickenings of domain and codomain, there are no branch points on $\partial\mathbb{D}$ and the degree $2r$ of the branch locus divisor Z is even.

Let $p_Z(z)$ be the monic degree $2r$ polynomial vanishing on Z. Then

$$(2.3.1) \qquad \mathbb{S} = \left\{ (z, w) \in \overline{\mathbb{D}} \times \mathbb{C} \text{ such that } w^2 = p_Z(z) \right\}, \quad \rho(z, w) = z.$$

Remark 2.3.5. Since $p_Z(z)$ has even degree, the boundary $\partial \mathbb{S} = \rho^{-1}(\partial \mathbb{D})$ of $\mathbb{S}$ consists of two connected components ∂' and ∂'', which we endow with the orientation induced from $\mathbb{S}$. We denote the resulting cycles in homology with the same symbols (cf. Notation 2.3.3).

Assume Z consists of $2r$ distinct points. By the Riemann-Hurwitz formula, $\mathbb{S}$ is biholomorphic to a compact Riemann surface of genus $r-1$ with two open disks removed.

Denote by $\mathbb{I} := [0, 2r + 1] \subseteq \mathbb{R}$, and let $\beta : \mathbb{I} \longrightarrow \overline{\mathbb{D}}$ be a differentiable imbedding such that $\partial \mathbb{D} \cap \beta(\mathbb{I}) = \{\beta(0), \beta(2r+1)\}$ and $Z = \{\beta(1), \ldots, \beta(2r)\}$. The subsets

$$\lambda_j := \rho^{-1}\left(\beta([j, j + 1]) \right), \ j = 1, \ldots, 2r - 1$$

are closed curves, which we orient subject to the requirements

$$(2.3.2) \qquad (\lambda_j, \lambda_{j+1}) = 1,$$

where $(\,,\,)$ denotes the intersection product with respect to the the natural orientation of $\mathbb{S}$, and the equality in homology

$$(2.3.3) \qquad \sum_{j=1}^{r} \lambda_{2j-1} = \partial'.$$

The 1-cycles $\{\lambda_j\}_{j=1}^{2r-1}$ form a basis for the first homology group $H_1(\mathbb{S})$.

Remark 2.3.6. In view of the long exact sequence in relative homology of the pair $(\mathbb{S}, \partial \mathbb{S})$, the kernel of the natural map $H_1(\mathbb{S}) \longrightarrow H_1(\mathbb{S}, \partial \mathbb{S})$ is one dimensional, generated by the cycle $\sum_{j=1}^{r} \lambda_{2j-1} = \partial'$. In order to complete the set $\{\lambda_j\}_{j=1}^{2r-1}$ to a system of generators of $H_1(\mathbb{S}, \partial \mathbb{S})$, we need to add a relative 1-cycle sent via the boundary map to a generator of $\mathrm{Ker}\{H_0(\partial \mathbb{S}) \longrightarrow H_0(\mathbb{S})\}$, namely a cycle joining the two components ∂', ∂'' of the boundary. We take the relative homology class of the curve

$$(2.3.4) \qquad \mu := \rho^{-1}\left(\beta([0, 1]) \right) \subseteq \mathbb{S}$$

oriented so that we have the first equality below. We have

$$(2.3.5) \qquad (\mu, \lambda_1) = 1 \text{ and } (\mu, \lambda_j) = 0, \ \forall\, 1 < j \leq 2r - 1.$$

The relative homology classes of the cycles $\{\lambda_1, \ldots, \lambda_{2r-1}, \mu\}$ form a set of generators for $H_1(\mathbb{S}, \partial \mathbb{S})$ subject to the only relation $\sum_{j=1}^{r} \lambda_{2j-1} = 0$.

2.3.2. *The family of coverings of the disc and its monodromy.* We identify the symmetric product $\mathbb{D}^{(2r)}$, parametrizing the effective divisors of degree $2r$ on the unit disk $\mathbb{D}$, with the space of monic polynomials of degree $2r$ whose roots have absolute values less than 1, by sending $v = (v_1, \ldots, v_{2r}) \in \mathbb{D}^{(2r)}$ to $p_v(X) = \prod_1^{2r}(X - v_l)$. The elementary symmetric functions of $(v_1, \ldots, v_{2r})$ give a system of coordinates for $\mathbb{D}^{(2r)}$, thus realizing it as a bounded open subset of $\mathbb{C}^{2r}$.

Notation 2.3.7. We denote a point in $\mathbb{D}^{(2r)} \subseteq \mathbb{C}^{2r}$ by the divisor v on $\mathbb{D}$ or by its associated monic polynomial p_v.

On $\overline{\mathbb{D}} \times \mathbb{D}^{(2r)}$ there is the divisor

$$\mathscr{L}_{2r} := \left\{ (z, p) \in \overline{\mathbb{D}} \times \mathbb{D}^{(2r)} \text{ such that } p(z) = 0 \right\}$$

and the double covering

$$\mathscr{S}_{2r} = \left\{ (z, p, w) \in \overline{\mathbb{D}} \times \mathbb{D}^{(2r)} \times \mathbb{C} \text{ such that } w^2 = p(z) \right\},$$

defining the family $\Phi_{2r} : \mathscr{S}_{2r} \longrightarrow \mathbb{D}^{(2r)}$ of (possibly singular) Riemann surfaces with boundary (for every fiber $\mathbb{S}_v$, the singularities are disjoint from the boundary):

(2.3.6)

$$\mathscr{S}_{2r} \xrightarrow{\;\;\Phi_{2r}\;\;} \mathbb{D}^{(2r)}$$

$$\rho_{2r} \searrow \qquad \nearrow p_2$$

$$\overline{\mathbb{D}} \times \mathbb{D}^{(2r)}.$$

The map ρ_{2r} is a double covering branched over $\mathscr{L}_{2r}$, and, for $v \in \mathbb{D}^{(2r)}$, the fibre $\mathbb{S}_v := \Phi_{2r}^{-1}(v)$ is the double covering $\mathbb{S}_v \longrightarrow \overline{\mathbb{D}}$ of equation $w^2 = p_v(z)$ branched precisely over the effective divisor v in $\mathbb{D}$.

Remark 2.3.8. By Remark 2.3.5, the boundary of every fibre of the map Φ_{2r} consists of two connected components. Since $\mathbb{D}^{(2r)}$ is contractible, we have a smooth trivialization $\partial \mathscr{S}_{2r} \simeq (S^1 \coprod S^1) \times \mathbb{D}^{(2r)}$, well defined up to isotopy.

The locus E of polynomials with vanishing discriminant is a divisor in $\mathbb{D}^{(2r)}$, and $\mathbb{D}_{\text{reg}}^{(2r)} := \mathbb{D}^{(2r)} \setminus E$ is the open subset corresponding to multiplicity free divisors, namely $2r$-tuples of distinct points in $\mathbb{D}$. The double covering $\mathbb{S}_v = \Phi_{2r}^{-1}(v)$ of $\overline{\mathbb{D}}$ introduced in Section 2.3.2 is nonsingular if and only if $v \in \mathbb{D}_{\text{reg}}^{(2r)}$.

We choose a base-point $\underline{v} \in \mathbb{D}_{\text{reg}}^{(2r)}$. The fundamental group $\pi_1(\mathbb{D}_{\text{reg}}^{(2r)}, \underline{v})$ is the classical braid group $\mathscr{B}^{2r}$ on $2r$ strands (see [4], §3.3). As in Section 2.3.1, a differentiable imbedding $\beta : \mathbb{I} \longrightarrow \mathbb{D}$, such that $\underline{v} = \{\beta(1), \ldots, \beta(2r)\}$, defines a basis $\{\lambda_j\}_{j=1}^{2r-1}$ of $H_1(\mathbb{S}_{\underline{v}})$, the relative class $\mu \in H_1(\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}})$, and the usual set $T_1, \ldots, T_{2r-1}$ of generators of $\mathscr{B}^{2r}$. If $v_i := \beta(i)$, the braid T_i exchanges v_i

with v_{i+1} by a half-turn. More precisely, let $\mathbb{D}^+, \mathbb{D}^-$ be the two open half-discs determined by β and its orientation; then T_i can be represented by two curves $\tau^+, \tau^- : [0, 1] \longrightarrow \mathbb{D}$ such that

$$(2.3.7) \qquad \begin{aligned} \tau^+(0) = \tau^-(1) = v_i, \qquad & \tau^+(1) = \tau^-(0) = v_{i+1}, \\ \tau^+((0,1)) \subseteq D^+, \qquad & \tau^-((0,1)) \subseteq D^-. \end{aligned}$$

We apply the Ehresmann fibration lemma to the restriction of the family Φ_{2r} to $\mathbb{D}_{\text{reg}}^{(2r)}$. We have monodromy homeomorphisms $M(T_i) : (\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}}) \longrightarrow (\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}})$ for $i = 1, \ldots, 2r - 1$, which restrict to the identity on the boundary $\partial \mathbb{S}_{\underline{v}}$. They are unique up to an isotopy which fixes the boundary pointwise.

Let $\gamma \in H_1(\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}})$ be a relative 1-cycle. Since the monodromy homeomorphisms fix the boundary, the difference $M(T_i)(\gamma) - \gamma$ is homologous to a cycle, denoted $\text{Var}_i(\gamma)$, disjoint from the boundary. This defines the classical *variation maps* (see [4, §2.1]):

$$\text{Var}_i : H_1(\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}}) \longrightarrow H_1(\mathbb{S}_{\underline{v}}), \qquad i = 1, \ldots, 2r - 1.$$

PROPOSITION 2.3.9. *The following holds*:

$$\text{Var}_i(\lambda_j) = \begin{cases} \lambda_i & \text{if } j = i - 1, \\ -\lambda_i & \text{if } j = i + 1, \\ 0 & \text{if } j \neq i - 1, i + 1, \end{cases} \qquad \text{Var}_i(\mu) = \begin{cases} 0 & \text{if } i \neq 1, \\ \lambda_1 & \text{if } i = 1. \end{cases}$$

Proof. The monodromy $M(T_i)$ is associated with the degeneration of $\mathbb{S}_{\underline{v}}$ in which the i-th and $(i+1)$-th ramification points come together and the covering acquires a node. It follows that $M(T_i)$ is a Dehn twist around λ_i. The Picard-Lefschetz formula ([4, §1.3]) gives

$$\text{if } c \in H_1(\mathbb{S}_{\underline{v}}, \partial \mathbb{S}_{\underline{v}}), \text{ then } \text{Var}_i(c) = (c, \lambda_i)\lambda_i.$$

We conclude by combining the above with (2.3.2) and (2.3.5). $\qquad \square$

2.3.3. *The local family.* Let $d = 2r$ be an even positive integer, and let $\mathfrak{a}$ be a partition of d, which we write

$$(2.3.8) \qquad \mathfrak{a} = (a_1, \ldots, a_{2\omega + \varepsilon}),$$

where $a_1, \ldots, a_{2\omega}$ are odd positive integers and $a_{2\omega+1}, \ldots, a_{2\omega+\varepsilon}$ are even positive integers; we set

$$(2.3.9) \qquad d_0 := 0, \ d_i := \sum_{j=1}^{i} a_j \text{ for } i = 1, \ldots, 2\omega + \varepsilon.$$

Clearly $d = d_{2\omega+\varepsilon}$.

Let σ be an effective divisor of degree d on a projective nonsingular curve C with *multiplicity type* $\mathfrak{a}$, namely

$$(2.3.10) \qquad \sigma = \sum_{i=1}^{2\omega+\varepsilon} a_i q_i \,,$$

where the points $q_1, \ldots, q_{2\omega+\varepsilon}$ of C are distinct.

Let $\mathcal{O}_C(\sigma)$ be the corresponding line bundle on C and let $s \in \Gamma(C, \mathcal{O}_C(\sigma))$ be the section vanishing at σ, well defined up to a nonzero scalar. We choose a square root of $\mathcal{O}_C(\sigma)$, that is, a line bundle L on C such that $L^{\otimes 2} \simeq \mathcal{O}_C(\sigma)$. The double cover C_σ of C branched over σ is the curve on the total space $\mathbb{V}(L) \xrightarrow{\pi} C$ of L defined by

$$\{y \in \mathbb{V}(L) \,:\, y^2 = \pi^* s\}.$$

Note that the topology, e.g., its being connected or not, depends on the choice of the square root L, and not only on σ.

From this point on, we work under the following

Assumption 2.3.10. The double covering C_σ is integral.

The effective divisors of degree $2r$ on the curve C are parametrized by the symmetric product $C^{(2r)}$, which is a nonsingular algebraic variety, stratified by the loci corresponding to divisors with a fixed multiplicity type. We denote by $C_{\mathrm{reg}}^{(2r)}$ the open subset consisting of multiplicity-free divisors and, for every subset $Y \subseteq C^{(2r)}$, we set $Y_{\mathrm{reg}} := Y \cap C_{\mathrm{reg}}^{(2r)}$. We have the divisor $Z := \{(c, u) \in C \times C^{(2r)} \,:\, c \in u\} \subseteq C \times C^{(2r)}$, the associated line bundle $\mathcal{O}(Z)$ on $C \times C^{(2r)}$ and its tautological section $S \in \Gamma(C \times C^{(2r)}, \mathcal{O}(Z))$ vanishing at Z. Given an open subset $V \subseteq C \times C^{(2r)}$, we set $Z_V := Z \cap V$ and denote by $\mathcal{O}_V(Z)$ and S_V the restrictions of the corresponding objects to V.

The following proposition follows readily from the fact that the squaring map $\mathrm{Pic}_C^r \longrightarrow \mathrm{Pic}_C^{2r}$ is étale.

PROPOSITION 2.3.11. *Let $\mathcal{U}$ be a connected and simply connected open neighborhood of σ in $C^{(2r)}$, and let $Z_\mathcal{U} := Z \cap (C \times \mathcal{U})$. Then for every line bundle L on C such that $L^{\otimes 2} \simeq \mathcal{O}_C(\sigma)$, there is a projective family $\Phi_\mathcal{U}$*
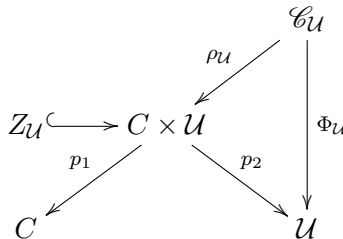

with the following properties:

(1) *For $u \in \mathcal{U}$, the curve $C_u := \Phi_{\mathcal{U}}^{-1}(u) \xrightarrow{\rho_u} C$ is a double covering of C ramified at the effective divisor $u = Z_{\mathcal{U}} \cap p_2^{-1}(u)$, and $C_\sigma := \Phi_{\mathcal{U}}^{-1}(\sigma) \xrightarrow{\rho_\sigma} C$ is the double covering of C ramified at σ corresponding to the choice of the square root L.*

(2) *The map $\rho_{\mathcal{U}}$ is a double covering branched over $Z_{\mathcal{U}}$.*

(3) *The restriction $\Phi_{\mathcal{U}_{\mathrm{reg}}} : \mathscr{C}_{\mathcal{U}_{\mathrm{reg}}} := \Phi_{\mathcal{U}}^{-1}(\mathcal{U}_{\mathrm{reg}}) \longrightarrow \mathcal{U}_{\mathrm{reg}}$ is a smooth family.*

(4) *If*
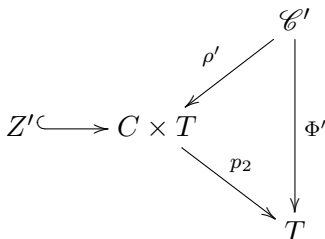
is a family of double coverings of C with ρ' ramified over the divisor Z' and, for $t_0 \in T$, there is an isomorphism
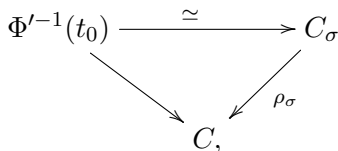
then, for a suitable neighborhood $V \subseteq T$ of t_0, the map $\theta : V \longrightarrow \mathcal{U}$ associating to $t \in T$ the branch locus of $\Phi'^{-1}(t) \longrightarrow C$ defines an isomorphism $\Phi'^{-1}(V) \simeq \mathscr{C}_{\mathcal{U}} \times_\theta V$ over V.

We now define the *distinguished neighborhoods* of σ in $C^{(2r)}$.

Choose a closed disc $\overline{\Delta} \subseteq C$ whose interior Δ contains the support of σ. Choose open discs $\Delta_1, \ldots, \Delta_{2\omega+\varepsilon} \subseteq \Delta \subseteq C$ so that

(1) $q_i \in \Delta_i$ for all i.

(2) $\overline{\Delta_i} \subseteq \Delta$ for all i, and $\overline{\Delta_i} \cap \overline{\Delta_j} = \emptyset$ for all $i \neq j$.

As in [Section 2.3.2](#), we have the a_i-th symmetric product $\Delta_i^{(a_i)}$ and its open subset $\Delta_{i,\mathrm{reg}}^{(a_i)}$ corresponding to a_i-tuples of distinct points. The set of effective divisors of degree $2r$ consisting of a_i points contained in Δ_i, where $i = 1, \ldots, 2\omega+\varepsilon$, defines a *distinguished* neighborhood of $\sigma \in C^{(2r)}$:

$$\mathcal{N} := \prod_i \Delta_i^{(a_i)} \subseteq \Delta^{(2r)} \subseteq C^{(2r)}.$$

Distinguished neighborhoods are contractible and give rise to a fundamental system of neighborhoods of $\sigma \in C^{(2r)}$. We also have the open subset

$$(2.3.11) \qquad \mathcal{N}_{\mathrm{reg}} := \mathcal{N} \cap C_{\mathrm{reg}}^{(2r)} = \prod_i \Delta_{i,\mathrm{reg}}^{(a_i)} \subseteq \Delta_{\mathrm{reg}}^{(2r)},$$

consisting of the simple, i.e., multiplicity-free, divisors in $\mathcal{N}$.

By Proposition 2.3.11, the choice of a square root L of the line bundle $\mathcal{O}_C(\sigma)$ yields the family $\Phi_{\Delta^{(2r)}} : \mathscr{C}_{\Delta^{(2r)}} \to \Delta^{(2r)}$, the smooth family $\Phi_{\Delta^{(2r)}_{\mathrm{reg}}} : \mathscr{C}_{\Delta^{(2r)}_{\mathrm{reg}}} \to \Delta^{(2r)}_{\mathrm{reg}}$, and their restrictions $\Phi_{\mathcal{N}} : \mathscr{C}_{\mathcal{N}} \to \mathcal{N}$ and $\Phi_{\mathcal{N}_{\mathrm{reg}}} : \mathscr{C}_{\mathcal{N}_{\mathrm{reg}}} \to \mathcal{N}_{\mathrm{reg}}$.

Our aim is the proof of Theorem 2.3.12 below. This result is the main step in the proof of Theorem 2.3.1 which, as we have seen at the beginning of Section 2.3, completes the proof of the main Theorem 2.1.4 of this section.

THEOREM 2.3.12. *Let C be a nonsingular projective curve of genus g, let $\sigma \in C^{(2r)}$ be an effective divisor of multiplicity type $\mathfrak{a} = (a_1, \ldots, a_{2\omega+\varepsilon})$, and let L be a square root of $\mathcal{O}_C(\sigma)$ such that the associated double covering $\rho_\sigma : C_\sigma \longrightarrow C$ is integral. Let $\mathcal{N}$ be a distinguished neighborhood of σ, let $\mathcal{N}_{\mathrm{reg}}$ be the open subset of simple divisors in $\mathcal{N}$, and let $j : \mathcal{N}_{\mathrm{reg}} \longrightarrow \mathcal{N}$ be the corresponding imbedding. Then*

$$(2.3.12)$$
$$\sum_{l=0}^{4g+2r-2} \dim\left(R^0 j_* \bigwedge^l R^1 \Phi_{\mathcal{N}_{\mathrm{reg}}*}\mathbb{Q}\right)_\sigma = \sum_{l=0}^{4g+2r-2} \dim \Gamma\left(\mathcal{N}_{\mathrm{reg}}, \bigwedge^l R^1 \Phi_{\mathcal{N}_{\mathrm{reg}}*}\mathbb{Q}\right)$$
$$= 2^{4g-2}\left(\prod_{i=1}^{2\omega+\varepsilon}(a_i+1)\right).$$

Remark 2.3.13. Recall the Definition 2.2.3 of singular type of an integral curve with A_k-singularities and of the two sets O and E. With the notation of Theorem 2.3.12, we have $E = \{\rho_\sigma^{-1}(q_i)\}$ for $i \in \{1, \ldots, 2\omega\}$ such that $a_i \neq 1$, and $O = \{\rho_\sigma^{-1}(q_i)\}$ for $2\omega+1 \leq i \leq 2\omega+\varepsilon$. With the convention that if $a_i = 1$, then the term $a_i - 1$ should be deleted from the singular type vector $\underline{k}$, we have that C_σ has singular type $\underline{k} = (a_{2\omega+1}-1, \ldots, a_{2\omega+\varepsilon}-1; a_1-1, \ldots, a_{2\omega}-1)$.

Then, the right-hand side of equation 2.3.12 equals the quantity

$$2^{2\widetilde{g}}\left(\prod_{c \in O}(2\delta_c+1)\right)\left(\prod_{c \in E}(\delta_c+1)\right)$$

associated with the singular curve C_σ (see Theorem 2.2.7).

For $i = 1, \ldots, 2\omega+\varepsilon$, we set $\delta_i := \delta_{\rho_\sigma^{-1}(q_i)}$, with the convention that $\delta_i = 0$ if $a_i = 1$, i.e., if $\rho_\sigma^{-1}(q_i)$ is a nonsingular point of C_σ. Clearly

$$(2.3.13) \quad \left(\prod_{c \in O}(2\delta_c+1)\right)\left(\prod_{c \in E}(\delta_c+1)\right) = \left(\prod_{i=1}^{2\omega}(\delta_i+1)\right)\left(\prod_{i=2\omega+1}^{2\omega+\varepsilon}(2\delta_i+1)\right).$$

By the Riemann-Hurwitz formula and the definition of the δ-invariant (§ 2.2), we have that

$$(2.3.14)$$
$$\widetilde{g} = 2g+r-1-\sum \delta_i = 2g+\frac{1}{2}\sum_{i=1}^{2\omega+\varepsilon}a_i-1-\frac{1}{2}\sum_{i=1}^{2\omega}(a_i-1)-\frac{1}{2}\sum_{i=\omega+1}^{2\omega+\varepsilon}a_i = 2g+\omega-1.$$

Now use the fact that

$$(2.3.15) \qquad a_i = \begin{cases} 2\delta_i & \text{if } a_i \text{ is even}, \\ 2\delta_i + 1 & \text{if } a_i \text{ is odd}, \end{cases}$$

and deduce the equality

$$2^{4g-2}\left(\prod_{i=1}^{2\omega+\varepsilon}(a_i+1)\right) = 2^{2\tilde{g}-2\omega}\prod_{i=1}^{2\omega}(2\delta_i+2)\prod_{i=2\omega+1}^{2\omega+\varepsilon}(2\delta_i+1)$$

$$= 2^{2\tilde{g}}\prod_{i=1}^{2\omega}(\delta_i+1)\prod_{i=2\omega+1}^{2\omega+\varepsilon}(2\delta_i+1),$$

whose right-hand side coincides, by (2.3.13), with

$$2^{2\tilde{g}}\left(\prod_{c\in O}(2\delta_c+1)\right)\left(\prod_{c\in E}(\delta_c+1)\right).$$

For $i = 1, \ldots, 2\omega + \varepsilon$, let $\underline{u}_i \in \Delta_{i,\text{reg}}^{(a_i)}$, and let $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_{2\omega+\varepsilon}) \in \mathcal{N}_{\text{reg}}$. We have the monodromy representation

$$\pi_1(\mathcal{N}_{\text{reg}}, \underline{u}) \longrightarrow \text{Aut}(H_1(C_{\underline{u}}))$$

and its exterior powers

$$\pi_1(\mathcal{N}_{\text{reg}}, \underline{u}) \longrightarrow \text{Aut}\left(\bigwedge^l H_1(C_{\underline{u}})\right).$$

The evaluation map at the base-point $\underline{u}$ gives an isomorphism

$$\Gamma\left(\mathcal{N}_{\text{reg}}, \bigwedge^l R^1\Phi_{\mathcal{N}_{\text{reg}}*}\mathbb{Q}\right) \xrightarrow{\simeq} \left(\bigwedge^l H_1(C_{\underline{u}})\right)^{\pi_1},$$

where $(-)^{\pi_1}$ denotes the subspace of invariants.

Remark 2.3.14. Since the family $\Phi_{\mathcal{N}_{\text{reg}}} : \mathscr{C}_{\mathcal{N}_{\text{reg}}} \longrightarrow \mathcal{N}_{\text{reg}}$ is the restriction to $\mathcal{N}_{\text{reg}}$ of the family $\Phi_{\Delta_{\text{reg}}^{(2r)}} : \mathscr{C}_{\Delta_{\text{reg}}^{(2r)}} \longrightarrow \Delta_{\text{reg}}^{(2r)}$, its monodromy representation is the composition

$$(2.3.16) \qquad \pi_1(\mathcal{N}_{\text{reg}}, \underline{u}) \longrightarrow \pi_1(\Delta_{\text{reg}}^{(2r)}, \underline{u}) \longrightarrow \text{Aut}(H_1(C_{\underline{u}})).$$

2.3.4. *Proof of Theorem* 2.3.12. *Step* 1: *splitting off the constant part.* We have the diagram

of Proposition 2.3.11 and the nonsingular branched double covering

$$(2.3.17) \qquad \rho_{\underline{u}} : \Phi_{\Delta^{(2r)}}^{-1}(\underline{u}) =: C_{\underline{u}} \longrightarrow C.$$

By the Riemann-Hurwitz formula we have $g(C_{\underline{u}}) = 2g+r-1$, where, we remind the reader, $r = \frac{\deg \sigma}{2} = \frac{\sum_i a_i}{2}$. We set

$$\hat{C} := \rho_{\underline{u}}^{-1}(C \setminus \Delta), \qquad \Xi_{\underline{u}} := \rho_{\underline{u}}^{-1}(\overline{\Delta}).$$

Remark 2.3.15. In view of Remark 2.3.5, the inverse image $\rho_{\underline{u}}^{-1}(\partial\overline{\Delta}) = \partial\,\Xi_{\underline{u}}$ consists of two connected components. There are two distinct possibilities for the restriction of the covering $\rho_{\underline{u}}$ to $\hat{C}$. The former is that this restricted covering is disconnected and thus biholomorphic to two copies of $C \setminus \Delta$. This is the case if the square root L of $\mathcal{O}_C(\sigma)$ is a trivial line bundle on $C \setminus \Delta$. The latter, corresponding to the case in which L is a nontrivial line bundle on $C \setminus \Delta$, is that $\hat{C}$ is connected, in which case $\hat{C} = C' \setminus (U_1 \coprod U_2)$ is obtained by removing two discs U_1, U_2 from a connected compact Riemann surface C' of genus $2g - 1$.

Since the line bundle associated with a divisor on C supported on Δ is trivial on $C \setminus \Delta$, we have a biholomorphism (of surfaces with boundaries) $\rho_{\Delta^{(2r)}}^{-1}((C \setminus \Delta) \times \Delta^{(2r)}) \simeq \hat{C} \times \Delta^{(2r)}$, and the family $\Phi_{\Delta^{(2r)}} : \mathscr{C}_{\Delta^{(2r)}} \longrightarrow \Delta^{(2r)}$ is obtained by glueing the family $\rho_{\Delta^{(2r)}}^{-1}(\overline{\Delta} \times \Delta^{(2r)}) \longrightarrow \Delta^{(2r)}$ to the constant family $\hat{C} \times \Delta^{(2r)} \longrightarrow \Delta^{(2r)}$ along the boundary $(S^1 \coprod S^1) \times \Delta^{(2r)}$; the same clearly applies to its restrictions $\Phi_{\mathcal{N}}, \Phi_{\mathcal{N}_{\mathrm{reg}}}$.

The long exact sequence of relative cohomology of the pair $\Xi_{\underline{u}} \subseteq C_{\underline{u}}$, the vanishing $H_2(\Xi_{\underline{u}}) = 0$, and the fact that $H_0(\Xi_{\underline{u}}) \longrightarrow H_0(C_{\underline{u}})$ is an isomorphism give the exact sequence

$$(2.3.18)$$

$$0 \longrightarrow H_2(C_{\underline{u}}) \longrightarrow H_2(C_{\underline{u}}, \Xi_{\underline{u}}) \longrightarrow H_1(\Xi_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}}, \Xi_{\underline{u}}) \longrightarrow 0$$

$$\qquad\qquad\qquad \uparrow \simeq \qquad\qquad\qquad\qquad\qquad\qquad \uparrow \simeq$$

$$\qquad\qquad H_2(\hat{C}, \partial\hat{C}) \qquad\qquad\qquad\qquad\qquad H_1(\hat{C}, \partial\hat{C}),$$

where the vertical arrows indicate the excision isomorphisms.

Case 1: $\hat{C}$ *is disconnected*. Here $\dim H_2(C_{\underline{u}}, \Xi_{\underline{u}}) = 2$ and $\dim H_1(C_{\underline{u}}, \Xi_{\underline{u}}) \simeq H_1(C)^{\oplus 2} = 4g$; define

$$(2.3.19) \qquad H_{\mathfrak{a},\mathrm{disc}} := \mathrm{Im}\{H_1(\Xi_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}})\}.$$

It follows from the sequence (2.3.18) that $\dim H_{\mathfrak{a},\mathrm{disc}} = 2r - 2$, and we have an exact sequence

$$(2.3.20) \qquad 0 \longrightarrow H_{\mathfrak{a},\mathrm{disc}} \longrightarrow H_1(C_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}}, \Xi_{\underline{u}}) \longrightarrow 0.$$

Remark 2.3.16. In this case, in order to satisfy Assumption 2.3.10, we must have $\omega \geq 1$. In fact, if $\omega = 0$, every singular point of C_σ has two branches.

Let $\nu : \widetilde{C_\sigma} \longrightarrow C_\sigma$ be the normalization map. Since the inverse image by ν of every singular points consists of two points, the composition $\widetilde{C_\sigma} \xrightarrow{\nu} C_\sigma \longrightarrow C$ is an étale covering, which must be trivial if $\hat{C}$ is disconnected. This implies that $\widetilde{C_\sigma}$ is disconnected and C_σ is reducible against Assumption 2.3.10. See also [44, §11].

Remark 2.3.17. Given a basis of $H_1(C)$, it is possible to represent its elements by cycles contained in $C \setminus \Delta$. By taking pre-images of these cycles via $\rho_{\underline{u}}$, we get $4g$ linearly independent homology classes which split the exact sequence (2.3.20). Since, as we have already observed, the family $\Phi_{\Delta^{(2r)}} : C_{\Delta^{(2r)}} \longrightarrow \Delta^{(2r)}$ is obtained by glueing the constant family with the family $\rho_{\Delta^{(2r)}}^{-1}(\overline{\Delta} \times \Delta^{(2r)}) \longrightarrow \Delta^{(2r)}$, the direct sum decomposition

$$(2.3.21) \qquad\qquad H_1(C_{\underline{u}}) = H_{\mathfrak{a},\mathrm{disc}} \oplus H_1(C)^{\oplus 2}$$

is invariant under the action of $\pi_1(\Delta_{\mathrm{reg}}^{(2r)}, \underline{u})$, which is trivial on the second summand.

Case 2: $\hat{C}$ *is connected.* In this case $\dim H_2(C_{\underline{u}}, \Xi_{\underline{u}}) = 1$ and $\dim H_1(C_{\underline{u}}, \Xi_{\underline{u}}) = 4g - 1$; the sequence (2.3.18) takes the form

$$(2.3.22) \qquad 0 \longrightarrow H_1(\Xi_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}}, \Xi_{\underline{u}}) = H_1(\hat{C}, \partial\hat{C}) \longrightarrow 0.$$

As we already observed in Remark 2.3.15, $\hat{C} = C' \setminus (U_1 \coprod U_2)$ is obtained by removing two discs U_1, U_2 from a connected compact Riemann surface C' of genus $2g - 1$; it is readily seen that the map $H_1(C') \longrightarrow H_1(C', \partial\overline{U_1} \coprod \partial\overline{U_2})$ is injective. We have the excision isomorphism $H_1(C', \partial\overline{U_1} \coprod \partial\overline{U_2}) = H_1(\hat{C}, \partial\hat{C})$, by which we identify $H_1(C')$ with a subspace of $H_1(\hat{C}, \partial\hat{C})$. Let $\hat{\gamma} \in H_1(\hat{C}, \partial\hat{C})$ be the class of a path in $\hat{C}$ joining the two connected components of its boundary. It is then easy to see that $H_1(\hat{C}, \partial\hat{C}) = H_1(C') \oplus \mathrm{Span}\,\hat{\gamma}$. By using the excision isomorphism $H_1(C_{\underline{u}}, \Xi_{\underline{u}}) = H_1(\hat{C}, \partial\hat{C})$, we obtain an isomorphism $H_1(C_{\underline{u}}, \Xi_{\underline{u}}) = H_1(C') \oplus \mathrm{Span}\,\hat{\gamma}$.

The natural map $H_1(\hat{C}) \longrightarrow H_1(C')$ is clearly surjective, as every class in $H_1(C')$ can be represented by cycles contained in $\hat{C}$. By using this fact, we choose a (noncanonical) splitting $H_1(C') \longrightarrow H_1(\hat{C})$. An easy argument, based on the Mayer-Vietoris exact sequence associated with the decomposition $C_{\underline{u}} = \hat{C} \cup \Xi_{\underline{u}}$, shows that the map $H_1(\hat{C}) \longrightarrow H_1(C_{\underline{u}})$ is injective. Via the composition $H_1(C') \longrightarrow H_1(\hat{C}) \longrightarrow H_1(C_{\underline{u}})$, we may then identify $H_1(C')$ with a subspace of $H_1(C_{\underline{u}})$. The lack of canonicity of this identification will be harmless for what follows.

2.3.5. *Proof of Theorem* 2.3.12. *Step* 2: *construction of an adapted basis.* Let us choose a differentiable imbedding $\beta : \mathbb{I} = [0, 2r + 1] \longrightarrow \overline{\Delta}$ with the following properties:

(1) $\beta(\mathbb{I}) \cap \partial\overline{\Delta} = \{\beta(0), \beta(2r+1)\}$.

(2) $\underline{u}_i = \{\beta(d_{i-1}+1), \ldots, \beta(d_i)\}$ for $i = 1, \ldots, 2\omega + \varepsilon$, with d_i defined in (2.3.9).

(3) For every $i = 1, \ldots, 2\omega + \varepsilon$, the inverse images $\beta^{-1}(\beta(\mathbb{I}) \cap \overline{\Delta_i})$ are closed sub-intervals of $\mathbb{I}$.

As in Sections 2.3.1 and 2.3.2, β defines the cycles $\lambda_i \in H_1(\Xi_{\underline{u}}), \mu \in H_1(\Xi_{\underline{u}}, \partial\Xi_{\underline{u}})$, and the set $\{T_i\}$ of generators of $\mathscr{B}^{2r}$.

The open imbedding $\mathcal{N}_{\text{reg}} \longrightarrow \Delta_{\text{reg}}^{(2r)}$ induces the group homomorphism $\pi_1(\mathcal{N}_{\text{reg}}, \underline{u}) \longrightarrow \pi_1(\Delta_{\text{reg}}^{(2r)}, \underline{u}) = \mathscr{B}^{2r}$. It is evident from the definition of Δ_i and β that, if $d_i < j < d_{i+1}$, then T_j can be represented by a pair of curves as in (2.3.7), whose image is entirely contained in Δ_i, and is hence contained in the image of the homomorphism above, whereas this is not possible if $j \in \{d_1, \ldots, d_{2\omega+\varepsilon-1}\}$. This observation readily implies the following lemma; the missing details of the proof are left to the reader.

LEMMA 2.3.18. *The map*

$$\pi_1(\mathcal{N}_{\text{reg}}, \underline{u}) \longrightarrow \pi_1(\Delta_{\text{reg}}^{(2r)}, \underline{u}) = \mathscr{B}^{2r}$$

is injective. Its image is the subgroup $\mathscr{B}^{\mathfrak{a}}$ of $\mathscr{B}^{2r}$ generated by the elements T_j's for $j \in \{1, \ldots, 2r-1\} \setminus \{d_1, \ldots, d_{2\omega+\varepsilon-1}\}$.

Remark 2.3.19. It follows from Lemma 2.3.18 that if $\mathcal{N}' \subseteq \mathcal{N}$ is another distinguished neighborhood and $\underline{u} \in \mathcal{N}'$, then the natural map $\pi_1(\mathcal{N}'_{\text{reg}}, \underline{u}) \longrightarrow \pi_1(\mathcal{N}_{\text{reg}}, \underline{u})$ is an isomorphism and

$$\Gamma\left(\mathcal{N}_{\text{reg}}, \bigwedge^l R^1 \Phi_{\mathcal{N}_{\text{reg}} *} \mathbb{Q}\right) \longrightarrow \Gamma\left(\mathcal{N}'_{\text{reg}}, \bigwedge^l R^1 \Phi_{\mathcal{N}'_{\text{reg}} *} \mathbb{Q}\right)$$

is an isomorphism. Hence the natural maps

$$\Gamma\left(\mathcal{N}_{\text{reg}}, \bigwedge^l R^1 \Phi_{\mathcal{N}_{\text{reg}} *} \mathbb{Q}\right)$$

$$\longrightarrow \Gamma\left(\mathcal{N}'_{\text{reg}}, \bigwedge^l R^1 \Phi_{\mathcal{N}'_{\text{reg}} *} \mathbb{Q}\right) \longrightarrow \left(R^0 j_* \bigwedge^l R^1 \Phi_{\mathcal{N}_{\text{reg}} *} \mathbb{Q}\right)_\sigma$$

are isomorphisms.

In the case in which $\hat{C}$ is disconnected, the kernel of the map $H_1(\Xi_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}})$ is generated by the element $\sum_{l=1}^d \lambda_{2l-1}$; see (2.3.3). Since, by Remark 2.3.16, $\omega \geq 1$, we may use this relation to eliminate λ_{a_1} so that the set $\{\lambda_i\}$, for $i = 1, \ldots, 2r-1, i \neq a_1$, is a basis for the space $H_{\mathfrak{a},\text{disc}} = \text{Im } H_1(\Xi_{\underline{u}}) \longrightarrow H_1(C_{\underline{u}})$. This choice is suggested by Lemma 2.3.18 since $T_{a_1} \notin \mathscr{B}^{\mathfrak{a}}$. It will be evident in Section 2.3.6 that this choice is computationally quite convenient.

In force of Remarks 2.3.17 and 2.3.14 and Lemma 2.3.18, we are reduced to compute the dimension of the subspace of invariants of $\bigwedge^\bullet H_{\mathfrak{a},\mathrm{disc}}$ for the action of the group $\mathscr{B}^{\mathfrak{a}} \subseteq \mathscr{B}^{2r} \simeq \pi_1(\Delta_{\mathrm{reg}}^{(2r)}, \underline{u})$ defined in Lemma 2.3.18.

We now deal with the case in which $\hat{C}$ is connected; we resume the notation introduced at the end of Section 2.3.4. Recall, in particular, the relative cycle $\hat{\gamma}$, the identification of $H_1(C')$ with a subspace of $H_1(C_{\underline{u}}, \Xi_{\underline{u}})$, and the noncanonical one with a subspace of $H_1(C_{\underline{u}})$. We lift the relative class $\hat{\gamma}$ to a homology class $\lambda_0 \in H_1(C_{\underline{u}})$ by joining $\hat{\gamma}$ with a representative of the relative cycle μ defined by (2.3.4) in $\Xi_{\underline{u}}$. Setting $H_{\mathfrak{a},\mathrm{conn}} := H_1(\Xi_{\underline{u}}) \bigoplus \mathrm{Span}\,\lambda_0$, the decomposition

$$(2.3.23) \qquad\qquad H_1(C_{\underline{u}}) = H_{\mathfrak{a},\mathrm{conn}} \bigoplus H_1(C')$$

is $\pi_1(\Delta_{\mathrm{reg}}^{(2r)}, \underline{u}) \simeq \mathscr{B}^{2r}$-invariant. Note that, by (2.3.5), the action on λ_0 is given by

$$(2.3.24) \qquad\qquad T_i(\lambda_0) = \lambda_0 \text{ if } i \neq 1 \text{ and } T_1(\lambda_0) = \lambda_0 + \lambda_1.$$

Since, by construction, the cycles in the subspace $H_1(C')$ may be chosen to be entirely contained in $\hat{C}$, the action of $\mathscr{B}^{2r}$ on the summand $H_1(C')$ is trivial, and, by Remark 2.3.14 and Lemma 2.3.18, we are reduced to compute the dimension of the subspace of invariants of $\bigwedge^\bullet H_{\mathfrak{a},\mathrm{conn}}$ for the action of the group $\mathscr{B}^{\mathfrak{a}} \subseteq \mathscr{B}^{2r} \simeq \pi_1(\Delta_{\mathrm{reg}}^{(2r)}, \underline{u})$.

In either case, a local coordinate $\zeta : \overline{\Delta} \longrightarrow \mathbb{D}$, defined on an open set containing $\overline{\Delta}$, identifies the family $\rho_{\Delta^{(2r)}}^{-1}(\overline{\Delta} \times \Delta^{(2r)}) \longrightarrow \Delta^{(2r)}$ with the family $\Phi_{2r} : \mathscr{S}_{2r} \longrightarrow \mathbb{D}^{(2r)}$ of Section 2.3.2 and the restriction $\rho_{\underline{u}|\Xi_{\underline{u}}} : \Xi_{\underline{u}} \longrightarrow \overline{\Delta}$ with the double covering $\mathbb{S}_{\underline{v}} \longrightarrow \overline{\mathbb{D}}$, where $\underline{v} := \zeta(\underline{u})$. By (2.3.9), the action of $\mathscr{B}^{\mathfrak{a}}$ on $H_{\mathfrak{a},\mathrm{conn}}$ and $H_{\mathfrak{a},\mathrm{disc}}$ is then given by

(2.3.25)
$$T_i(\lambda_j) = \lambda_j \text{ if } |i-j| \neq 1, \quad T_i(\lambda_{i+1}) = \lambda_{i+1} - \lambda_i, \quad T_i(\lambda_{i-1}) = \lambda_{i-1} + \lambda_i.$$

2.3.6. *Proof of Theorem 2.3.12. Step* 3: *computation of monodromy invariants.* Lemma 2.3.20 and Proposition 2.3.21 below summarize the linear algebra facts which we need to complete the proof of Theorem 2.3.12.

LEMMA 2.3.20. *Let U be a vector space of even dimension $2m$ with basis $c_1, \ldots, c_{2m}$, and denote by $\bigwedge^\bullet U$ its exterior algebra. Let $T_1, \ldots, T_{2m} \in \mathrm{Aut}(U)$ be defined by*

$$(2.3.26) \quad T_i(c_j) = c_j \text{ if } |i-j| \neq 1, \quad T_i(c_{i+1}) = c_{i+1} - c_i, \quad T_i(c_{i-1}) = c_{i-1} + c_i,$$

and denote their natural extensions to $\bigwedge^\bullet U$ again by T_i. For $I \subseteq \{1, \ldots, 2m\}$, let T_I be the subgroup of $\mathrm{Aut}(\bigwedge^\bullet U)$ generated by the T_i's with $i \in I$, and denote by $(\bigwedge^\bullet U)^{T_I} \subseteq \bigwedge^\bullet U$ the subspace of T_I-invariants.

(1) *For $I = \{1, 2, \ldots, 2m\}$, we have $\dim \left(\bigwedge^\bullet U \right)^{T_I} = m + 1$.*

(2) *For $I' = \{2, 3, \ldots, 2m\}$, we have $\dim \left(\bigwedge^\bullet U \right)^{T_{I'}} = 2m + 1$.*

(3) *For $I'' = \{2, \cdots, t, t+2, \ldots, 2m\}$, with t odd, we have $\dim \left(\bigwedge^\bullet U \right)^{T_{I''}} = (t+1)(2m - t + 1)$.*

Proof. For $a, b \in \{1, \ldots, 2m\}$, with $a \leq b$ and $a \equiv b\,(2)$, we set

$$c_{[a,b]} := c_a + c_{a+2} + \cdots + c_{b-2} + c_b.$$

It immediately follows from (2.3.26) that

$$(2.3.27) \qquad T_i \left(c_{[a,b]} \right) = \begin{cases} c_{[a,b]} & \text{if } i \neq a - 1, b + 1, \\ c_{[a,b]} - c_{a-1} & \text{if } i = a - 1, \\ c_{[a,b]} + c_{b+1} & \text{if } i = b + 1. \end{cases}$$

- Case $I = \{1, \ldots, 2m\}$. A direct computation using (2.3.27) shows that

$$\Omega := \sum_{s=1}^m c_{[1, 2s-1]} \wedge c_{2s} \in \bigwedge^2 U$$

is T^I-invariant and $\Omega^m \neq 0$. Hence $1, \Omega, \Omega^2, \ldots, \Omega^m$ are the desired $m + 1$ T^I-invariants.

- Case $I' = \{2, \ldots, 2m\}$. Since $T_{I'} < T_I$, the Ω^t's introduced above are $T_{I'}$-invariant. Furthermore, since $T_1 \notin T_{I'}$, it follows from (2.3.27) that $c_{[2, 2m]} \in U^{T_{I'}}$. Then $1, c_{[2, 2m]}, \Omega, c_{[2, 2m]} \wedge \Omega, \ldots, c_{[2, 2m]} \wedge \Omega^{m-1}, \Omega^m$, give the desired $2m + 1$ $T^{I'}$-invariants, which, being nonzero and of different degrees, are linearly independent.

- Case $I'' = \{2, \ldots, t, t+2, \ldots, 2m\}$ with t odd. In addition to the $T^{I'}$-invariant $c_{[2, 2m]} \in U$ introduced above, we have $c_{[1, t]} \in U^{T_{I''}}$, again by (2.3.27), since $T_{t+1} \notin T_{I''}$.

Let U_0 be the space spanned by $c_{[2, 2m]}$ and $c_{[1, t]}$, and set

$$U_1 := \operatorname{Span}\{c_2, c_3, \ldots, c_t\} \quad \text{and} \quad U_2 := \operatorname{Span}\{c_{t+2}, c_{t+3}, \ldots, c_{2m}\}.$$

It results from (2.3.26), and again from the fact that $T_{t+1} \notin T_{I''}$, that the direct sum decomposition $U = U_0 \oplus U_1 \oplus U_2$ is $T_{I''}$-invariant.

Let G_1 be the group generated by $\{T_2, T_3, \ldots, T_t\}$, and let G_2 be the group generated by $\{T_{t+2}, T_{t+3}, \ldots, T_{2m}\}$.

Applying case (1) of this Lemma to the vector spaces U_1 and U_2 with the groups G_1, G_2 respectively, gives

$$\Omega_1^k \in \left(\bigwedge^{2k} U_1 \right)^{G_1}, \qquad \forall\, 0 \leq k \leq \frac{1}{2}(t-1)$$

and

$$\Omega_2^l \in \left(\bigwedge^{2l} U_2\right)^{G_2}, \qquad \forall \, 0 \leq l \leq \frac{1}{2}(2m - t - 1).$$

Since G_1 acts trivially on U_2 and G_2 acts trivially on U_1, the $\frac{1}{4}(t+1)$ $\cdot(2m-t+1)$ elements $\Omega_1^k \otimes \Omega_2^l \in \bigwedge^{2k} U_1 \otimes \bigwedge^{2l} U_2$ are $T_{I''}$ invariant. They are, furthermore, linearly independent since they are nonzero and live in different summands of the direct sum decomposition of $\bigwedge^\bullet (U_1 \oplus U_2)$.

From the $T_{I''}$-isomorphism

$$\bigwedge^\bullet U \simeq \left(\bigwedge^\bullet U_0\right) \otimes \left(\bigwedge^\bullet U_1\right) \otimes \left(\bigwedge^\bullet U_2\right),$$

we conclude that $(\bigwedge^\bullet U)^{T_{I''}}$ is a free $\frac{1}{4}(t+1)(2m-t+1)$-rank module over the four-dimensional $T_{I''}$-invariant algebra $\bigwedge^\bullet U_0$, hence its dimension is $(t+1)(2m-t+1)$.

In all of the three cases considered, it is not hard to verify that there is no other invariant. $\qquad \square$

Let $\mathfrak{a}$ be a partition of d, with associated integers $a_i, \omega, \varepsilon, d_i$ as in (2.3.8), (2.3.9), and let $\mathscr{B}^\mathfrak{a}$ be the group of Lemma 2.3.18. Let
(2.3.28)
$V_{\mathfrak{a},\text{disc}}$ be the $\mathbb{Q}$-vector space generated by the set $I_{\text{disc}} = \{1, \ldots, d-1\} \setminus \{a_1\}$,

and let
(2.3.29)

$V_{\mathfrak{a},\text{conn}}$ be the $\mathbb{Q}$-vector space generated by the set $I_{\text{conn}} = \{0, \ldots, d-1\}$.

In either case, denote by $\{\lambda_i\}_{i \in I}$ the corresponding basis, with $I = I_{\text{disc}}$ or $I = I_{\text{conn}}$ and endow the vector spaces and their exterior algebras $\bigwedge^\bullet V_{\mathfrak{a},\text{disc}}$ and $\bigwedge^\bullet V_{\mathfrak{a},\text{conn}}$ with the $\mathscr{B}^\mathfrak{a}$-module structure defined by (2.3.25).

PROPOSITION 2.3.21. *Let $(-)^{\mathscr{B}^\mathfrak{a}}$ denote the subspace of $\mathscr{B}^\mathfrak{a}$-invariants. We have*

(2.3.30) $$\dim \left(\bigwedge^\bullet V_{\mathfrak{a},\text{disc}}\right)^{\mathscr{B}^\mathfrak{a}} = \frac{1}{4} \prod (a_i + 1),$$

(2.3.31) $$\dim \left(\bigwedge^\bullet V_{\mathfrak{a},\text{conn}}\right)^{\mathscr{B}^\mathfrak{a}} = \prod (a_i + 1).$$

Proof. We first prove (2.3.30), starting with the case $\varepsilon = 0$. We proceed by induction on ω.

Assume $\omega = 1$. Then $\mathfrak{a} = (a_1, a_2)$ and $\mathscr{B}^\mathfrak{a} = \mathscr{B}^{a_1} \times \mathscr{B}^{a_2}$ is the group generated by $T_1, \ldots, T_{a_1-1}, T_{a_1+1}, \ldots, T_{a_1+a_2-1}$. Since $T_{a_1} \notin \mathscr{B}^\mathfrak{a}$, it follows

from (2.3.25) that the direct sum decomposition $V_{\mathfrak{a},\mathrm{disc}} = W_1 \oplus W_2$ with

$$W_1 = \mathrm{Span}\{\lambda_1, \ldots, \lambda_{a_1-1}\}, \quad W_2 = \mathrm{Span}\{\lambda_{a_1+1}, \ldots, \lambda_{a_1+a_2-1}\},$$

is $\mathscr{B}^{\mathfrak{a}}$-invariant. Since furthermore, $\mathscr{B}^{a_1}$ acts trivially on $\bigwedge^{\bullet} W_2$ and $\mathscr{B}^{a_2}$ acts trivially on $\bigwedge^{\bullet} W_1$, we have

$$\left(\overset{\bullet}{\bigwedge} V_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}}} \simeq \left(\overset{\bullet}{\bigwedge} W_1\right)^{\mathscr{B}^{\mathfrak{a}}} \otimes \left(\overset{\bullet}{\bigwedge} W_2\right)^{\mathscr{B}^{\mathfrak{a}}}$$

$$\simeq \left(\overset{\bullet}{\bigwedge} W_1\right)^{\mathscr{B}^{a_1}} \otimes \left(\overset{\bullet}{\bigwedge} W_2\right)^{\mathscr{B}^{a_2}}.$$

We now apply Lemma 2.3.20(1) twice, first setting $U = W_1$ and $2m = a_1 - 1$ and then $U = W_2$ and $2m = a_2 - 1$, to find $\dim \left(\bigwedge^{\bullet} V_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}}} = \frac{1}{4}(a_1+1)(a_2+1)$.

• Assume the statement is proved for every multiplicity type $\mathfrak{a}$ with $\varepsilon = 0$ and $\omega \leq k$, and let $\mathfrak{a}' = (\mathfrak{a}, a_{2k+1}, a_{2k+2})$, with $\mathfrak{a} := (a_1, \ldots, a_{2k})$. Set $d' := d + a_{2k+1}$ and $d'' := d + a_{2k+1} + a_{2k+2}$. We have $\mathscr{B}^{\mathfrak{a}'} \simeq \mathscr{B}^{\mathfrak{a}} \times \mathscr{B}^{\mathfrak{b}}$, where $\mathscr{B}^{\mathfrak{b}} \simeq \mathscr{B}^{a_{2k+1}} \times \mathscr{B}^{a_{2k+2}}$ is the subgroup generated by

$$T_{d+1}, \ldots, T_{d'-1}, T_{d'+1}, \ldots, T_{d''-1}.$$

The subspace $V_{\mathfrak{a},\mathrm{disc}} = \mathrm{Span}\{\lambda_i\}_{\substack{i \in \{1, \ldots, d-1\}, \\ i \neq a_1}} \subseteq V_{\mathfrak{a}'}$ is $\mathscr{B}^{\mathfrak{a}'}$-invariant, and the subgroup $\mathscr{B}^{\mathfrak{b}}$ acts trivially on it by (2.3.25). Hence

$$\left(\bigwedge V_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}'}} = \left(\bigwedge V_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}}} \quad \text{and} \quad \dim \left(\bigwedge V_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}}} = \frac{1}{4} \prod_{i=1}^{2k}(a_i+1)$$

by the inductive hypothesis. The subspace $\mathrm{Span}\{\lambda_d, \ldots, \lambda_{d''-1}\}$, however, is not $\mathscr{B}^{\mathfrak{a}'}$ invariant as $T_{d-1}(\lambda_d) = \lambda_d - \lambda_{d-1}$. We correct this by introducing $\hat{\lambda}_d := \lambda_d + \lambda_{d-2} + \cdots + \lambda_{d_{2k-1}+1}$; by (2.3.25) we have $T_j(\hat{\lambda}_d) = \hat{\lambda}_d$ if $j \neq d_{2k-1}, d+1$. Since $T_{d_{2k-1}} \notin \mathscr{B}^{\mathfrak{a}'}$, while $T_{d+1}(\hat{\lambda}_d) = \hat{\lambda}_d + \lambda_{d+1}$, the subspace

$$W := \mathrm{Span}\{\hat{\lambda}_d, \lambda_{d+1}, \ldots, \lambda_{d''-1}\}$$

is $\mathscr{B}^{\mathfrak{a}'}$-invariant. The decomposition $V_{\mathfrak{a}',\mathrm{disc}} = V_{\mathfrak{a},\mathrm{disc}} \oplus W$ is hence $\mathscr{B}^{\mathfrak{a}'}$-invariant and $\mathscr{B}^{\mathfrak{a}}$ acts trivially on W. Since $T_d \notin \mathscr{B}^{\mathfrak{a}'}$, we can apply case (3) of Lemma 2.3.20 to $U = W$ with $t = a_{2k+1}$. The statement is now proved for every $\mathfrak{a}$ such that $\varepsilon = 0$.

Case $\varepsilon > 0$. Assume the statement is proved for every $\mathfrak{a}$ with $\varepsilon \leq k$. Let $\mathfrak{a}' := (\mathfrak{a}, a_{2\omega+\varepsilon})$ and let $d' := d + a_{2\omega+\varepsilon}$. Just as in the case above, we set

$$W := \mathrm{Span}\{\hat{\lambda}_d, \lambda_{d+1}, \ldots, \lambda_{d'-1}\},$$

where $\hat{\lambda}_d := \lambda_d + \lambda_{d-2} + \cdots + \lambda_{d_{2\omega-1}+1}$, we have a $\mathscr{B}^{\mathfrak{a}'}$-invariant decomposition $V_{\mathfrak{a}',\mathrm{disc}} = V_{\mathfrak{a},\mathrm{disc}} \oplus W$ with the property that $\mathscr{B}^{\mathfrak{a}}$ acts trivially on W and $\mathscr{B}^{a_{2\omega+\varepsilon}}$ acts trivially on $V_{\mathfrak{a},\mathrm{disc}}$; we may thus apply case (2) of Lemma 2.3.20 to $U = W$ with $2m = a_{2\omega+\varepsilon}$.

The proof of (2.3.31) goes along the same lines as that of (2.3.30), so we will skip some details. We proceed by induction on $\omega + \varepsilon$.

• Assume $\varepsilon = 1$, $\omega = 0$. Then $\mathfrak{a} = (a_1)$ with a_1 even, and $V_{\mathfrak{a},\mathrm{conn}}$ is generated by $\lambda_0, \ldots, \lambda_{a_1-1}$ with the action of the group generated by $T_1, \ldots, T_{a_1-1}$. This is, up to an obvious renumbering, precisely case (2) of Lemma 2.3.20, which gives $\dim(\bigwedge^\bullet V_{\mathfrak{a},\mathrm{conn}})^{\mathscr{B}^{\mathfrak{a}}} = a_1 + 1$.

• Assume instead $\omega = 1$, $\varepsilon = 0$. Then $\mathfrak{a} = (a_1, a_2)$ with a_1, a_2 odd, $d = a_1 + a_2$, and $V_{\mathfrak{a},\mathrm{conn}}$ is generated by $\lambda_0, \ldots, \lambda_{d-1}$ with the action of the group generated by $T_1, \ldots, T_{a_1-1}, T_{a_1+1}, \ldots, T_{d-1}$. This is, up to an obvious renumbering, case (3) of Lemma 2.3.20 with $t = a_1$, $2m = a_1 + a_2$, and we obtain $\dim(\bigwedge^\bullet V_{\mathfrak{a},\mathrm{conn}})^{\mathscr{B}^{\mathfrak{a}}} = (a_1 + 1)(a_2 + 1)$.

• Assume the statement is proved for all $\mathfrak{a}$ with $\omega + \varepsilon \leq k$. Given $\mathfrak{a}'$ with $\omega + \varepsilon = k + 1$, one needs to consider two cases:

(i) $\mathfrak{a}' := (\mathfrak{a}, a)$ *with a even.* Defining $\hat{\lambda}_d := \lambda_d + \lambda_{d-2} + \cdots + \lambda_{d_0}$, we have a $\mathscr{B}^{\mathfrak{a}'}$-invariant decomposition $V_{\mathfrak{a}',\mathrm{conn}} = V_{\mathfrak{a},\mathrm{conn}} \oplus W$, with $W := \mathrm{Span}\{\hat{\lambda}_d, \lambda_{d+1}, \ldots, \lambda_{d+a-1}\}$, and we proceed as above, applying case (2) of Lemma 2.3.20 to $U = W$ with $2m = a$.

(ii) $\mathfrak{a}' := (\mathfrak{a}, a', a'')$ *with a', a'' odd.* Defining $\hat{\lambda}_d := \lambda_d + \lambda_{d-2} + \cdots + \lambda_{d_0}$, we have a $\mathscr{B}^{\mathfrak{a}'}$-invariant decomposition $V_{\mathfrak{a}',\mathrm{conn}} = V_{\mathfrak{a},\mathrm{conn}} \oplus W$, with $W := \mathrm{Span}\{\hat{\lambda}_d, \lambda_{d+1}, \ldots, \lambda_{d+a'+a''-1}\}$, and we proceed as above, applying case (3) of Lemma 2.3.20 to $U = W$ with $2m = a$. $\qquad\square$

Proof of Theorem 2.3.12. In the case in which $\hat{C}$ is disconnected, the decomposition (2.3.21) $H_1(C_{\underline{u}}) = H_{\mathfrak{a},\mathrm{disc}} \oplus H_1(C)^{\oplus 2}$, the fact that $\pi_1(\mathcal{N}_{\mathrm{reg}}, \underline{u})$ acts trivially on the $4g$-dimensional space $H_1(C)^{\oplus 2}$, the identification of $\pi_1(\mathcal{N}_{\mathrm{reg}}, \underline{u})$ with $\mathscr{B}^{\mathfrak{a}}$ acting on $H_{\mathfrak{a},\mathrm{disc}}$ as described in (2.3.25), and case (1) of Proposition 2.3.21 applied to $H_{\mathfrak{a},\mathrm{disc}}$ imply that

$$\left(\bigwedge^\bullet H_1(C_{\underline{u}})\right)^{\pi_1(\mathcal{N}_{\mathrm{reg}},\underline{u})} = \left(\bigwedge^\bullet H_{\mathfrak{a},\mathrm{disc}}\right)^{\mathscr{B}^{\mathfrak{a}}} \otimes \bigwedge^\bullet \left(H_1(C)^{\oplus 2}\right)$$

and

$$\dim\left(\bigwedge^\bullet H_1(C_{\underline{u}})\right)^{\pi_1(\mathcal{N}_{\mathrm{reg}},\underline{u})} = 2^{4g}\left(\frac{1}{4}\prod(a_i+1)\right) = 2^{4g-2}\prod(a_i+1)$$

In a completely analogous way, in the case in which $\hat{C}$ is connected, the decomposition (2.3.23) $H_1(C_{\underline{u}}) = H_{\mathfrak{a},\mathrm{conn}} \oplus H_1(C')$, the fact that $\pi_1(\mathcal{N}_{\mathrm{reg}}, \underline{u})$ acts trivially on the $4g - 2$-dimensional space $H_1(C')$, the identification of $\pi_1(\mathcal{N}_{\mathrm{reg}}, \underline{u})$ with $\mathscr{B}^{\mathfrak{a}}$ acting on $H_{\mathfrak{a},\mathrm{conn}}$ as described in (2.3.25), and case (2) of

Proposition 2.3.21 applied to $H_{\mathfrak{a},\mathrm{conn}}$, imply that

$$\left(\bigwedge^{\bullet} H_1(C_{\underline{u}})\right)^{\pi_1(\mathcal{N}_{\mathrm{reg}},\underline{u})} = \left(\bigwedge^{\bullet} H_{\mathfrak{a},\mathrm{conn}}\right)^{\mathcal{B}^{\mathfrak{a}}} \otimes \left(\bigwedge^{\bullet} H_1(C')\right)$$

and

$$\dim\left(\bigwedge^{\bullet} H_1(C_{\underline{u}})\right)^{\pi_1(\mathcal{N}_{\mathrm{reg}},\underline{u})} = 2^{4g-2}\prod(a_i+1). \qquad \square$$

2.3.7. *Proof of Theorem 2.3.1: Back to the spectral curve.* We resume the notation of the statement of Theorem 2.3.1. Let $\mathcal{N}$ be a distinguished neighborhood of $\Theta(s) \in C^{(2d)}$. For a small enough neighborhood N of $s \in \mathcal{A}_{\mathrm{ell}}$, we have $\Theta(N) \subseteq \mathcal{N}$ and $\Theta(N \cap \mathcal{A}_{\mathrm{reg}}) \subseteq \mathcal{N}_{\mathrm{reg}}$. Let $\Phi_{\mathcal{N}}$ be the family constructed in Proposition 2.3.11 associated with the choice of the square root $\mathcal{O}_C(D)$ of $\mathcal{O}_C(\Theta(s)) \simeq \mathcal{O}_C(2D)$. By point (4) in Proposition 2.3.11, the restriction of the spectral curve family to $N \cap \mathcal{A}_{\mathrm{reg}}$ is the pullback via Θ of $\Phi_{\mathcal{N}_{\mathrm{reg}}}$. Hence, by Corollary 1.3.6 and by the base change theorem for proper maps, we have the isomorphisms of local systems on $N \cap \mathcal{A}_{\mathrm{reg}}$:

$$R^l\chi_{\mathrm{reg}*}\mathbb{Q} \simeq \Theta^*\left(\bigwedge^l R^1\Phi_{\mathcal{N}_{\mathrm{reg}}*}\mathbb{Q}\right).$$

As the stalk $(R^0 j_* R^l\chi_{\mathrm{reg}*}\mathbb{Q})_s$ is the direct limit over the set of neighborhoods N of s in $\mathcal{A}$ of the space of monodromy invariants of the local system $R^l\chi_{\mathrm{reg}*}\mathbb{Q}$ in $N \cap \mathcal{A}_{\mathrm{reg}}$, the statement follows from Theorem 2.3.12 and Remark 2.3.13. $\qquad \square$

As shown just after the statement of Theorem 2.3.1, we have also completed the proof of Theorem 2.1.4. This latter result has the following consequence.

COROLLARY 2.3.22. *Let $s \in \mathcal{A}_{\mathrm{ell}}$, and let $C_s, \widetilde{g}, O, E$ be as in the statement of Theorem 2.3.1. Then*

(1) *The spectral sequence* (2.2.5) *degenerates at E_1.*
(2) *The Poincaré polynomial of the fiber $\chi^{-1}(s) \simeq \overline{\mathrm{Pic}}_{C_s}^0$ is*

$$\sum_l t^l b_l(\chi^{-1}(s)) = (1+t)^{2\widetilde{g}}\left(\prod_{c\in O}\left(1+t+\ldots+t^{2\delta_c}\right)\right)\left(\prod_{c\in E}\left(1+t^2+\ldots+t^{\delta_c}\right)\right). \tag{2.3.32}$$

Proof. It follows from Theorems 2.2.7 and 2.3.1 that

$$\sum_{p,q}\dim E_{\infty}^{p,q} = \sum_l b_l(\chi^{-1}(s))$$

$$= 2^{2\widetilde{g}}\left(\prod_{c\in O}(2\delta_c+1)\right)\left(\prod_{c\in E}(\delta_c+1)\right) = \sum_{p,q}\dim E_1^{p,q};$$

hence all the differentials in the spectral sequence (2.2.5) are forced to vanish, proving point (1). Point (2) follows immediately by the equality

$$(R^0 j_* R^l \chi_{\mathrm{reg}_*} \mathbb{Q})_s = b_l(\overline{\mathrm{Pic}_{C_s}})$$

of Theorem 2.1.4, keeping track of the cohomological degrees of the monodromy invariants.   □

2.4. *The cases* SL_2 *and* PGL_2. We now extend the main result found for χ, Corollary 2.1.5, to $\hat{\chi}$. In order to do this, we need to discuss how $\mathcal{M}$, $\hat{\mathcal{M}}$, $\check{\mathcal{M}}$ and the relative maps χ, $\hat{\chi}$, $\check{\chi}$ introduced in Section 1.3.1 are related. As in Section 1.2.2, we denote by $\Gamma := \mathrm{Pic}^0_C[2] \simeq \mathbb{Z}_2^{2g}$ the group of points of order two in Pic^0_C, by $\mathcal{M}^0 \subseteq \mathcal{M}$ be the subset of stable Higgs bundles with traceless Higgs field

$$\mathcal{M}^0 = \{(E, \phi) \text{ with } \mathrm{tr}(\phi) = 0\}.$$

We denote by $\chi^0 : \mathcal{M}^0 \longrightarrow \mathcal{A}^0$ the restriction of the Hitchin map, where, we recall, $\mathcal{A}^0 := H^0(C, 2D) \subseteq \mathcal{A}$. There are natural maps, easily seen to be isomorphisms of algebraic varieties:

$$sq : H^0(C, D) \times \mathcal{M}^0 \longrightarrow \mathcal{M}, \qquad sq' : H^0(C, D) \times \mathcal{A}^0 \longrightarrow H^0(C, D) \times \mathcal{A}^0 = \mathcal{A}$$

given by

$$sq : ((E, \phi), s) \longmapsto \left(E, \phi + \frac{s}{2} \otimes 1_E\right), \qquad sq' : (v, u) \longmapsto \left(v, u + \frac{v^{\otimes 2}}{4}\right),$$

making the following into a commutative diagram:

$$(2.4.1) \qquad \begin{array}{ccccc} \mathcal{M}^0 & \longleftarrow & H^0(C, D) \times \mathcal{M}^0 & \xrightarrow{\ sq\ }_{\simeq} & \mathcal{M} \\ \downarrow{\scriptstyle \chi^0} & & \downarrow{\scriptstyle \mathrm{Id} \times \chi^0} & & \downarrow{\scriptstyle \chi} \\ \mathcal{A}^0 & \xleftarrow{\ p\ } & H^0(C, D) \times \mathcal{A}^0 & \xrightarrow{\ sq'\ }_{\simeq} & \mathcal{A}. \end{array}$$

Remark 2.4.1. It follows from the commutative diagram (2.4.1) that we have an isomorphism

$$\chi_* \mathbb{Q}_{\mathcal{M}} \simeq sq'_* p^* \chi^0_* \mathbb{Q}_{\mathcal{M}^0}.$$

Remark 2.4.2. We clearly have (see Proposition 1.3.2)

$$\mathcal{A}^0_{\mathrm{reg}} = \mathcal{A}_{\mathrm{reg}} \cap \mathcal{A}^0 = \{s \in H^0(C, 2D) \,|\, s \text{ has simple zeros}\}$$

and

$$sq'^{-1}(\mathcal{A}_{\mathrm{reg}}) = H^0(C, D) \times \mathcal{A}^0_{\mathrm{reg}}, \qquad sq^{-1}(\mathcal{M}_{\mathrm{reg}}) = H^0(C, D) \times \mathcal{M}^0_{\mathrm{reg}}.$$

Similarly,

$$\mathcal{A}^0_{\mathrm{ell}} = \mathcal{A}_{\mathrm{ell}} \cap \mathcal{A}^0 = \{s \in H^0(C, 2D) |\, C_s \text{ is integral}\}$$

and

$$sq'^{-1}(\mathcal{A}_{\mathrm{ell}}) = H^0(C, D) \times \mathcal{A}^0_{\mathrm{ell}}, \qquad sq^{-1}(\mathcal{M}_{\mathrm{ell}}) = H^0(C, D) \times \chi^{0^{-1}}(\mathcal{A}^0_{\mathrm{ell}}).$$

Diagram (2.4.1) and Remark 2.4.2 reduce the study of $\chi : \mathcal{M} \longrightarrow \mathcal{A}$ to that of $\chi^0 : \mathcal{M}^0 \longrightarrow \mathcal{A}^0$. We can safely identify $sq'^*\chi_*\mathbb{Q}_\mathcal{M}$ with $p^*\chi^0_*\mathbb{Q}_{\mathcal{M}^0}$, and the main Theorem 2.1.4 and its corollary 2.1.5 hold for χ^0 on $\mathcal{A}^0_\text{ell}$; namely, if $j : \mathcal{A}^0_\text{reg} \longrightarrow \mathcal{A}^0_\text{ell}$ is the open imbedding, then

$$(2.4.2) \qquad IC\left(R^l\chi^0_{\text{reg}*}\mathbb{Q}_{\mathcal{M}^0_\text{reg}}\right)_{|\mathcal{A}^0_\text{ell}} \simeq \left(R^0j_*R^l\chi^0_{\text{reg}*}\mathbb{Q}_{\mathcal{M}^0_\text{reg}}\right)[\dim \mathcal{A}^0],$$
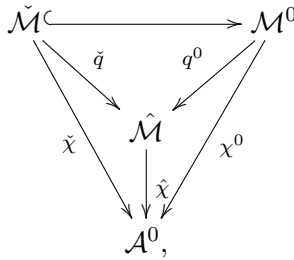
and

$$\chi^0_*\mathbb{Q}_{\mathcal{M}^0|\mathcal{A}^0_\text{ell}} \simeq \bigoplus_l \left(R^0j_*R^l\chi^0_{\text{reg}*}\mathbb{Q}_{\mathcal{M}^0_\text{reg}}\right)[-l].$$

For the other groups SL_2 and PGL_2, we have

$$\check{\mathcal{M}} = \lambda_D^{-1}((\Lambda,0)), \qquad \hat{\mathcal{M}} = \check{\mathcal{M}}/\Gamma = \mathcal{M}^0/\text{Pic}^0_C,$$

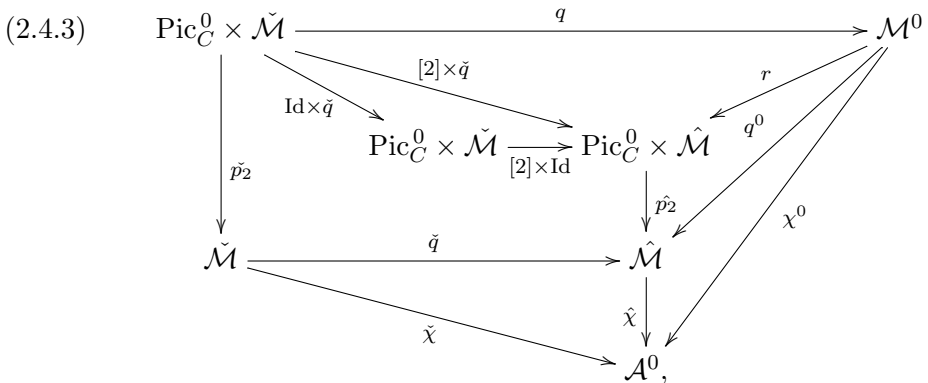and the corresponding Hitchin maps

where $\check{q}$ and q^0 are the two quotient maps.

The following is readily verified.

PROPOSITION 2.4.3. *The map*

$$q : \text{Pic}^0_C \times \check{\mathcal{M}} \longrightarrow \mathcal{M}^0, \qquad (L,(E,\phi)) \longmapsto (E \otimes L, \phi \otimes 1_L).$$

is an unramified Galois covering with group Γ, and there is a commutative diagram

where $r : \mathcal{M}^0 \longrightarrow \text{Pic}^0_C \times \hat{\mathcal{M}}$ sends (E,ϕ) to $(\det E \otimes \Lambda^{-1}, q^0(E,\phi))$ and $[2] \times \check{q}$ sends $(L,(E,\phi))$ to $(L^{\otimes 2}, \check{q}(E,\phi))$.

Remark 2.4.4. The map $[2] \times \check{q} : \text{Pic}_C^0 \times \check{\mathcal{M}} \longrightarrow \text{Pic}_C^0 \times \hat{\mathcal{M}}$ is the quotient map relative to the diagonal action of $\Gamma \times \Gamma$ on $\text{Pic}_C^0 \times \check{\mathcal{M}}$.

Proposition 2.4.3 implies that

$$H^*(\mathcal{M}) \simeq H^*(\mathcal{M}^0) \simeq \left(H^*(\check{\mathcal{M}}) \otimes H^*(\text{Pic}_C^0)\right)^\Gamma \simeq H^*(\hat{\mathcal{M}}) \otimes H^*(\text{Pic}_C^0).$$

The last isomorphism follows from the fact that the action of Γ on $H^*(\text{Pic}_C^0)$ is trivial, as it is the restriction to a subgroup of the action of the connected group Pic_C^0.

Before stating Theorem 2.4.5, which gives a refinement of the isomorphism above at the level of derived categories, we make some general remarks on actions of finite abelian groups and the splitting they induce on complexes.

For ease of exposition, until further notice, we work with the constructible derived category of sheaves of *complex* vector spaces. Let K be an object of D_A and suppose that a finite abelian group Γ acts on the right on K, i.e., that we are given a representation $\Gamma \to (\text{Aut}_{D_A}(K))^{op}$. It then follows from [19, 2.24] (see also [39, Lemma 3.2.5]) that there is a character decomposition

$$(2.4.4) \qquad\qquad\qquad K \simeq \bigoplus_{\zeta \in \hat{\Gamma}} K_\zeta,$$

where $\hat{\Gamma}$ denotes the group of characters of Γ.

Suppose Γ acts on the left on an algebraic variety M, and let $M \xrightarrow{q} M/\Gamma$ be the quotient map. Since q is finite, the derived direct image complex $q_* \mathbb{C}_M$ is a sheaf. Clearly, Γ acts on $q_* \mathbb{C}_M$ on the right via pull-backs, and (2.4.4) above boils down to the canonical decomposition in the Abelian category of sheaves

$$\bigoplus_{\zeta \in \hat{\Gamma}} L_\zeta \simeq q_* \mathbb{C}_M.$$

Let $h : M \longrightarrow A$ be a proper map which is Γ-equivariant. We have the commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{\ q\ } & M/\Gamma \\
{\scriptstyle h}\downarrow & \swarrow {\scriptstyle h'} & \\
A & &
\end{array}$$

and, by using $h'_* q_* = h_*$, we get the canonical identification.

$$(2.4.5) \qquad\qquad\qquad \bigoplus_{\zeta \in \hat{\Gamma}} h'_* L_\zeta \simeq h_* \mathbb{C}_M.$$

Clearly, $h_* \mathbb{C}_M$ is endowed with the Γ-action, and (2.4.5) is just its character decomposition, namely $(h_* \mathbb{C}_M)_\zeta = h'_* L_\zeta$.

In particular, taking the trivial representation $\rho = 1$, we have $L_1 = (q_* \mathbb{C}_M)^\Gamma \simeq \mathbb{C}_{M/\Gamma}$, and we thus identify the direct image $h'_* \mathbb{C}_{M/\Gamma} = h'_*((q_* \mathbb{C}_M)^\Gamma)$

with the canonical direct summand (which we may call the Γ-invariant part) $(h_*\mathbb{C}_M)^\Gamma := (h_*\mathbb{C}_M)_1$ of $h_*\mathbb{C}_M$.

Let us go back to our situation where $\Gamma \simeq \mathbb{Z}_2^{2g}$. In this case, the characters are all $\{\pm 1\}$-valued, and we can safely return to rational coefficients.

From the discussion above and the diagram (2.4.3), it follows that

$$(\check{q} \circ \tilde{p}_2)_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}} = (q^0 \circ q)_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}}$$

contains $q_*^0 \mathbb{Q}_{\mathcal{M}^0}$ and $(\check{q} \circ \tilde{p}_2)_* \mathbb{Q}_{\check{\mathcal{M}}}$.

THEOREM 2.4.5. *There are canonical isomorphisms in $D_{\mathcal{A}^0}$:*

$$(2.4.6) \quad \chi_*^0 \mathbb{Q}_{\mathcal{M}^0} \simeq \bigoplus_{i \in \mathbb{N}} \overset{i}{\bigwedge} H^1(C) \otimes \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}}[-i] \simeq \bigoplus_{i \in \mathbb{N}} \overset{i}{\bigwedge} H^1(C) \otimes (\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})^\Gamma [-i].$$

Proof. Consider the diagram (2.4.3). As noticed in Remark 2.4.4, the map $[2] \times \check{q}$ is the quotient by the action of $\Gamma \times \Gamma$. Consider the character decompositions

$$[2]_* \mathbb{Q}_{\mathrm{Pic}_C^0} \simeq \bigoplus_{\zeta \in \hat{\Gamma}} L_\zeta, \qquad \check{q}_* \mathbb{Q}_{\check{\mathcal{M}}} \simeq \bigoplus_{\zeta \in \hat{\Gamma}} M_\zeta.$$

The Künneth formula gives the following canonical isomorphisms in $D_{\mathrm{Pic}_C^0 \times \hat{\mathcal{M}}}$:

$$(2.4.7) \qquad ([2] \times \check{q})_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}} \simeq [2]_* \mathbb{Q}_{\mathrm{Pic}_C^0} \boxtimes \check{q}_* \mathbb{Q}_{\check{\mathcal{M}}} \simeq \bigoplus_{(\zeta, \zeta') \in \hat{\Gamma} \times \hat{\Gamma}} L_\zeta \boxtimes M_{\zeta'},$$

$$(2.4.8) \qquad ([2] \times \mathrm{Id})_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \hat{\mathcal{M}}} \simeq \bigoplus_{\zeta \in \hat{\Gamma}} L_\zeta \boxtimes \mathbb{Q}_{\hat{\mathcal{M}}} \subseteq ([2] \times \check{q})_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}},$$

and

$$(2.4.9) \qquad r_* \mathbb{Q}_{\mathcal{M}^0} \simeq \bigoplus_{\zeta \in \hat{\Gamma}} L_\zeta \boxtimes M_\zeta \subseteq ([2] \times \check{q})_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}},$$

this latter since q is the quotient by the diagonal action $\Gamma \longrightarrow \Gamma \times \Gamma$.

Noting that the map $[2]$ is a finite covering of a product of circles, we have canonical isomorphisms

$$(2.4.10) \qquad (\hat{p}_2)_* L_\zeta = \left(\bigoplus_i H^i(\mathrm{Pic}_C^0, L_\zeta)[-i] \right) \otimes \mathbb{Q}_{\hat{\mathcal{M}}}$$

$$= \begin{cases} 0 & \text{if } \zeta \neq 1, \\ \bigoplus_i \overset{i}{\bigwedge} H^1(C) \otimes \mathbb{Q}_{\hat{\mathcal{M}}}[-i] & \text{if } \zeta = 1. \end{cases}$$

Applying the functor $\hat{p}_{2*}$ to $([2] \times \check{q}))_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}}$ to $([2] \times \mathrm{Id}))_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \hat{\mathcal{M}}}$ and to $r_* \mathbb{Q}_{\mathcal{M}^0}$, we obtain canonical isomorphisms in $D_{\hat{\mathcal{M}}}$:

$$(2.4.11) \qquad (\hat{p}_2 \circ ([2] \times \check{q}))_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \check{\mathcal{M}}} \simeq \bigoplus_{i \in \mathbb{N}, \zeta \in \hat{\Gamma}} \overset{i}{\bigwedge} H^1(C) \otimes M_\zeta[-i],$$

$$(2.4.12) \qquad (\hat{p}_2 \circ ([2] \times \mathrm{Id}))_* \mathbb{Q}_{\mathrm{Pic}_C^0 \times \hat{\mathcal{M}}} \simeq \bigoplus_{i \in \mathbb{N}} \overset{i}{\bigwedge} H^1(C) \otimes \mathbb{Q}_{\hat{\mathcal{M}}}[-i],$$

$$(2.4.13) \qquad q_*^0 \mathbb{Q}_{\mathcal{M}^0} = (\hat{p}_2 \circ r)_* \mathbb{Q}_{\mathcal{M}^0} \simeq \bigoplus_{i \in \mathbb{N}} \overset{i}{\bigwedge} H^1(C) \otimes \mathbb{Q}_{\hat{\mathcal{M}}}[-i].$$

Taking the direct image $\hat{\chi}_*$ of the isomorphisms (2.4.11), (2.4.12), and (2.4.13), and using the fact that $\check{q}_* \mathbb{Q}_{\check{\mathcal{M}}} \simeq \bigoplus_{\zeta \in \hat{\Gamma}} M_\zeta$ with $M_1 \simeq \mathbb{Q}_{\hat{\mathcal{M}}}$, we find the canonical isomorphisms in $D_{\mathcal{A}^0}$ we are seeking for. □

The map $\hat{\chi}$ is projective, and it will be shown in Corollary 5.1.3 that the class α, defined by equation 1.2.10, is the cohomology class of a $\hat{\chi}$-ample line bundle on $\hat{\mathcal{M}}$. We can thus apply the results of Section 1.4.3. We have the Deligne isomorphims that depend on α,

$$\phi_\alpha : \bigoplus_{p \geq 0} \hat{\mathcal{P}}^p[-p] \xrightarrow{\simeq} \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}}[\dim \hat{\mathcal{M}}],$$

underlying an even finer decomposition (see (1.4.11)). The cohomology groups $H^*(\hat{\mathcal{M}})$ are endowed with the direct sum decomposition (1.4.14): $H^*(\hat{\mathcal{M}}) = \sum Q^{i,j}$.

Similarly the class

$$\tilde{\alpha} := \alpha \otimes 1 + 1 \otimes \sum_i \varepsilon_i \varepsilon_{i+g} \in H^*(\hat{\mathcal{M}}) \otimes H^*(\mathrm{Pic}_C) = H^*(\mathcal{M}),$$

introduced in (1.2.6), is the class of a relatively ample line bundle on $\mathcal{M}$.

We now determine the Deligne splitting $\phi_{\tilde{\alpha}}$ associated with χ and $\tilde{\alpha}$. Denote by $S_{\mathrm{Pic}_C^0} : \mathrm{Pic}_C^0 \longrightarrow pt$ the "structural map." By using the canonical splitting in D_{pt}

$$\phi_J : S_{\mathrm{Pic}_C^0 *} \mathbb{Q}_{\mathrm{Pic}_C^0} \xrightarrow{\simeq} \left(\bigoplus_{i \geq 0} \overset{i}{\bigwedge} H^1(C)[-i] \right),$$

the canonical isomorphism of Theorem 2.4.5 takes the form

$$\chi_*^0 \mathbb{Q}_{\mathcal{M}^0} \simeq \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}} \boxtimes S_{\mathrm{Pic}_C^0 *} \mathbb{Q}_{\mathrm{Pic}_C^0} \simeq \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}} \boxtimes \left(\bigoplus_{i \in \mathbb{N}} \overset{i}{\bigwedge} H^1(C)[-i] \right).$$

Note that this splitting does not depend on the choice of $\sum_i \varepsilon_i \varepsilon_{i+g}$ and that the operation of cupping with this class is diagonal (with respect to the splitting).

Let

$$\phi_{\tilde{\alpha}}^0 : \bigoplus_{p\geq 0} \mathcal{P}^p[-p] \xrightarrow{\simeq} \chi_*^0 \mathbb{Q}_{\mathcal{M}^0}[\dim \mathcal{A}^0]$$

be the Deligne splitting for χ^0 relative to (the restriction of) $\tilde{\alpha}$. Since, as pointed out above, the action of $\sum_i \varepsilon_i \varepsilon_{i+g}$ is diagonal,

$$\phi_\alpha \otimes \phi_J : \left(\bigoplus_{p\geq 0} \hat{\mathcal{P}}^p[-p] \right) \otimes \left(\bigoplus_{i\geq 0} \bigwedge^i H^1(C)[i] \right)$$

$$\longrightarrow \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}}[\dim \mathcal{A}^0] \otimes S_{\mathrm{Pic}_C^0 *} \mathbb{Q}_{\mathrm{Pic}_C^0} \simeq \chi_*^0 \mathbb{Q}_{\mathcal{M}^0}[\dim \mathcal{A}^0]$$

satisfies the properties stated in Fact 1.4.2; hence it is the isomorphism $\phi_{\tilde{\alpha}}^0$. Since, by Remark 2.4.1, we have the natural isomorphism $\chi_* \mathbb{Q}_{\mathcal{M}} \simeq sq'_* p^* \chi_*^0 \mathbb{Q}_{\mathcal{M}^0}$, and p is smooth with contractible fibres of dimension $\dim \mathcal{A} - \dim \mathcal{A}^0$, the isomorphism

$$sq'_* p^*(\phi_\alpha \otimes \phi_J) = sq'_* p^*(\phi_{\tilde{\alpha}}^0) : \chi_* \mathbb{Q}_{\mathcal{M}}[\dim \mathcal{A}^0]$$

$$\longrightarrow \bigoplus_{r\geq 0} sq'_* p^* \left(\bigoplus_{p+i=r} \hat{\mathcal{P}}^p \otimes \bigwedge^i H^1(C) \right)[-r]$$

is, up to the shift $\dim \mathcal{A}^0 - \dim \mathcal{A}$, the Deligne isomorphism $\phi_{\tilde{\alpha}}$ associated with χ and $\tilde{\alpha}$.

In particular, we have, for all k and p, a canonical isomorphism:

$$(2.4.14) \qquad H_{\leq p}^k(\mathcal{M}) = \bigoplus_{j\geq 0} \left(H_{\leq p-j}^{k-j}(\hat{\mathcal{M}}) \otimes \bigwedge^j H^1(C) \right).$$

The isomorphism of Theorem 2.4.5 can be understood geometrically as follows.

Given the nonsingular double (branched) cover $C_s \longrightarrow C$, the Prym variety $\mathrm{Prym}(C_s) \subseteq \mathrm{Pic}_{C_s}^0$ is defined as

$$(2.4.15) \qquad \mathrm{Prym}(C_s) := \{ \mathcal{F} \in \mathrm{Pic}_{C_s}^0 \mid \mathrm{Nm}(\mathcal{F}) = \mathcal{O}_C \},$$

where $\mathrm{Nm} : \mathrm{Pic}_{C_s}^0 \longrightarrow \mathrm{Pic}_C^0$ is the norm map; see [3, App. B, §1]. Clearly, the image by pullback of the subgroup Γ is contained in $\mathrm{Prym}(C_s)$, and we have the quotient isogeny

$$\mathrm{Prym}(C_s) \longrightarrow \mathrm{Prym}(C_s)/\Gamma.$$

The open subset $\check{\mathcal{M}}_{\mathrm{reg}} := \check{\mathcal{M}} \cap \mathcal{M}_{\mathrm{reg}}$ is a torsor for the Abelian scheme $\mathscr{P}_{\mathrm{reg}}^-$ over $\mathcal{A}_{\mathrm{reg}}^0$ whose fibre over $s \in \mathcal{A}_{\mathrm{reg}}^0$ is $\mathrm{Prym}(C_s)$. Similarly, the open subset $\hat{\mathcal{M}}_{\mathrm{reg}} := \check{\mathcal{M}}_{\mathrm{reg}}/\Gamma$ is a torsor for the Abelian scheme $\mathscr{P}_{\mathrm{reg}}^-/\Gamma$ over $\mathcal{A}_{\mathrm{reg}}^0$ whose fibre over $s \in \mathcal{A}_{\mathrm{reg}}^0$ is $\mathrm{Prym}(C_s)/\Gamma \simeq \mathrm{Prym}(C_s)^\vee$.

The involution ι (see §1.3.2) on the family of spectral curves $u : \mathscr{C}_\mathcal{A} \to \mathcal{A}$ gives a $\mathbb{Z}/2\mathbb{Z}$-action on the local system $R^1 u_{\mathrm{reg}*} \mathbb{Q}_{\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}}}$ and a corresponding decomposition into (± 1)- eigenspaces $\mathcal{V}^\pm$. The pullback from C gives a canonical

isomorphism of local systems

$$\mathcal{V}^+ = H^1(C) \otimes \mathbb{Q}_{\mathcal{A}_{\mathrm{reg}}},$$

between the local system of invariants and the constant sheaf with stalk $H^1(C)$, so that

$$R^1 \chi_{\mathrm{reg}*} \mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}} \simeq R^1 u_{\mathrm{reg}*} \mathbb{Q}_{\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}}} = \left(H^1(C) \otimes \mathbb{Q}_{\mathcal{A}_{\mathrm{reg}}} \right) \bigoplus \mathcal{V}^-.$$

It follows from Corollary 1.3.6 that, for every l,

$$(2.4.16) \qquad R^l \chi_{\mathrm{reg}*} \mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}} \simeq \bigwedge^l R^1 u_{\mathrm{reg}*} \mathbb{Q}_{\mathscr{C}_{\mathcal{A}_{\mathrm{reg}}}} = \bigoplus_{a+b=l} \bigwedge^a H^1(C) \otimes \bigwedge^b \mathcal{V}^-.$$

Clearly, the analogous statement for the restriction to $\mathcal{A}_{\mathrm{reg}}^0$ holds true. Comparing with Theorem 2.4.5, we see that

$$(2.4.17) \qquad R^l \hat{\chi}_{\mathrm{reg}*} \mathbb{Q}_{\hat{\mathcal{M}}} \simeq \bigwedge^l \mathcal{V}^- \quad \text{and} \quad \hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}|\mathcal{A}_{\mathrm{reg}}^0} \simeq \bigoplus_l \bigwedge^l \mathcal{V}^-[-l].$$

From (2.4.16), we have

$$IC\left(R^l \chi_{\mathrm{reg}*}^0 \mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}} \right) \simeq \bigoplus_{a+b=l} IC\left(\bigwedge^a H^1(C) \otimes \bigwedge^b \mathcal{V}^- \right)$$

$$\simeq \bigoplus_{a+b=l} \bigwedge^a H^1(C) \otimes IC\left(\bigwedge^b \mathcal{V}^- \right)$$

while, from the first equality in (2.4.2),

$$IC\left(R^l \chi_{\mathrm{reg}*}^0 \mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}^0} \right)_{|\mathcal{A}_{\mathrm{ell}}^0} \simeq \left(R^0 j_* R^l \chi_{\mathrm{reg}*}^0 \mathbb{Q}_{\mathcal{M}_{\mathrm{reg}}^0} \right) [\dim \mathcal{A}^0]$$

$$\simeq \bigoplus_{a+b=l} \bigwedge^a H^1(C) \otimes R^0 j_* \bigwedge^b \mathcal{V}^-.$$

Comparing with the second equality of in (2.4.2) and Theorem 2.4.5, we finally obtain

COROLLARY 2.4.6. *Let* $j : \mathcal{A}_{\mathrm{reg}}^0 \hookrightarrow \mathcal{A}_{\mathrm{ell}}^0$ *be the open imbedding. Over the locus* $\mathcal{A}_{\mathrm{ell}}^0$, *we have*

$$IC\left(\bigwedge^l \mathcal{V}^- \right)_{|\mathcal{A}_{\mathrm{ell}}^0} \simeq \left(R^0 j_* \bigwedge^l \mathcal{V}^- \right) [\dim \mathcal{A}^0]$$

and

$$\left(\hat{\chi}_* \mathbb{Q}_{\hat{\mathcal{M}}} \right)_{|\mathcal{A}_{\mathrm{ell}}^0} \simeq \bigoplus_l \left(R^0 j_* \bigwedge^l \mathcal{V}^- \right) [-l].$$

In particular, over the locus $\mathcal{A}_{\mathrm{ell}}^0$, *the map* $\hat{\chi}$ *satisfies* Assumption 1.4.5.

3. **Preparatory results**

3.1. *Placing the generators in the right perversity.* Here we prove the following theorem.

THEOREM 3.1.1. *We have*

$$\varepsilon_i \in H^1_{\leq 1}(\mathcal{M}), \ \ \forall\, 1 \leq i \leq 2g.$$

In each of the three cases $\mathrm{GL}_2, \mathrm{SL}_2,$ *and* $\mathrm{PGL}_2,$ *we have*

$$\alpha \in H^2_{\leq 2}(\mathcal{M}), \ \psi_i \in H^3_{\leq 2}(\mathcal{M}), \ \ \forall\, 1 \leq i \leq 2g.$$

Furthermore, if $g > 2$, *or* $g \geq 2$ *and* $\deg D > 2g - 2$,

$$\beta \in H^4_{\leq 2}(\mathcal{M}).$$

Proof. By the isomorphism (2.4.14), it is enough to work in the case of GL_2. Recalling that $\deg \varepsilon_i = 1$, $\deg \alpha = 2$, $\deg \psi_i = 3$, and $\deg \beta = 4$, Proposition 1.4.12 implies that

$$\varepsilon_i \in H^1_{\leq 1}(\mathcal{M}), \ \alpha \in H^2_{\leq 2}(\mathcal{M}), \psi_i \in H^3_{\leq 3}(\mathcal{M}), \ \text{and}\ \beta \in H^4_{\leq 4}(\mathcal{M}).$$

By Thaddeus' Proposition 5.1.2 in the appendix, we have that α does not vanish over the general fiber, while ψ_i and β do. Theorem 1.4.8 implies $\alpha \in H^2_{\leq 2}(\mathcal{M}), \psi_i \in H^3_{\leq 2}(\mathcal{M})$ and $\beta \in H^4_{\leq 3}(\mathcal{M})$. The same proposition shows that in order to conclude that $\beta \in H^4_{\leq 2}(\mathcal{M})$, we need to prove that β vanishes over a generic line $\Lambda \subseteq \mathcal{A}$.

Set, for simplicity of notation, $M_\Lambda := \chi^{-1}(\Lambda), M_{\Lambda_{\mathrm{reg}}} := \chi^{-1}(\Lambda_{\mathrm{reg}})$, where $\Lambda_{\mathrm{reg}} := \Lambda \cap \mathcal{A}_{\mathrm{reg}}$. Since, by Lemma 2.1.3, the generic line avoids $\mathcal{A} \setminus \mathcal{A}_{\mathrm{ell}}$ unless $g = 2$ and $D = K_C$, we have that Assumption 1.4.5 holds for $\chi_{|M_\Lambda} : M_\Lambda \to \Lambda$ due to Corollary 2.1.5. We thus have Fact 1.4.6. Let $j : \Lambda_{\mathrm{reg}} \to \Lambda$ be the open immersion.

Since Λ and Λ_{reg} are affine and one dimensional, their cohomology groups in degree ≥ 2 with coefficients in constructible sheaves are zero. In particular, the Leray spectral sequences for $\chi_{|M_\Lambda}$ and $\chi_{|M_{\Lambda_{\mathrm{reg}}}}$ are necessarily E_2-degenerate. The restriction map in cohomology yields a map of Leray spectral sequences and thus a commutative diagram of short exact "edge" sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(j_* R^3) & \longrightarrow & H^4(M_\Lambda) & \longrightarrow & H^0(j_* R^4) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & H^1(R^3) & \longrightarrow & H^4(M_{\Lambda_{\mathrm{reg}}}) & \longrightarrow & H^0(R^4) & \longrightarrow & 0.
\end{array}
$$

(As in §1.4.4, R^l stands for the local system $R^l \chi_{\mathrm{reg}*}\mathbb{Q}$.) The arrow r' is, in turn, arising from the edge sequence of the Leray spectral sequence for the map j and is thus injective.

Below, we prove that the restriction of the class $\beta \in H^4(\mathcal{M})$ to $\mathcal{M}_{\mathrm{reg}}$ vanishes. The sought-after conclusion $\beta_{|M_\Lambda} = 0$ follows from this by a simple diagram chasing.

The class β is a multiple of the second Chern class $c_2(\mathcal{M})$. This can be seen by formally calculating the total Chern class of $\mathcal{M}$ using $\mathbb{E}$. The result $c(T\mathcal{M}) = (1 - \beta)^{2g-2}$ formally agrees with the formula for the total Chern class of $T\mathcal{N} \oplus T^*\mathcal{N}$ that was calculated in [43, Cor. 2]. Every linear function on $\mathcal{A}$ gives a Hamiltonian vector field on $\mathcal{M}$, tangent to the fibres of χ. These Hamiltonian vector fields trivialize the relative tangent bundle of $\mathcal{M}_{\mathrm{reg}}$. The tangent bundle $T\mathcal{M}_{\mathrm{reg}}$ is an extension of the trivial bundle $\chi^*_{\mathrm{reg}} T\mathcal{A}_{\mathrm{reg}}$ by the relative tangent bundle. It follows that the Chern classes of $T\mathcal{M}_{\mathrm{reg}}$ vanish. □

Remark 3.1.2. In fact, although the argument above cannot be applied, we have that $\beta \in H^4_{\leq 2}(\mathcal{M})$, and that β vanishes over the generic line, also in the case $g = 2$ and $D = K_C$. This fact is proved in Proposition 4.3.7.

3.2. *Vanishing of the refined intersection form.* The purpose of this section is to establish Corollary 3.2.4, a fact we need in Section 4.3 as one of the pieces in the proof of the equality of the weight and perverse Leray filtrations in the case $D = K_C$. We need the following result proved in [29, Th. 1.1].

THEOREM 3.2.1. *The natural map*

$$H^{6g-6}_c(\check{\mathcal{M}}_{\mathrm{Dol}}) \longrightarrow H^{6g-6}(\check{\mathcal{M}}_{\mathrm{Dol}})$$

from compactly supported cohomology-to-cohomology is the zero map.

Remark 3.2.2. Note that $H^d(\check{\mathcal{M}}_{\mathrm{Dol}}) = H^d(\hat{\mathcal{M}}_{\mathrm{Dol}}) = 0$ for every $d > 6g - 6$, since, by (1.2.9), $\check{\mathcal{M}}_{\mathrm{Dol}}$ and $\hat{\mathcal{M}}_{\mathrm{Dol}}$ are homeomorphic to $\check{\mathcal{M}}_{\mathrm{B}}$ and $\hat{\mathcal{M}}_{\mathrm{B}}$ respectively, which are affine complex varieties of dim $6g - 6$.

Let us recall that the refined intersection forms on the fibres of a map (see [12, §3.4] for a general discussion) is the composite of the two maps

$$(3.2.1) \qquad H_{12g-12-r}(\check{\chi}^{-1}(0)) \longrightarrow H^r(\check{\mathcal{M}}_{\mathrm{Dol}}) \longrightarrow H^r(\check{\chi}^{-1}(0))$$

and arises by taking r-th cohomology of the adjunction maps

$$(3.2.2) \qquad i_! i^! \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}} \longrightarrow \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}} \longrightarrow i_* i^* \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}}$$

and using the canonical isomorphisms $H^r(\mathcal{A}, i_! i^! \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}}) \simeq H^r(\check{\mathcal{M}}_{\mathrm{Dol}}, \check{\mathcal{M}}_{\mathrm{Dol}} \setminus \check{\chi}^{-1}(0)) \simeq H_{12g-12-r}(\check{\chi}^{-1}(0))$ and $H^r(\mathcal{A}, i_* i^* \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}}) \simeq H^r(\check{\chi}^{-1}(0))$. The first map in (3.2.1) is the ordinary push-forward in homology (followed by Poincaré duality) and the second map is the restriction map.

By the decomposition theorem (see formula (1.4.4)), the complex $\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}}$ splits into the direct sum of its perverse cohomology sheaves. Combining this

fact with (3.2.2), we obtain one refined intersection form for each perversity a:

$$\iota_a^r \; : \; H_{12g-12-r,a}(\check{\chi}^{-1}(0)) \longrightarrow H_a^r(\check{\chi}^{-1}(0)).$$

The following fact will be used in the proof of the next corollary.

THEOREM 3.2.3. *The map* $\iota_{3g-3}^{3g-3} : H_{6g-6,3g-3}(\check{\chi}^{-1}(0)) \longrightarrow H_{3g-3}^{6g-6}(\check{\chi}^{-1}(0))$ *is an isomorphism.*

Proof. See [12, Th. 2.1.10] (where a different numbering convention is adopted). □

COROLLARY 3.2.4. *The perverse Leray filtration on the middle-dimensional groups satisfies*

$$H_{\leq 3g-3}^{6g-6}(\check{\mathcal{M}}_{\mathrm{Dol}}) = 0, \qquad H_{\leq 3g-3}^{6g-6}(\hat{\mathcal{M}}_{\mathrm{Dol}}) = 0.$$

Proof. Since $\hat{\chi}_*\mathbb{Q}_{\hat{\mathcal{M}}_{\mathrm{Dol}}} = (\check{\chi}_*\mathbb{Q}_{\check{\mathcal{M}}_{\mathrm{Dol}}})^\Gamma$, it suffices to prove the statement for $\check{\mathcal{M}}_{\mathrm{Dol}}$.

For every r, the first map in (3.2.1) factors as follows:

$$(3.2.3) \qquad H^r(\check{\mathcal{M}}_{\mathrm{Dol}}, \check{\mathcal{M}}_{\mathrm{Dol}} \setminus \check{\chi}^{-1}(0)) \longrightarrow H_c^r(\check{\mathcal{M}}_{\mathrm{Dol}}) \longrightarrow H^r(\check{\mathcal{M}}_{\mathrm{Dol}})$$

so that, by Theorem 3.2.1, it is the zero map. It follows that the refined intersection form (3.2.1),

$$H_{6g-6}(\check{\chi}^{-1}(0)) \longrightarrow H^{6g-6}(\check{\chi}^{-1}(0)),$$

vanishes. This, in turn, implies that all the graded refined intersection forms ι_a^{3g-3} are zero.

If we combine the vanishing of ι_{3g-3}^{3g-3} with Theorem 3.2.3, then we deduce that

$$(3.2.4) \qquad\qquad H_{3g-3}^{6g-6}(\check{\chi}^{-1}(0)) = 0.$$

Proposition 1.4.12 implies that $H_{\leq 3g-4}^{6g-6}(\check{\mathcal{M}}_{\mathrm{Dol}}) = 0$. In order to conclude, we need to prove that $H_{3g-3}^{6g-6}(\check{\mathcal{M}}_{\mathrm{Dol}}) = 0$.

Since the restriction map to the fiber is compatible with any splitting coming from the decomposition theorem, in view of the vanishing (3.2.4), it is enough to show that the restriction map $H^{6g-6}(\check{\mathcal{M}}_{\mathrm{Dol}}) \longrightarrow H^{6g-6}(\check{\chi}^{-1}(0))$ is an isomorphism. In fact it follows from [49, §3] that $\check{\chi}^{-1}(0)$ — being the downward flow [28, Ths. 3.1 and 5.2] of a $\mathbb{C}^\times$-action on $\check{\mathcal{M}}_{\mathrm{Dol}}$ — is a deformation retract of $\check{\mathcal{M}}_{\mathrm{Dol}}$. □

3.3. *Bi-graded* $\mathfrak{sl}_2(\mathbb{Q})$-*modules.* We collect here some linear algebra considerations that will be used in the sequel of the paper. Let $\mathbb{H} = \bigoplus_{d,w \geq 0} H_w^d$ be a finite dimensional bi-graded vector space. We say that d is the degree and

w is the weight. We employ the following notation:

$$\mathbb{H}_w := \bigoplus_{d \geq 0} \mathbb{H}_w^d, \qquad \mathbb{H}^d := \bigoplus_{w \geq 0} \mathbb{H}_w^d.$$

Let Y be a nilpotent endomorphism of $\mathbb{H}$ that is bi-homogeneous of type $(2, 2)$, i.e., $Y : \mathbb{H}_*^* \to \mathbb{H}_{*+2}^{*+2}$. Let $w_o \in \mathbb{Z}^{\geq 0}$ be such that for every $l \geq 0$, we have hard-Lefschetz-type isomorphisms

$$Y^l : \mathbb{H}_{w_o - l} \xrightarrow{\simeq} \mathbb{H}_{w_o + l}.$$

Note that we must then have that $\mathbb{H}_w = \{0\}$ for every $w > 2w_o$.

It is well known that we can turn $\mathbb{H}$ into an $\mathfrak{sl}_2(\mathbb{Q})$-module in a natural way by means of a unique pair (X, H) of homogeneous endomorphism of $\mathbb{H}$ or respective types $(-2, -2)$ and $(0, 0)$ subject to $[X, Y] = H$. In this case H is just the "w-grading" operator: $Hu = (w - w_0)u$ if $u \in \mathbb{H}_w$.

Given a bi-homogeneous element $u \in \mathbb{H}_w^d$, we define

$$(3.3.1) \qquad\qquad \Delta(u) := d - w.$$

Note that the action of $\mathfrak{sl}_2(\mathbb{Q})$ leaves Δ invariant.

We define the primitive space $\mathbb{P} := \operatorname{Ker} X \subseteq \mathbb{H}$, and we obtain the primitive decomposition

$$(3.3.2) \qquad\qquad \mathbb{H} = \bigoplus_{j \geq 0} Y^j \cdot \mathbb{P}.$$

Note that we have

$$Y^j \cdot \mathbb{P} = \operatorname{Ker} X^{j+1} \cap \operatorname{Im} Y^j.$$

Since X is homogeneous, the space $\mathbb{P}$ is also bi-graded. We set $\mathbb{P}_w := \mathbb{P} \cap \mathbb{H}_w$, $\mathbb{P}^d := \mathbb{P} \cap \mathbb{H}^d$, and $\mathbb{P}_w^d := \mathbb{P} \cap \mathbb{H}_w^d$. We have $\mathbb{P}_w = \{0\}$ for every $w > w_o$ and

$$\mathbb{P} = \bigoplus_{d,w} \mathbb{P}_w^d = \bigoplus_{w \geq 0} \mathbb{P}_w = \bigoplus_{d \geq 0} \mathbb{P}^d.$$

For every fixed weight w, the primitive decomposition can be rewritten as follows:

$$\mathbb{H}_w = \bigoplus_{j \geq 0} Y^j \cdot \mathbb{P}_{w-2j}, \qquad \mathbb{H}_w^d = \bigoplus_{j \geq 0} Y^j \cdot \mathbb{P}_{w-2j}^{d-2j}.$$

We denote by Π the operator of projection onto $\mathbb{P}$; clearly, if $u \in \mathbb{H}_w^d$, then $\Pi(u) \in \mathbb{P}_w^d$ and $\Pi(u) = u + \sum_{j>0} Y^j u_j$ with $u_j \in \mathbb{H}_{w-2j}^{d-2j}$.

Given a subset $S \subseteq \mathbb{H}$, we define the associated Y-string

$$\langle S \rangle_Y = \bigoplus_{j \geq 0} Y^j \cdot \langle S \rangle_{\mathbb{Q}} \subseteq \mathbb{H}.$$

In particular, $\langle \mathbb{P} \rangle_Y = \mathbb{H}$. If u is a bi-homogeneous element, then Δ is constant on the Y-string $\langle u \rangle_Y$ generated by u.

Let $0 \neq p_w \in \mathbb{P}_w$. Then $\langle p_w \rangle_Y = \langle p_w, Y \cdot p_w, \ldots, Y^{w_o-w} \cdot p_w \rangle_{\mathbb{Q}} \subseteq \mathbb{H}$ is isomorphic to the irreducible $\mathfrak{sl}_2(\mathbb{Q})$-submodule of dimension $(w_o - w) + 1$. Hence, the isotypical decomposition of the $\mathfrak{sl}_2(\mathbb{Q})$-module $\mathbb{H}$ is

$$\mathbb{H} = \bigoplus_{0 \leq w \leq w_o} \langle \mathbb{P}_w \rangle_Y.$$

We define the *isobaric* decomposition of $\mathbb{H}$ as the direct sum decomposition obtained by grouping terms according to the powers of Y in the isotypical decomposition given above, namely

$$(3.3.3) \qquad \mathbb{H} = \bigoplus_{0 \leq w \leq w_o} \bigoplus_{0 \leq j \leq w_o-w} Y^j \cdot \mathbb{P}_w.$$

The proof of the following proposition is completely elementary and safely left to the reader. (For point (4), just remark that if $u \in \mathbb{H}_w^d \cap \mathrm{Im} Y$, then $u = Yv$, with $v \in \mathbb{H}_{w-2}^{d-2}$.)

PROPOSITION 3.3.1. *Let $M \subseteq \mathbb{H}$ be a subset of bi-homogeneous elements such that $\langle M \rangle_Y = \mathbb{H}$. Then*

(1) *The set $\Pi(M) \subseteq \mathbb{P}$ obtained by projecting M to the primitive space is also a Y-generating subset of bi-homogeneous elements with the same bi-degrees, and its linear span is $\mathbb{P}$:*

$$\langle \Pi(M) \rangle_Y = \mathbb{H}, \qquad \langle \Pi(M) \rangle_{\mathbb{Q}} = \mathbb{P}.$$

(2) *If $T' \subseteq \Pi(M)$ is a linearly independent set, then it can be completed to a basis $T \subseteq \Pi(M)$ for $\mathbb{P}$ that is also Y-generating: $\langle T \rangle_{\mathbb{Q}} = \mathbb{P}$ and $\langle T \rangle_Y = \mathbb{H}$. Let $T \subseteq \Pi(M)$ be a basis for $\mathbb{P}$.*

(3) *Let $T_w^d := T \cap \mathbb{H}_w^d$, $T^d := T \cap \mathbb{H}^d$, $T_w := T \cap \mathbb{H}_w$; then*

$$\mathbb{P}_w^d = \langle T_w^d \rangle_{\mathbb{Q}}, \qquad \langle \mathbb{P}_w \rangle_{\mathbb{Q}} = \langle T_w \rangle_{\mathbb{Q}}, \qquad \langle \mathbb{P}^d \rangle_{\mathbb{Q}} = \langle T^d \rangle_{\mathbb{Q}}.$$

(4) *If $m \in M$ has bidegree (d, w), then there are $C_{j,t} \in \mathbb{Q}$ for $j > 0, t \in T_{w-2j}^{d-2j}$ such that*

$$\Pi(m) = m + \sum_{j,t} C_{j,t} Y^j t.$$

In particular, if $\Pi(m) \neq 0$, then $\Delta(m) = \Delta(\Pi(m))$.

4. W=P

This section contains the proof of the main result of this paper: the identification of the perverse Leray filtration associated to the Hitchin map with the weight filtration of the cohomology of the character variety in the case $D = K_C$ (§4.3), or, if $\deg D > 2g - 2$, with the abstract weight filtration, defined in Definition 1.2.11. The latter case is easier and we deal with it first.

Let $\mathbb{H} := \bigoplus_{d \geq 0} H^d(\hat{\mathcal{M}})$, and let $Y := \alpha \cup : \mathbb{H}_w^d \longrightarrow \mathbb{H}_{w+2}^{d+2}$ be the operation of cupping with α.

In virtue of the curious hard Lefschetz Theorem 1.2.12, we are in the situation described in Section 3.3, with $w_0 = g-1+\deg D$. The monomials $\psi^{\underline{t}}\beta^s$ (as defined in Proposition 4.2.1) give a set of bi-homogeneous elements that is Y-generating, for $\mathbb{H}$, and, by Proposition 3.3.1, the elements $\Pi(\psi^{\underline{t}}\beta^s)$ span $\mathbb{P}$. The perverse filtration is denoted by $\mathbb{H}_{\leq}$. As noticed in Corollary 5.1.3, α is relatively ample and it defines a Deligne decomposition of $\mathbb{H}$, whose summands are denoted, as in Section 1.4.3, by $Q^{i,j}$ and, when we want to emphasize the cohomological degree, by $Q^{i,j;d} := Q^{i,j} \cap \mathbb{H}^d$. We set $Q := \bigoplus_{i,d} Q^{i,0;d}$. Clearly,

$$(4.1.1) \qquad\qquad \langle Q \rangle_Y = \mathbb{H}.$$

PROPOSITION 4.1.1. *Let $u \in \mathbb{H}_w^d$, i.e., the weight $w(u) = w$.*

(1) *If $\Delta(u) \leq \operatorname{codim}\mathcal{A}^0 \setminus \mathcal{A}_{\text{ell}}^0$, then $u \in \mathbb{H}_{\leq w}^d$, i.e., the perversity $p(u) \leq w$.*

(2) *If $u \in \mathbb{P}_w^d$ and $\Delta(u) \leq \operatorname{codim}\mathcal{A}^0 \setminus \mathcal{A}_{\text{ell}}^0$, then, more precisely, $u \in Q^{w,0;d} \subseteq \mathbb{H}_{\leq w}^d$.*

Proof. Since the monomials $u = \alpha^r\psi^{\underline{t}}\beta^s$ are additive generators, it is enough to prove (1) for these monomials. By virtue of Lemma 1.4.4, we are further reduced to the case $u = \psi^{\underline{t}}\beta^s$. Keeping in mind the upper bound $p \leq 2$ on the perversity of β and ψ given by Theorem 3.1.1, we can apply the perversity test given by Proposition 1.4.11 (where the set Y in loc. cit. is the present set $\mathcal{A}^0 \setminus \mathcal{A}_{\text{ell}}^0$) and obtain that $u \in \mathbb{H}_{\leq 2(s+t)}^{4s+3t}$ as soon as $(4s+3t)-(2s+2t) - 1 = \Delta(\psi^{\underline{t}}\beta^s) - 1 < \operatorname{codim}\mathcal{A}^0 \setminus \mathcal{A}_{\text{ell}}^0$. This proves (1)

In view of Proposition 3.3.1, in order to prove the second statement, it is enough to prove that $\Pi(\psi^{\underline{t}}\beta^s) \in Q^{2(s+t),0;3t+4s}$.

By using induction on $r := s + t$, we first show that $\Pi(\psi^{\underline{t}}\beta^s) \in \mathbb{H}_{\leq 2(s+t)}$.

- For $r = 0$ there is nothing to prove.
- Suppose we know that $\Pi(\psi^{\underline{t'}}\beta^{s'}) \in \mathbb{H}_{\leq 2(s'+t')}$ for all $(s',\underline{t'})$ with $s' + t' \leq r - 1$. If $(s,\underline{t})$ is such that $s + t = r$, we may write (see again Proposition 3.3.1, part (4))

$$(4.1.2) \quad \Pi(\psi^{\underline{t}}\beta^s) = \psi^{\underline{t}}\beta^s + \sum_{(s_i,\underline{t_i})} C_{s_i,\underline{t_i}}\alpha^{j_i}\Pi(\psi^{\underline{t_i}}\beta^{s_i}), \text{ with } j_i = r - s_i - t_i > 0.$$

By (1), we have that $\psi^{\underline{t}}\beta^s \in \mathbb{H}_{\leq 2(s+t)}$. By the inductive hypothesis, we have that $\Pi(\psi^{\underline{t_i}}\beta^{s_i}) \in \mathbb{H}_{\leq 2(s_i+t_i)}$ so that, by Lemma 1.4.4, we have that $\alpha^{j_i}\Pi(\psi^{\underline{t_i}}\beta^{s_i}) \in \mathbb{H}_{\leq 2r}$. The conclusion on the sum $\Pi(\psi^{\underline{t}}\beta^s) \in \mathbb{H}_{\leq 2r}$ follows.

By definition of primitivity, $\alpha^{w_0-2(s+t)+1}\Pi(\psi^{\underline{t}}\beta^s) = 0$. Since $\Pi(\psi^{\underline{t}}\beta^s) \in \mathbb{H}_{\leq 2(s+t)}$, the nonmixing Lemma 1.4.3, coupled with the equality above, implies that $\Pi(\psi^{\underline{t}}\beta^s) \in Q^{2(s+t),0}$. □

4.2. *The case* $\deg D > 2g - 2$, G = PGL$_2$, GL$_2$. In this section we assume $n := \deg D + 2 - 2g > 0$.

Recall the relations (1.2.11) between the generators of the cohomology ring. Theorem 1.2.10 readily implies the following.

PROPOSITION 4.2.1. *For nonnegative integers* $t_1, \ldots, t_{2g}$, *let us write*[1] $\psi^{\underline{t}}$ *for* $\psi_1^{t_1} \ldots \psi_{2g}^{t_{2g}}$ *and let us set* $t := \sum_{i=1}^{2g} t_i$. *Then*

(1) *If* $n := \deg D + 2 - 2g > 0$, *then* $\beta^s \psi^{\underline{t}} = 0$ *for* $2s + t \geq \deg D$.
(2) *If* $D = K_C$, *then* $\beta^s \psi^{\underline{t}} = 0$ *for* $2s + t \geq 2g - 2$, *unless* $\psi^{\underline{t}} = \gamma^r$, *with* $r + s = g - 1$.

Proof. Let us assume $n > 0$. The monomial $\beta^s \psi^{\underline{t}}$ is a sum of terms of the form $A_i \gamma^i \beta^s$ with $A_i \in \Lambda_0^{t-2i}$. From the relations in Theorem 1.2.10, it follows that

$$\text{if } \gamma^i \beta^s \in I_{n+t-2i}^{g+2i-t}, \text{ then } A_i \gamma^i \beta^s = 0.$$

By inequalities (1.2.12) and Remark 1.2.9, $\gamma^i \beta^s \in I_{n+t-2i}^{g+2i-t}$ if $2i + 2s \geq 2(g-t+2i) - 2 + n + t - 2i$, that is, if $2s + t \geq 2g - 2 + n = d$. If $n = 0$, we proceed as above, noting that Remark 1.2.9 fails exactly in the case $i = 0$, in which case we find that $\beta^{g-1}, \gamma\beta^{g-2}, \ldots, \gamma^{g-1} \neq 0$. □

THEOREM 4.2.2. *The abstract weight filtration* $W_\bullet'$ *on* $H^*(\hat{\mathcal{M}})$ (*Definition* 1.2.11) *coincides with the perverse Leray filtration associated with the Hitchin map* $\hat{\chi}$. *For every integer* i, *we have*

$$(4.2.1) \qquad H_{\leq i}^*(\hat{\mathcal{M}}) = W_i' H^*(\hat{\mathcal{M}}) = \langle \alpha^r \psi^{\underline{t}} \beta^s \rangle_{2(r+s+t) \leq i}.$$

More precisely, the isobaric decomposition (3.3.3) *coincides with the Deligne decomposition* (1.4.14) *in Section* 1.4.3 *associated to* α: *for every* w, *we have* $\mathbb{P}_w = Q^{w,0}$.

Proof. Since the statement about the equality of the filtrations follows at once from the second on the equality of the internal direct sum decompositions, we prove the latter one. By Proposition 4.2.1, we have that $\psi^{\underline{t}} \beta^s = 0$ as soon as $\Delta(\psi^{\underline{t}} \beta^s) = t + 2s \geq \deg D$. It follows that we only need to consider the monomials $\psi^{\underline{t}} \beta^s$ with $t + s < \deg D$. By Lemma 2.1.3, $\text{codim } \mathcal{A}^0 \setminus \mathcal{A}_{\text{ell}}^0 = \deg D$. We can then apply Proposition 4.1.1 and deduce that

$$(4.2.2) \qquad \Pi(\psi^{\underline{t}} \beta^s) \in Q^{2(s+t),0;\,4s+3t}.$$

Since $\{\Pi(\psi^{\underline{t}} \beta^s)\}$ is a set of generators for the primitive space $\mathbb{P}$, and since Π strictly preserves the weights by construction, we deduce that $\mathbb{P}_w \subseteq Q^{w,0}$ for all w. Since, by (3.3.2), $\langle \mathbb{P} \rangle_Y = \mathbb{H}$ and, by (4.1.1), $\langle Q \rangle_Y = \mathbb{H}$, it follows that $\mathbb{P}_w$ and $Q^{w,0}$ coincide for all w. □

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[1]As the ψ classes have degree 3 they anticommute, so we could assume $t_i = 0, 1$.

THEOREM 4.2.3. *The abstract weight filtration $W_\bullet'$ on $H^*(\mathcal{M})$ coincides with the perverse filtration associated with the Hitchin map. More precisely, the isobaric decomposition coincides with the Deligne decomposition associated to $\tilde{\alpha} = \alpha \otimes 1 + 1 \otimes (\sum \varepsilon_i \varepsilon_{i+g})$.*

Proof. As cupping with i increases the perversity exactly by one, the statement follows from the isomorphism (2.4.14) and Lemma 1.4.4.   □

4.3. The case $D = K_C$, $\mathrm{G} = \mathrm{PGL}_2, \mathrm{GL}_2$. In this section, we set

$$\mathbb{H} = \bigoplus_{d \geq 0} H^d(\hat{\mathcal{M}}_{\mathrm{Dol}}).$$

LEMMA 4.3.1. *We have the following:*

$$\alpha^r \psi^t \beta^s \in \mathbb{H}^{2r+3t+4s}_{\leq 2(r+t+s)}, \quad \Pi(\psi^t \beta^s) \in Q^{2(t+s),0;\,3t+4s}$$

unless $\psi^t \beta^s = \gamma^v\,\beta^s$ with $v + s = g - 1$ (cf. (2) in Proposition 4.2.1).

Proof. Since, by Lemma 1.4.4, cupping with α increases the perversity by at most 2, we may suppose $r = 0$. Note that we are excluding precisely the classes in the statement of Proposition 4.2.1 (case $D = K_C$). By this same proposition, we may thus assume that $2s + t < 2g - 2 = \mathrm{codim}\,\mathcal{A}^0 \setminus \mathcal{A}^0_{\mathrm{ell}} + 1$, where the last equality results from Lemma 2.1.3. The result follows from Proposition 4.1.1.   □

Remark 4.3.2. The argument above breaks when dealing with the classes $\gamma^r \beta^{g-1-r}$ that we have excluded from the statement. To check that $\gamma^r \beta^{g-1-r} \in \mathbb{H}^{2r+4g-4}_{\leq 2g-2+2r}$ we should consider a linear subspace of dimension $(2r + 4g - 4) - (2g - 2 + 2r) - 1 = 2g - 3$, which is exactly the codimension of the "bad locus" $\mathcal{A}^0 \setminus \mathcal{A}^0_{\mathrm{ell}}$. On the other hand, a general linear subspace of dimension one less, i.e., $2g - 4$, misses the bad locus, and thus yields, by Theorem 1.4.8, the following upper bound on the perversity

$$(4.3.1) \qquad\qquad \gamma^r \beta^{g-1-r} \in \mathbb{H}^{2r+4g-4}_{\leq 2g-1+2r}.$$

While this upper bound is not sufficient for our purposes, it is used in what follows.

Remark 4.3.3. The relations $\rho^c_{1,s,g-1-s} \in I^g_0$ in (1.2.11) show that for all r, the class $\gamma^r \beta^{g-1-r}$ is a multiple of $\alpha^r \beta^{g-1}$.

As pointed out several times, in view of Lemma 1.4.4, cupping with α is harmless for us and we are reduced to prove that $\beta^{g-1} \in \mathbb{H}^{4g-4}_{\leq 2g-2}$. The remainder of the analysis is devoted to improve the upper bound (4.3.1), by one unit, i.e., to proving that $\beta^{g-1} \in \mathbb{H}^{4g-4}_{\leq 2g-1}$.

LEMMA 4.3.4. *For every s in the range $0 \leq s \leq g - 1$, we have*

$$\beta^s \in \mathbb{P}^{4s}_{2s}.$$

In particular, the classes β^s are not divisible by α.

Proof. Recall that $\mathbb{H}^d = 0$ for every $d > 6g - 6$ (see Remark 3.2.2). Clearly, since weights w are strictly multiplicative, $\beta^s \in \mathbb{H}^{4s}_{2s}$. Since, in the terminology of Section 3.3, $\mathbb{H}$ is a bi-graded $\mathfrak{sl}_2(\mathbb{Q})$-module with $w_0 = 3g - 3$, we have that $\alpha^{3g-3-2s}\beta^s \neq 0$. On the other hand, $\alpha^{3g-3-2s+1}\beta^s \in \mathbb{H}^{6g-4} = \{0\}$. These are precisely the conditions defining primitivity. $\square$

Set

(4.3.2) $\quad \mathbb{J} := \{\beta^s\psi^{\underline{t}} \in \mathbb{H}$ such that $\Delta(\beta^s\psi^{\underline{t}}) \leq 2g - 3$ if t is odd and

$$\Delta(\beta^s\psi^{\underline{t}}) \leq 2g - 4 \text{ if } t \text{ is even}\}$$

and

$$\widetilde{\mathbb{H}} := \langle\, \mathbb{J}\, \rangle_\alpha = \langle\, \Pi(\mathbb{J})\, \rangle_\alpha.$$

By Remark 4.3.3, if $r \geq 1$, then $\gamma^r\beta^{g-1-r}$ is divisible by α. In this case, the projection to the primitive space $\Pi(\gamma^r\beta^{g-1-r}) = 0$. By Lemma 4.3.4, we have $\Pi(\beta^{g-1}) = \beta^{g-1}$ and $\gamma^r\beta^{g-1-r} \in \langle\beta^{g-1}\rangle_\alpha$. Since $\Delta(\beta^s\psi^{\underline{t}}) = 2s + t$, point (2) of the statement of Proposition 4.2.1 can be rephrased by saying that, unless $\beta^s\psi^{\underline{t}} \in \langle\beta^{g-1}\rangle_\alpha$, we have that $\beta^s\psi^{\underline{t}} \in \widetilde{\mathbb{H}}$. Thus we have an $\mathfrak{sl}_2(\mathbb{Q})$-*invariant decomposition*

(4.3.3) $$\mathbb{H} = \widetilde{\mathbb{H}} \bigoplus \langle\beta^{g-1}\rangle_\alpha.$$

LEMMA 4.3.5. *The following facts hold:*
(1) $\mathbb{H}^d = \widetilde{\mathbb{H}}^d$ *unless $d + 4 - 4g$ is even nonnegative.*
(2) $\dim \mathbb{H}^{4g-4+2k} = \dim \widetilde{\mathbb{H}}^{4g-4+2k} + 1$ *for $0 \leq k \leq g - 1$.*
(3) *In the range of point (1), $\mathbb{P}^d_w = Q^{w,0;\,d}$. In particular, all the nonzero summands $Q^{i,j,d}$ in the Deligne decomposition satisfy $d - i - 2j \leq 2g - 3$ if d is odd and $d - i - 2j \leq 2g - 4$ if d is even.*
(4) *In the range of point (2), there is at most one nonzero, necessarily one-dimensional, summand $Q^{i,j,d}$ satisfying $d - i - 2j > 2g - 4$.*

Proof. Points (1) and (2) follow immediately from the fact that the α-string $\langle\beta^{g-1}\rangle_\alpha$ contains the classes $\beta^{g-1}, \alpha\beta^{g-1}, \ldots, \alpha^{g-1}\beta^{g-1}$ whose cohomological degrees are $4g - 4, 4g - 2 \ldots, 6g - 6$. Note that the monomials $\beta^s\psi^{\underline{t}}$ in $\mathbb{J}$ are precisely those to which Lemma 4.3.1 applies; hence

$$\mathbb{P}^d_w \cap \widetilde{\mathbb{H}}^d_w \subseteq Q^{w,0;\,d}$$

and, for $j \geq 0$,

$$(\alpha^j \mathbb{P}^{d-2j}_{w-2j}) \cap \widetilde{\mathbb{H}}^d_w \subseteq \alpha^j Q^{w-2j,0;\,d-2j} = Q^{w-2j,j;\,d}.$$

Combining this fact with the α-decompositions (3.3.3) and (1.4.14) in Section 1.4.3, and with the points (1) and (2) that we just proved, we immediately obtain points (3) and (4)   □

LEMMA 4.3.6. *Either* $\beta^{g-1} \in Q^{2g-2,0;\,4g-4}$ *or* $\beta^{g-1} \in Q^{2g-1,0;\,4g-4}$.

Proof. By (4.3.1), with $r = 0$, we have that $\beta^{g-1} \in \mathbb{H}^{4g-4}_{\leq 2g-1}$. There is hence at least one nonzero summand $Q^{i_0,j_0;\,4g-4}$ in the Deligne decomposition, satisfying

$$(4.3.4) \qquad\qquad i_0 + 2j_0 \leq 2g - 1.$$

Suppose $j_0 \neq 0$; by (1.4.18), we have

$$Q^{i_0,j_0;\,4g-4} = \alpha^{j_0} Q^{i_0,0;\,4g-4-2j_0}.$$

Since $Q^{i_0,0;\,4g-4-2j_0} \neq \{0\}$, we have, by point (3) of Lemma 4.3.5, $(4g-4-2j_0) - i_0 - 2j_0 \leq 2g - 4$, which contradicts the inequality (4.3.4) above, showing that $j_0 = 0$. By Corollary 1.4.13, we then have

$$\beta \in \mathbb{H}^{4g-4}_{\leq 2g-1} = Q^{2g-2,0;4g-4} \oplus Q^{2g-1,0;4g-4} \qquad\qquad\qquad □$$

PROPOSITION 4.3.7. *We have that*

$$\beta^{g-1} \in Q^{2g-2,0;\,4g-4}.$$

Proof. Suppose that the statement is false. By Lemma 4.3.6, the space $Q^{2g-2,0;\,4g-4} = 0$ and the class $\beta^{g-1} \in Q^{2g-1,0;\,4g-4}$. From the property of the Deligne decomposition expressed by (1.4.18), it follows that, for $j \leq g-2$, we have $0 \neq \alpha^j \beta^{g-1} \in Q^{2g-1,j;\,4g-4+2j}$. By using the decomposition (4.3.3) and Lemma 4.3.5, it follows that, for every even nonnegative integer $d < 6g - 6$,

$$(4.3.5) \qquad \mathbb{H}^d = \widetilde{\mathbb{H}}^d = \bigoplus_{d-i-2j\leq 2g-4} Q^{i,j;\,d} \quad \text{if } d < 4g - 4$$

and

$$(4.3.6)$$

$$\mathbb{H}^d = \left(\bigoplus_{d-i-2j\leq 2g-4} Q^{i,j;\,d} \right) \oplus Q^{2g-1,j_0;\,d}, \text{ with } j_0 = d/2 - 2g + 2, \text{ if } d \geq 4g-4.$$

In this latter case, $Q^{2g-1,j_0;\,d} = \langle \alpha^{j_0} \beta^{g-1} \rangle_{\mathbb{Q}}$.

Applying one of the defining properties of the Deligne decomposition, i.e., the second equation in (1.4.17) (with $f = 3g-3$ and $i = 2g-1$), to $\alpha^{g-2}\beta^{g-1} \in Q^{2g-1,g-2;\,6g-8}$, we have the following upper bound for the perversity

$$\alpha^{g-1}\beta^{g-1} = \alpha(\alpha^{g-2}\beta^{g-1}) \in \mathbb{H}^{6g-6}_{\leq 4g-5}.$$

From Corollary 3.2.4, it follows that

$$\mathbb{H}^{6g-6}_{\leq 3g-3} = \{0\}.$$

It follows that there exists $1 \leq r \leq g - 2$ such that

$$(4.3.7) \qquad \alpha^{g-1}\beta^{g-1} \in \mathbb{H}^{6g-6}_{\leq 3g-3+r} \text{ and } \alpha^{g-1}\beta^{g-1} \notin \mathbb{H}^{6g-6}_{\leq 3g-4+r}.$$

From this and from point (4) of Lemma 4.3.5, it follows that $\alpha^{g-1}\beta^{g-1}$ must belong to the unique summand $Q^{3g-3-r,r;\,6g-6}$ with $1 \leq r \leq g-2$.

Since $r \geq 1$, the relation (1.4.18) gives

$$Q^{3g-3-r,r;\,6g-6} = \alpha^r Q^{3g-3-r,0;\,6g-6-2r}.$$

On the other hand, (4.3.6) together with $d = 6g - 6 - 2r$ shows that $Q^{3g-3-r,0;\,6g-6-2r} = 0$. $\qquad \square$

Remark 4.3.8. For $g = 2$, the previous argument shows that $\beta \in \mathbb{H}^4_{\leq 2}$, as anticipated in Remark 3.1.2.

Proposition 4.3.7 allows us to complete point (3) in Lemma 4.3.5:

$$\mathbb{P}^d_w = Q^{w,0;\,d} \text{ for all } d, w.$$

We finally summarize what we proved in the following theorem, which is the main result of this paper:

THEOREM 4.3.9. *The non-Abelian Hodge theorem for* PGL_2, (*resp.* GL_2) *identifies*

- *the perverse Leray filtration with the weight filtration: for every integer i, we have*

$$H^*_{\leq i}(\hat{\mathcal{M}}_{\mathrm{Dol}}) \simeq W_{2i}H^*(\hat{\mathcal{M}}_{\mathrm{B}}) = W_{2i+1}H^*(\hat{\mathcal{M}}_{\mathrm{B}}),$$
$$H^*_{\leq i}(\mathcal{M}_{\mathrm{Dol}}) \simeq W_{2i}H^*(\mathcal{M}_{\mathrm{B}}) = W_{2i+1}H^*(\mathcal{M}_{\mathrm{B}});$$

- *the relative hard Lefschetz theorem* (1.4.9) *relative to the Hitchin map* $\hat{\chi}$ (*resp.* χ) *and to the relatively ample class α (resp. $\tilde{\alpha} = \alpha \otimes 1 + 1 \otimes (\sum \varepsilon_i \varepsilon_{i+g})$) with the curious hard Lefschetz Theorem 1.2.3:*

$$\begin{array}{ccc} H^*_{3g-3-i}(\hat{\mathcal{M}}_{\mathrm{Dol}}) & \xrightarrow{\simeq} & \mathrm{Gr}^W_{6g-6-2i}H^*(\hat{\mathcal{M}}_{\mathrm{B}}) \\ \simeq \downarrow \alpha^i & & \simeq \downarrow \alpha^i \\ H^{*+2i}_{3g-3+i}(\hat{\mathcal{M}}_{\mathrm{Dol}}) & \xrightarrow{\simeq} & \mathrm{Gr}^W_{6g-6+2i}H^{*+2i}(\hat{\mathcal{M}}_{\mathrm{B}}) \end{array}$$

and

$$\begin{array}{ccc} H^*_{4g-3-i}(\mathcal{M}_{\mathrm{Dol}}) & \xrightarrow{\simeq} & \mathrm{Gr}^W_{8g-6-2i}H^*(\mathcal{M}_{\mathrm{B}}) \\ \simeq \downarrow \tilde{\alpha}^i & & \simeq \downarrow \tilde{\alpha}^i \\ H^{*+2i}_{4g-3+i}(\mathcal{M}_{\mathrm{Dol}}) & \xrightarrow{\simeq} & \mathrm{Gr}^W_{8g-6+2i}H^{*+2i}(\mathcal{M}_{\mathrm{B}}); \end{array}$$

- *the Deligne Q-splitting* (1.4.14) *associated with the relatively ample class α (resp. $\tilde{\alpha}$) with the isobaric splitting* (3.3.3).

Proof. It follows from Lemma 4.3.1 that every cohomology class of the additive basis, with the possible exception of those in the α-chain of β^{g-1} (see Remark 4.3.3), satisfies "W=P." Lemma 4.3.7 show that also β^{g-1} satisfies the condition "W=P," and so do the classes in its α-chain, thus proving the first statement for PGL$_2$. The two other statements follow similarly. The extension to GL$_2$ follows immediately from the isomorphism (2.4.14) and Lemma 1.4.4, as cupping with ε_i increases the perversity exactly by one. □

Remark 4.3.10. We have made a heavy use and made explicit Deligne's splitting of the direct image complex via the use of a relatively-ample line bundle. This general splitting mechanism is described in [20]. The same paper details the construction of two additional splittings. As the simple example of the ruled surface $P^1 \times P^1 \to P^1$ already shows, in general, the three splittings differ. It is possible to show that, in the case of all three Hitchin maps considered in this paper, all three splittings coincide when viewed in cohomology.

Remark 4.3.11. Let $u \in Q^{i,j}$. Such a class has perversity $p := i + 2j$ when viewed as a cohomology class for the Higgs moduli space. The main result of this paper, i.e., $P = W$, shows that the non-Abelian Hodge theorem turns this class into a (p,p)-class for the split Hodge-Tate mixed Hodge structure on the associated character variety.

4.4. SL$_2$. In this section, for the sake of notational simplicity, we will denote simply by $\check{\mathcal{M}}$ the moduli space $\check{\mathcal{M}}_{\mathrm{Dol}}$ of stable Higgs bundles on C of rank 2 and fixed determinant of degree 1. Let $\check{\chi} : \check{\mathcal{M}} \to \mathcal{A}^0$ be the Hitchin map. The action of $\Gamma = \mathrm{Pic}^0_C[2] \simeq \mathbb{Z}_2^{2g}$ on $\check{\mathcal{M}}$ by tensorization preserves the map $\check{\chi}$, and, as discussed in Section 2.4, we have a direct sum decomposition according to the characters of Γ:

$$(4.4.1) \qquad \check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}} \simeq \bigoplus_{\kappa \in \hat{\Gamma}} (\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa = (\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})^\Gamma \bigoplus (\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})_{\mathrm{var}} ,$$

where we set $(\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})_{\mathrm{var}} = \bigoplus_{0 \neq \kappa \in \hat{\Gamma}} (\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa$. Taking cohomology, (4.4.1) gives

$$(4.4.2) \qquad H^*(\check{\mathcal{M}}) = \bigoplus_{\kappa \in \hat{\Gamma}} H^*(\check{\mathcal{M}})_\kappa = H^*(\check{\mathcal{M}})^\Gamma \bigoplus H^*_{\mathrm{var}}(\check{\mathcal{M}}),$$

where $H^*(\check{\mathcal{M}})_\kappa = H^*(\mathcal{A}^0, (\chi_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa)$ is the subspace of $H^*(\check{\mathcal{M}})$ where Γ acts via the character κ and $H^*_{\mathrm{var}}(\check{\mathcal{M}}) := \bigoplus_{0 \neq \kappa \in \hat{\Gamma}} H^*(\check{\mathcal{M}})_\kappa = H^*(\mathcal{A}^0, (\chi_* \mathbb{Q}_{\check{\mathcal{M}}})_{\mathrm{var}})$ is the *variant* part of $H^*(\check{\mathcal{M}})$.

Recall from [35, formula after (7.13)] that

$$(4.4.3) \quad \dim H^{4g+2d-5}_{\mathrm{var}}(\check{\mathcal{M}}) = \begin{cases} (2^{2g} - 1) \binom{2g-2}{2g-2d-1} & \text{if } d = 1, \ldots, g-1, \\ 0 & \text{otherwise .} \end{cases}$$

For $\gamma \in \Gamma \subseteq \mathrm{Pic}_C^0$, let L_γ be the corresponding order 2 line bundle, and let i_γ be the "squaring" map

$$(4.4.4) \qquad i_\gamma : H^0(C, K_C \otimes L_\gamma) \longrightarrow H^0(C, 2K_C) = \mathcal{A}^0, \qquad i_\gamma(a) = a \otimes a,$$

with image $\mathcal{A}_\gamma^0 := \mathrm{Im}(i_\gamma) \subset \mathcal{A}^0$. By the Riemann-Roch theorem, $\dim(\mathcal{A}_0^0) = g$ and, if $\gamma \in \Gamma^* = \Gamma \setminus \{0\}$, then $\dim(\mathcal{A}_\gamma^0) = g - 1$. Points in $\cup_{\gamma \in \Gamma^*} \mathcal{A}_\gamma^0$ are called *endoscopic points*. Set

$$\mathcal{A}_{ne}^0 := \mathcal{A}^0 \setminus \cup_{\gamma \in \Gamma \setminus 0} \mathcal{A}_\gamma^0, \text{ and } \check{\mathcal{M}}_{ne} := \check{\chi}^{-1}(\mathcal{A}_{ne}^0).$$

Our goal is to prove the following proposition.

PROPOSITION 4.4.1. *Let $s \in \mathcal{A}_{ne}^0$, and let $\check{\mathcal{M}}_s := \check{\chi}^{-1}(s)$ denote the fiber of the Hitchin fibration over s. The group Γ acts trivially on $H^*(\check{\mathcal{M}}_s)$.*

Proposition 4.4.1 immediately implies

COROLLARY 4.4.2. *The variant complex $(\check{\chi}_* \mathbb{Q}_{\check{\mathcal{M}}})_{\mathrm{var}}$ is supported on $\cup_{\gamma \in \Gamma^*} \mathcal{A}_\gamma^0 = \mathcal{A}^0 \setminus \mathcal{A}_{ne}^0$.*

Proof. Taking the cohomology sheaves of the decomposition (4.4.1), we have that Γ acts as multiplication by the character $\kappa \in \hat{\Gamma}$ on $\mathcal{H}^i((\chi_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa)$. By Proposition 4.4.1,

$$\text{if } s \in \mathcal{A}_{ne}^0 \text{ and } \kappa \neq 0, \text{ then } \mathcal{H}^i((\chi_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa)_s = 0 \text{ for all } i;$$

therefore the restriction of $\bigoplus_{\kappa \in \hat{\Gamma} \setminus \{0\}} (\chi_* \mathbb{Q}_{\check{\mathcal{M}}})_\kappa$ to $\mathcal{A}_{ne}^0$ vanishes. $\square$

In order to prove Proposition 4.4.1, we show that the action of Γ on $\check{\mathcal{M}}_s$, for $s \in \mathcal{A}_{ne}^0$, is the restriction to Γ of an action of a connected group $\mathrm{Prym}_{C_s/C}$; as such Γ acts trivially on $H^*(\check{\mathcal{M}}_s)$. We begin with some preliminary considerations on the norm map.

Fix $s \in \mathcal{A}_{ne}^0$, and let $\pi : C_s \to C$ be the corresponding spectral cover. When $s \in \mathcal{A}_0^0$, we have that $C_s = C_1 \cup C_2$ is reducible; otherwise C_s is an integral curve. We denote by $\mathrm{Pic}_{C_s}^0$ the connected component of the identity of Pic_{C_s}. Denote by $\nu : \widetilde{C}_s \to C_s$ the normalization. Define the norm map $\mathrm{Nm}_{C_s/C} : \mathrm{Pic}_{C_s}^0 \to \mathrm{Pic}_C^0$ by

$$(4.4.5) \qquad \mathrm{Nm}_{C_s/C} := \mathrm{Nm}_{\widetilde{C}_s/C} \circ \nu^*$$

where, for a divisor D on the nonsingular curve $\widetilde{C}_s$, the norm map

$$\mathrm{Nm}_{\widetilde{C}_s/C}(\mathcal{O}(D)) = \mathcal{O}((\pi \circ \nu)_* D)$$

is the classical one. Consequently $\mathrm{Nm}_{C_s/C} : \mathrm{Pic}^0(C_s) \to \mathrm{Pic}^0(C)$ is a group homomorphism.

By Proposition 3.8 in [31], we have the following alternative formula for the norm map:

$$(4.4.6) \qquad \mathrm{Nm}_{C_s/C}(\mathcal{L}) = \det(\pi_*(\mathcal{L})) \otimes \det(\pi_*(\mathcal{O}))^{-1}.$$

As $\mathrm{Nm}_{C_s/C} : \mathrm{Pic}^0_{C_s} \to \mathrm{Pic}^0_C$ is a group homomorphism, the kernel $\mathrm{Prym}_{C_s/C} := \mathrm{Nm}^{-1}_{C_s/C}(\mathcal{O}_C)$ is a subgroup of $\mathrm{Pic}^0_{C_s}$.

LEMMA 4.4.3. *If $s \in \mathcal{A}^0_{ne}$, then the group* $\mathrm{Prym}_{C_s/C} = \mathrm{Nm}^{-1}_{C_s/C}(\mathcal{O}_C)$ *is connected.*

Proof. The case of an integral spectral curve C_s is treated in [44, §11]. The argument is easily adapted to the case of a reducible and reduced curve. For a proof more in the spirit of the present paper, see [31, Th. 1.2]. □

LEMMA 4.4.4. *Let $\pi : X \to Y$ be a degree two map from a reduced projective curve X to a nonsingular projective curve Y. Let $\mathcal{E}$ be a rank 1 torsion free sheaf on X and $\mathcal{L}$ an invertible one. Then*

$$(4.4.7) \qquad \det(\pi_*(\mathcal{L} \otimes \mathcal{E})) = \det(\pi_*(\mathcal{E})) \otimes \mathrm{Nm}_{X/Y}(\mathcal{L}).$$

Proof. First we note that by Theorem 2.2.2, we have that there is a unique partial normalization $\pi' : X' \to X$ and an invertible sheaf $\mathcal{L}'$ on X' such that $\mathcal{E} = \pi'_*(\mathcal{L}')$. The multiplicativity of the norm map and (4.4.6) imply

$$\det((\pi \circ \pi')_*((\pi')^*(\mathcal{L}) \otimes \mathcal{L}')) = \det((\pi \circ \pi')_*(\mathcal{L}')) \otimes \mathrm{Nm}_{X'/Y}((\pi')^*(\mathcal{L})).$$

This together with $\mathrm{Nm}_{X'/Y} \circ (\pi')^* = \mathrm{Nm}_{X/Y}$ yields the result. □

Proof of Proposition 4.4.1. We first prove that $\mathrm{Prym}_{C_s/C}$ acts on the fiber $\check{\mathcal{M}}_s$. Recall that $\check{\mathcal{M}}_s$ can be identified with pure rank 1 torsion-free sheaves $\mathcal{E}$ on C_s for which the corresponding Higgs bundle $(\pi_*(\mathcal{E}), \phi_\mathcal{E})$ is stable, and $\det(\pi_*(\mathcal{E})) \cong \Lambda$. (Note that $\mathrm{tr}(\phi_\mathcal{E}) = 0$ is automatic as $s \in H^0(C; 2K_C)$.) Now, $\mathcal{L} \in \mathrm{Prym}_{C_s/C}$ acts on $\mathcal{E}$ as $\mathcal{E} \mapsto \mathcal{L} \otimes \mathcal{E}$, where the result is again a pure rank 1 torsion-free sheaf on C_s. It follows that $(\pi_*(\mathcal{L} \otimes \mathcal{E}), \phi_{\mathcal{L} \otimes \mathcal{E}})$ is a rank 2 Higgs bundle. By Lemma 4.4.4, we have that

$$(4.4.8) \qquad \det(\pi_*(\mathcal{L} \otimes \mathcal{E})) = \Lambda.$$

Finally, we prove that tensoring with an element of $\mathrm{Prym}_{C_s/C}$ preserves stability. Assume there is a rank 1 Higgs subbundle $(L_\mathcal{F}, \phi_\mathcal{F})$ of $(\pi_*(\mathcal{L} \otimes \mathcal{E}), \phi_{\mathcal{L} \otimes \mathcal{E}})$ corresponding to the torsion-free sheaf $\mathcal{F} \subset \mathcal{L} \otimes \mathcal{E}$ on C_s. Then the spectral curve $C_\mathcal{F} = \mathrm{supp}(\mathcal{F})$ of $(L_\mathcal{F}, \phi_\mathcal{F})$ must be a subscheme of C_s, so that $C_\mathcal{F} \to C$ is degree 1. Thus $C_s = C_1 \cup C_2$ must be reducible and $\mathcal{F}$ be supported on one of the components, say $C_\mathcal{F} = C_1$, with $\pi_1 := \pi|_{C_1} : C_1 \overset{\cong}{\to} C$ an isomorphism. As $L_\mathcal{F} = \pi_*(\mathcal{F})$ we can identify it with $\mathcal{F}|_{C_1}$. Finally since

$\mathcal{L} \in \operatorname{Prym}_{C_s/C} \subset \operatorname{Pic}^0_{C_s}$, we have that $\deg(\mathcal{L}|_{C_1}) = 0$, and so
$$\deg(L_{\mathcal{L}^{-1} \otimes \mathcal{F}}) = \deg(\mathcal{L}^{-1}|_{C_1} \otimes \mathcal{F}|_{C_1}) = \deg(\mathcal{F}|_{C_1}) = \deg(L_{\mathcal{F}}).$$
To summarize, $(\pi_*(\mathcal{E}), \phi_{\mathcal{E}})$ and $(\pi_*(\mathcal{L} \otimes \mathcal{E}), \phi_{\mathcal{L} \otimes \mathcal{E}})$ have the same degree sub Higgs-bundles. It follows that if $(\pi_*(\mathcal{E}), \phi_{\mathcal{E}})$ is stable, then so is $(\pi_*(\mathcal{L} \otimes \mathcal{E}), \phi_{\mathcal{L} \otimes \mathcal{E}})$.

We thus proved that the connected group scheme $\operatorname{Prym}_{C_s/C}$ acts on $\check{\mathcal{M}}_s$. For any order 2 line bundle $L \in \Gamma = \operatorname{Pic}^0_C[2]$, we have $\operatorname{Nm}_{C_s/C}(\pi_s^*(L)) = L^2 = \mathcal{O}_C$; therefore $\pi^*(L) \in \operatorname{Prym}_{C_s/C}$, and consequently $\Gamma \subset \operatorname{Prym}_{C_s/C}$ acts trivially on $H^*(\check{\mathcal{M}}_s)$. $\qquad\square$

Remark 4.4.5. For a more detailed calculation of the group of components of Prym varieties of spectral covers, see [31].

We now introduce the notation $H^k_{\leq p, \mathrm{var}} := P_p \cap H^k_{\mathrm{var}}(\check{\mathcal{M}})$ and $H^k_{p, \mathrm{var}} := \operatorname{Gr}^P_p(H^k_{\mathrm{var}}(\check{\mathcal{M}}))$.

THEOREM 4.4.6. *The perverse Leray filtration on $H^*_{\mathrm{var}}(\check{\mathcal{M}})$ satisfies*
$$0 = H^k_{\leq k-2g+1, \mathrm{var}}(\check{\mathcal{M}}) \subseteq H^k_{\leq k-2g+2, \mathrm{var}}(\check{\mathcal{M}}) = H^k_{\mathrm{var}}(\check{\mathcal{M}}).$$

Proof. Since $\dim \mathcal{A}^0 \setminus \mathcal{A}^0_{ne} = g - 1$, a general $(2g - 3)$-dimensional linear subspace Λ^{2g-3} of the $(3g - 3)$-dimensional affine base $\mathcal{A}^0$ lies entirely inside $\mathcal{A}^0_{ne}$. By Proposition 4.4.1, the restriction of a class in $H^*_{\mathrm{var}}(\check{\mathcal{M}})$ to $H^*(\check{\mathcal{M}}_{ne})$, and thus to $H^*(\check{\mathcal{M}}|_{\Lambda^{2g-3}})$, is trivial. This fact, coupled with the test for perversity given by Theorem 1.4.8, implies the inclusion $H^k_{\mathrm{var}}(\check{\mathcal{M}}) \subseteq H^k_{\leq k-2g+2, \mathrm{var}}(\check{\mathcal{M}})$.

We are left with proving that

(4.4.9) $$H^k_{\leq k-2g+1, \mathrm{var}}(\check{\mathcal{M}}) = 0.$$

Let k be the smallest integer such that $H^k_{\leq k-2g+1, \mathrm{var}}(\check{\mathcal{M}}) \neq 0$. We thus have that
$$H^{k'}_{\leq k'-2g+1, \mathrm{var}}(\check{\mathcal{M}}) = 0, \qquad \forall\, k' < k.$$
By combining this vanishing with the equality established above, we deduce that

(4.4.10) $$H^{k'}_{k'-2g+2, \mathrm{var}}(\check{\mathcal{M}}) \cong H^{k'}_{\mathrm{var}}(\check{\mathcal{M}}).$$

The class α is Γ-invariant, so that cupping with the powers of α respects the Γ-decomposition. In particular, by relative hard Lefschetz, we see that cupping with the appropriate power of α yields an isomorphism of graded groups

(4.4.11) $$H^{k'}_{k'-2g+2, \mathrm{var}}(\check{\mathcal{M}}) \cong H^{10g-10-k'}_{8g-8-k', \mathrm{var}}(\check{\mathcal{M}}).$$

In view of (4.4.3), we have that $\dim H^{k'}_{\mathrm{var}}(\check{\mathcal{M}}) = \dim H^{10g-10-k'}_{\mathrm{var}}(\check{\mathcal{M}})$ so that, by (4.4.11) and by (4.4.10), we have $H^{10g-10-k'}_{8g-8-k', \mathrm{var}}(\check{\mathcal{M}}) \cong H^{10g-10-k'}_{\mathrm{var}}(\check{\mathcal{M}})$,

and consequently

$$(4.4.12) \qquad H^{10g-10-k'}_{\leq 8g-9-k',\mathrm{var}}(\check{\mathcal{M}}) = 0.$$

By our choice of k, we have that $H^k_{\leq k-2g+1,\mathrm{var}}(\check{\mathcal{M}}) \neq 0$, so that there is $l < k$ such that

$$H^k_{l-2g+2,\mathrm{var}}(\check{\mathcal{M}}) \neq 0.$$

As above, the relative hard Lefschetz yields

$$H^{10g-10+k-2l}_{8g-8-k',\mathrm{var}}(\check{\mathcal{M}}) \neq 0.$$

In view of the fact that $k > l > 2l - k$, this contradicts (4.4.12) with the choice of $k' = 2l - k$, and (4.4.9) follows. $\qquad \square$

Theorem 4.4.6 determines the perverse Leray filtration on the Γ-variant part $H^*_{\mathrm{var}}(\check{\mathcal{M}})$. In the course of the proof we have proved that for $2d < g$, the Lefschetz map

$$(4.4.13) \qquad \cup \alpha^{g-2d} : H^{4g+2d-5}_{\mathrm{var}}(\check{\mathcal{M}}) \longrightarrow H^{6g-5-2d}_{\mathrm{var}}(\check{\mathcal{M}})$$

is an isomorphism.

Now we determine the mixed Hodge structure on the cohomology of the character variety $\check{\mathcal{M}}_{\mathrm{B}}$. Notice that the direct sum decomposition $H^*(\check{\mathcal{M}}_{\mathrm{B}}) \simeq H^*(\check{\mathcal{M}}_{\mathrm{B}})^\Gamma \oplus H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$, being associated with the algebraic action of a group, is a decomposition into a direct sum of Mixed Hodge structures.

Theorem 4.4.7.

$$(4.4.14) \quad 0 = W_{2k-4g+3}H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) \subset W_{2k-4g+4}H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) = H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$$

and

$$(4.4.15) \quad 0 = F^{k-2g+2}H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) \subset F^{k-2g+1}H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) = H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}).$$

Proof. As the invariant part $H^*(\check{\mathcal{M}}_{\mathrm{B}})^\Gamma \cong H^*(\hat{\mathcal{M}}_{\mathrm{B}})$, we have

$$\begin{aligned}
(4.4.16) \quad E_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}; x, y) &:= \sum_{d,i,j} x^i y^j (-1)^k \dim \left(\mathrm{Gr}^W_{i+j} H^k_{c,\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) \right)^{ij}_{\mathbb{C}} \\
&= E(\check{\mathcal{M}}_{\mathrm{B}}; x, y) - E(\hat{\mathcal{M}}_{\mathrm{B}}; x, y) \\
&= (2^{2g} - 1)(xy)^{2g-2} \left(\frac{(xy-1)^{2g-2} - (xy+1)^{2g-2}}{2} \right) \\
&= \sum_{i=1}^{g-1} (2^{2g} - 1) \, \binom{2g-2}{2i-1} (xy)^{2g-3+2i},
\end{aligned}$$

where $E(\hat{\mathcal{M}}_{\mathrm{B}}; x, y)$ is given by the right-hand side of [32, (1.1.3)] and $E(\check{\mathcal{M}}_{\mathrm{B}}; x, y)$ is given by [41, (4.6)] with $q = xy$.

We first observe that

$$E_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}; 1/x, 1/y) = (xy)^{6g-6} E_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}; x, y)$$

is palindromic. Consequently, by Poincaré duality the corresponding expression on ordinary cohomology

$$\sum_{d,i,j} x^i y^j (-1)^k \dim(\mathrm{Gr}^W_{i+j} H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})_{\mathbb{C}})^{ij} = E_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}; x, y)$$

is thus also given by (4.4.16). Now we note that (4.4.16) only depends on xy; i.e., every term is of the form $x^p y^p$. Additionally, by (4.4.3), if $H^k_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}) \neq 0$, then k is odd, and every nontrivial $\dim \left(\mathrm{Gr}^W_{i+j} H^d(\check{\mathcal{M}}_{\mathrm{B}})_{\mathbb{C}} \right)^{i,j}$ will contribute with a negative coefficient; thus there is no cancellation and the only nontrivial terms are of the form $(\mathrm{Gr}^W_{2p} H^d_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})_{\mathbb{C}})^{p,p}$. It follows that the mixed Hodge structure on $H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}; x, y)$ is of Hodge-Tate type; thus (4.4.15) follows from (4.4.14).

We now determine the weights on $H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$. Again, as $H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$ is only nontrivial in odd cohomology, weights cannot cancel each other since they all contribute with a negative coefficient. The possible weights therefore are $4g - 2, 4g + 2, \ldots, 8g - 10$ (twice the degrees in xy of the monomials appearing in (4.4.16)), with multiplicities that turn out to be equal to

$$\dim H^{4g-3}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}), \dim H^{4g-1}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})) , \ldots, \dim H^{6g-7}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$$

respectively. (These appear as the coefficients in (4.4.16).) To conclude, we need to show that they will be the weights on the cohomologies

$$H^{4g-3}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}), H^{4g-1}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}}), \ldots, H^{6g-7}_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$$

respectively. This follows from (4.4.13) by an argument similar to the proof of (4.4.6). □

COROLLARY 4.4.8. *We have that $P = W$ on $H^*_{\mathrm{var}}(\check{\mathcal{M}})$ and consequently on $H^*(\check{\mathcal{M}})$.*

Remark 4.4.9. Note also that this implies a complete description of the ring $H^*(\check{\mathcal{M}})$. We already know the ring structure on the invariant part. Now, α acts on $H^*_{\mathrm{var}}(\check{\mathcal{M}})$ as described in (4.4.13). Also, β and ψ_i act trivially, since their weights are such that when multiplied with any class in $H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$, it would provide a class with a degree and weight that does not exist in $H^*_{\mathrm{var}}(\check{\mathcal{M}}_{\mathrm{B}})$. Finally by degree reasons variant classes multiply to 0. This implies a complete description of the ring structure on $H^*(\check{\mathcal{M}})$.

Remark 4.4.10. The determination of the perverse filtration on $H^*_{\mathrm{var}}(\check{\mathcal{M}})$ using the symmetry provided by the group scheme $Prym$ is inspired by Ngô's approach in [44], [45]. More connections to his work is discussed in [27, §5.2].

5. **Appendix**

In this appendix we recall a result of M. Thaddeus, Proposition 5.1.2 below, concerning the restriction of the generators α, ψ_i, β defined in Section 1.2.2 to a general fibre of the Hitchin fibration for $\check{\mathcal{M}}$. In view of Theorem 1.4.8, this yields an upper bound for their perversity, which is used in Theorem 3.1.1. Since these results, contained in Thaddeus' Master thesis ([51]), have not been published, we report here the original proof.

Let $s \in \mathcal{A}_{\mathrm{reg}}^0$, let $\pi := \pi_s : C_s \to C$ be the corresponding spectral curve covering, and let $\iota := \iota_s : C_s \longrightarrow C$ be the involution exchanging the two sheets of the covering (see §1.3.2). The following easy to prove facts are used in the course of the proof of Proposition 5.1.2.

- The involution ι induces ι_* on $H_1(C_s)$, which splits into the ± 1-eigenspaces
$$H_1(C_s) = H_1(C_s)^+ \oplus H_1(C_s)^-.$$
We denote by $\Pi^{\pm}$ the corresponding projections.

Analogously,
$$H^1(C_s) = H^1(C_s)^+ \oplus H^1(C_s)^-$$
and, via Poincaré duality,
$$H^1(C_s)^+ \simeq (H_1(C_s)^+)^{\vee}, \ H^1(C_s)^- \simeq (H_1(C_s)^-)^{\vee}.$$
The projections are still denoted by $\Pi^{\pm}$.

- We let $[C] \in H_2(C) \simeq H^0(C)$, $[C_s] \in H_2(C_s) \simeq H^0(C_s)$ be the fundamental classes and $[c] \in H_0(C) \simeq H^2(C)$, $[c_s] \in H_0(C_s) \simeq H^2(C_s)$ the classes of a point in C, resp. C_s.

We have

(5.1.1) $\pi_*([c_s]) = [c], \ \pi_*([C_s]) = 2[C], \ \mathrm{Ker}\,\pi_* = H^1(C_s)^-,$

the last equality due to the fact that $\pi \circ \iota = \pi$ and π_* is surjective.

- In terms of the identification $H^1(C_s)^+ \simeq H^1(C)$ given by the pull-back map π^*, the map

(5.1.2) $\pi_* : H^1(C_s) \longrightarrow H^1(C) \simeq H^1(C_s)^+$

is identified with the projection Π^+.

Remark 5.1.1. We have isomorphisms $\chi^{-1}(s) \simeq \mathrm{Pic}_{C_s}^0$, $\check{\chi}^{-1}(s) \simeq \mathrm{Prym}_{C_s}$ and $\hat{\chi}^{-1}(s) \simeq \mathrm{Prym}_{C_s}/\Gamma$. (See Theorem 1.3.4 for χ and §2.4 for $\check{\chi}$ and $\hat{\chi}$.) We have the canonical isomorphisms
$$H^1(\mathrm{Pic}_{C_s}^0) \simeq H_1(C_s), \qquad H^1(\mathrm{Prym}_{C_s}) \simeq H_1(C_s)^-,$$
in terms of which the restriction map $H^1(\mathrm{Pic}_{C_s}^0) \longrightarrow H^1(\mathrm{Prym}_{C_s})$ is identified with the projection $\Pi^- : H_1(C_s) \longrightarrow H_1(C_s)^-.$

PROPOSITION 5.1.2.
(1) *The restrictions of the classes ψ_i, β to a general fibre of the Hitchin maps $\chi, \hat{\chi}$, and $\check{\chi}$ vanish.*
(2) *The restriction of the class α to a general fibre of the Hitchin maps χ is nonzero. When restricted to a general fibre of $\hat{\chi}$ and $\check{\chi}$, the class α is an ample class.*

Proof. For notational simplicity, we denote the fundamental classes $[C] \in H^0(C)$, $[C_s] \in H^0(C_s)$, $[\mathrm{Pic}^0_{C_s}] \in H^0(\mathrm{Pic}^0_{C_s})$, and $[\mathrm{Prym}_{C_s}] \in H^0(\mathrm{Prym}_{C_s})$ by 1 so that, for example, the second equality in (5.1.1) above reads $\pi_*(1) = 2$.

Let $s \in \mathcal{A}^0_{\mathrm{reg}}$. As discussed in Section 1.3.1, the map $\check{\chi}$ is the restriction of χ to $\check{\mathcal{M}}$, and $\hat{\chi}$ is derived from $\check{\chi}$ by passing to the quotient $\hat{\mathcal{M}} = \check{\mathcal{M}}/\Gamma$. It is therefore enough to prove the first statement for the map χ We denote the product map $\pi \times \mathrm{id} : C_s \times \mathrm{Pic}^0_{C_s} \longrightarrow C \times \mathrm{Pic}^0_{C_s}$ simply by π.

Let $\mathcal{L}$ be the Poincaré line bundle on $C_s \times \mathrm{Pic}^0_{C_s}$. The restriction of $\mathbb{E}$ to $C \times \chi^{-1}(s) \simeq C \times \mathrm{Pic}^0_{C_s}$ is isomorphic to $\pi_*\mathcal{L}$ (see §1.3.2). The Grothendieck-Riemann-Roch theorem (see [24, Th. 15.2]) gives the following equality in $H^*(C \times \mathrm{Pic}^0_{C_s})$:
(5.1.3)
$$\mathrm{ch}(\mathbb{E}_{|C \times \chi^{-1}(s)})\mathrm{td}(C \times \mathrm{Pic}^0_{C_s}) = \mathrm{ch}(\pi_*\mathcal{L})\mathrm{td}(C \times \mathrm{Pic}^0_{C_s}) = \pi_*(\mathrm{ch}(\mathcal{L})\mathrm{td}(C_s \times \mathrm{Pic}^0_{C_s})).$$

By the Künneth formula and Poincaré duality,
$$H^2(C_s \times \mathrm{Pic}^0_{C_s}) \simeq \left(H^0(C_s) \otimes H^2(\mathrm{Pic}^0_{C_s})\right)$$
$$\oplus \left(H^1(C_s) \otimes H_1(C_s)\right) \oplus \left(H^2(C_s) \otimes H^0(\mathrm{Pic}^0_{C_s})\right)$$
and we have the natural isomorphism
(5.1.4)
$$\mathrm{End}\, H^1(C_s) \simeq H^1(C_s) \otimes H_1(C_s) \simeq H^1(C_s) \otimes H^1(\mathrm{Pic}^0_{C_s}) \subseteq H^2(C_s \times \mathrm{Pic}^0_{C_s}).$$

We say that a class in $H^*(C \times \mathrm{Pic}^0_{C_s})$ has type (a, b) if it is in the Künneth summand $H^a(C) \otimes H^b(\mathrm{Pic}^0_{C_s})$, and similarly for classes in $H^*(C_s \times \mathrm{Pic}^0_{C_s})$. Clearly the cup product of a class of type (a, b) with one of type (c, d) is of type $(a + c, b + d)$; in particular, it vanishes if $a + c > 2$.

We set, for simplicity, $g' := g(C_s)$. We fix a symplectic basis $\delta_1, \ldots, \delta_{2g'}$ for $H^1(C_s)$, and we identify its dual basis, $\delta_1^\vee, \ldots, \delta_{2g'}^\vee$, with a basis for $H^1(\mathrm{Pic}^0_{C_s})$. In terms of the isomorphism (5.1.4) above, the first Chern class of $\mathcal{L}$ is represented by the identity in $\mathrm{End}\, H^1(C_s)$ (see [3, VIII.2]), namely $c_1(\mathcal{L}) = \sum_i \delta_i \otimes \delta_i^\vee$; hence it is of type $(1,1)$. Let $[c_s] \in H^2(C_s)$, $[c] \in H^2(C)$ be as defined above.

A direct computation gives $c_1^2(\mathcal{L}) = 2[c_s] \otimes (\sum_i \delta_i^\vee \wedge \delta_{i+g'}^\vee) = 2[c_s] \otimes \theta$, where $\theta := \sum_i \delta_i^\vee \wedge \delta_{i+g'}^\vee \in H^2(\mathrm{Pic}^0_{C_s})$ denotes the cohomology class of the theta divisor on $\mathrm{Pic}^0_{C_s}$.

Since $c_1(\mathcal{L})$ is of type $(1,1)$, we have $c_1^r(\mathcal{L}) = 0$ if $r \geq 3$; hence

$$\text{ch}(\mathcal{L}) = 1 + c_1(\mathcal{L}) + \frac{c_1^2(\mathcal{L})}{2} = 1 + \sum_i \delta_i \otimes \delta_i^\vee + [c_s] \otimes \theta.$$

Since the tangent bundle of a torus is trivial, we have that $\text{td}(\text{Pic}_{C_s}^0) = 1$, while $\text{td}(C_s) = 1 + (1-g')[c_s]$ and $\text{td}(C) = 1 + (1-g)[c]$. By the multiplicativity properties of the Todd class, we have that

$$\text{td}(C_s \times \text{Pic}_{C_s}^0) = 1 \otimes 1 + (1-g')[c_s] \otimes 1, \qquad \text{td}(C \times \text{Pic}_{C_s}^0) = 1 \otimes 1 + (1-g)[c] \otimes 1,$$

so that

$$\text{ch}(\mathcal{L})\text{td}(C_s \times \text{Pic}_{C_s}^0) = 1 \otimes 1 + \sum_i \delta_i \otimes \delta_i^\vee + [c_s] \otimes \theta + (1-g')[c_s] \otimes 1.$$

Plugging this into the Grothendieck-Riemann-Roch theorem formula (5.1.3), we get

$$(5.1.5) \qquad \text{ch}(\mathbb{E}_{|C \times \chi^{-1}(s)})(1 + (1-g)[c])$$
$$= \pi_* \left(1 \otimes 1 + \sum_i \delta_i \otimes \delta_i^\vee + [c_s] \otimes \theta + (1-g')[c_s] \otimes 1 \right).$$

Applying (5.1.1) gives $\pi_*(1 \otimes 1) = 2 \otimes 1$, $\pi_*([c_s] \otimes \theta) = [c] \otimes \theta$, and $\pi_*([c_s] \otimes 1) = [c] \otimes 1$. Combining the third equality of (5.1.1) with the isomorphism $H^1(C) \simeq H^1(C_s)^+$, we get $\pi_*(\sum_i \delta_i \otimes \delta_i^\vee) = \sum_i \Pi^+(\delta_i) \otimes \delta_i^\vee$.

By plugging the above equalities in equation (5.1.5) and equating the components of degree 2, we deduce that

$$(5.1.6) \quad c_1(\mathbb{E}_{|C \times \chi^{-1}(s)}) + 2(1-g)[c] \otimes 1 = \sum_i \Pi^+(\delta_i) \otimes \delta_i^\vee + (1-g')[c] \otimes 1.$$

Since the product of $[c] \otimes 1$ with a class not of type $(0,2)$ vanish, we obtain

$$(5.1.7) \qquad c_1^2(\mathbb{E}_{|C \times \chi^{-1}(s)}) = \left(\sum_i \Pi^+(\delta_i) \otimes \delta_i^\vee \right)^2,$$

which has type $(2,2)$. Equating the components of degree 4 in equation (5.1.5) and using the fact that, by type consideration, the product $c_1(\mathbb{E}_{|C \times \chi^{-1}(s)}) \cdot ([c] \otimes 1) = 0$, we have

$$(5.1.8) \qquad \frac{1}{2}\left(c_1^2(\mathbb{E}_{|C \times \chi^{-1}(s)}) - 2c_2(\mathbb{E}_{|C \times \chi^{-1}(s)}) \right) = [c] \otimes \theta,$$

from which we deduce that also $c_2(\mathbb{E}_{|C \times \chi^{-1}(s)})$ has type $(2,2)$. From the equality, true in general for rank 2 vector bundles,

$$c_2(\text{End}\,\mathbb{E}_{|C \times \chi^{-1}(s)}) = 4c_2(\mathbb{E}_{|C \times \chi^{-1}(s)}) - c_1^2(\mathbb{E}_{|C \times \chi^{-1}(s)}),$$

it follows that $c_2(\text{End}\,\mathbb{E}_{|C \times \chi^{-1}(s)})$ has type $(2,2)$, which, by the very definition (see the defining equation 1.2.10 in §1.2.2) of β and ψ_i, means that these classes vanish on $\chi^{-1}(s)$.

The proof of the second statement is immediately reduced to the case of $\check{\chi}$. By Remark 5.1.1, the restriction of the first Chern class of $\mathcal{L}$ to $C_s \times \mathrm{Prym}_{C_s}$ is $\sum_i \delta_i \otimes \Pi^-(\delta_i^\vee)$, and the Grothendieck-Riemann-Roch theorem now reads

$$(5.1.9) \qquad \mathrm{ch}(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)})(1 + (1 - g)[c])$$

$$= \pi_* \left(1 \otimes 1 + \sum_i \delta_i \otimes \Pi^-(\delta_i^\vee) + [c_s] \otimes \theta + (1 - g')[c_s] \otimes 1 \right),$$

where θ denotes now the cohomology class of the restriction of the theta divisor to Prym_{C_s}. As in the first part of the proof, we apply Formulæ 5.1.1, giving $\pi_*(1 \otimes 1) = 2 \otimes 1$, $\pi_*([c_s] \otimes \theta) = [c] \otimes \theta$, $\pi_*([c_s] \otimes 1) = [c] \otimes 1$, and $\pi_*(\sum_i \delta_i \otimes \delta_i^\vee) = \sum_i \Pi^+(\delta_i) \otimes \Pi^-(\delta_i^\vee) = 0$.

By plugging the above equalities in equation (5.1.5), we deduce that

$$(5.1.10) \qquad c_1(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) + 2(1 - g)[c] = (1 - g')[c],$$

so that $c_1^2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) = 0$ and $\left(c_1^2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) - 2c_2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) \right) = 2[c] \otimes \theta$. Hence, $c_2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)})) = -[c] \otimes \theta$, and finally,

$$(5.1.11) \quad c_2(\mathrm{End}\, \mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) = 4c_2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) - 4c_1^2(\mathbb{E}_{|C \times \check{\chi}^{-1}(s)}) = -4[c] \otimes \theta,$$

which shows (cf. (1.2.10) in §1.2.2) that the restriction of α to $\check{\chi}^{-1}(s)$ equals 4θ. $\qquad \square$

COROLLARY 5.1.3. *The class α is ample on $\hat{\mathcal{M}}$.*

Proof. The variety $\hat{\mathcal{M}} = \mathcal{M}^0/\mathrm{Pic}_C^0$ is quasiprojective as $\mathcal{M}$ is by [46, Prop. 7.4]. Additionally it follows from Theorem 1.2.10 that $\dim H^2(\hat{\mathcal{M}}) = 1$. As we proved above that α is ample on the generic fiber of $\hat{\chi}$ it must be ample on $\hat{\mathcal{M}}$ as well. $\qquad \square$

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STONY BROOK UNIVERSITY, STONY BROOK, NY
E-mail: mde@math.sunysb.edu

UNIVERSITY OF OXFORD, OXFORD, UK
E-mail: hausel@maths.ox.ac.uk

UNIVERSITÀ DI BOLOGNA, BOLOGNA, ITALY
E-mail: migliori@dm.unibo.it