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COHOMOLOGY OF LARGE SEMIPROJECTIVE  
HYPERKÄHLER VARIETIES

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## COHOMOLOGY OF LARGE SEMIPROJECTIVE HYPERKÄHLER VARIETIES

*by*

Tamás Hausel & Fernando Rodriguez Villegas

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*À Gérard Laumon à l'occasion de son 60<sup>ème</sup> anniversaire*

**Abstract.** — In this paper we survey geometric and arithmetic techniques to study the cohomology of semiprojective hyperkähler manifolds including toric hyperkähler varieties, Nakajima quiver varieties and moduli spaces of Higgs bundles on Riemann surfaces. The resulting formulae for their Poincaré polynomials are combinatorial and representation theoretical in nature. In particular we will look at their Betti numbers and will establish some results and state some expectations on their asymptotic shape.

**Résumé (Cohomologie des variétés hyperkähleriennes semiprojectives grandes).** — Dans cet article, nous passons en revue les techniques géométriques et arithmétiques pour étudier la cohomologie des variétés hyperkähleriennes semiprojectives, en particulier les variétés hyperkähleriennes toriques, les variétés de carquois de Nakajima et les espaces de modules de fibrés de Higgs sur les surfaces de Riemann. Les formules obtenues pour leurs polynômes de Poincaré sont de nature combinatoire et liées à la théorie des représentations. En particulier, nous étudions leurs nombres de Betti et nous établissons des résultats et formulons quelques hypothèses sur leur comportement asymptotique.

At the conference “De la géométrie algébrique aux formes automorphes: une conférence en l’honneur de Gérard Laumon” the first author gave a talk, whose subject is well-documented in the survey paper [Ha4]. Here, instead, we will discuss techniques, both geometrical and arithmetic, for obtaining information on the cohomology of semiprojective hyperkähler varieties and we will report on some observations on the asymptotic behaviour of their Betti numbers in certain families of examples.

We call  $X$  a smooth quasi-projective variety with a  $\mathbb{C}^\times$ -action *semiprojective* when the fixed point set  $X^{\mathbb{C}^\times}$  is projective and for every  $x \in X$  and as  $\lambda \in \mathbb{C}^\times$  tends to 0 the limit  $\lim_{\lambda \rightarrow 0} \lambda x$  exists.

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Varieties with these assumptions were originally studied by Simpson in [Si2, §11] and varieties with similar assumptions were studied by Nakajima in [Na3, §5.1]. The terminology semiprojective in this context appeared in [HS], which concerned semiprojective toric varieties and toric hyperkähler varieties. In particular, a large class of hyperkähler varieties, which arise as a hyperkähler quotient of a vector space by a gauge group, are semiprojective. These include Hilbert schemes of  $n$ -points on  $\mathbb{C}^2$ , Nakajima quiver varieties and moduli spaces of Higgs bundles on Riemann surfaces.

It turns out that despite their simple definition we can say quite a lot about the geometry and cohomology of semiprojective varieties. We can construct a Bialynicki-Birula stratification (§1.2), which in §1.3 will give a perfect Morse stratification in the sense of Atiyah-Bott. This way we will be able to deduce that the cohomology of a semiprojective variety is isomorphic with the cohomology of the fixed point set  $X^{\mathbb{C}^\times}$  with some cohomological shifts. Also, the opposite Bialynicki-Birula stratification will stratify a projective subvariety  $\mathcal{C} \subset X$  of the semiprojective variety, the so-called *core*, which turns out to be a deformation retract of  $X$ . This way we can deduce that the cohomology  $H^*(X; \mathbb{C})$  is always pure. Furthermore, we can compactify  $\overline{X} = X \amalg Z$  with a divisor  $Z$ , to get an orbifold  $\overline{X}$ . Finally in §1.4 we will look at a version of a weak form of the Hard Lefschetz theorem satisfied by semiprojective varieties.

We will also discuss arithmetic approach to obtain information on the cohomology of our hyperkähler varieties. It turns out that the algebraic symplectic quotient construction of our hyperkähler varieties will enable us to use a technique we call *arithmetic harmonic analysis* to count the points of our hyperkähler varieties over finite fields. With this technique we can effectively determine the Betti numbers of the toric hyperkähler varieties and Nakajima quiver varieties as well as formulate a conjectural expression for the Betti numbers of the moduli space of Higgs bundles.

To test the range in which the Weak Hard Lefschetz theorem of §1.4 might hold, we will look at the graph of Betti numbers for our varieties when their dimension is very large. The resulting pictures are fairly similar and we observe that asymptotically they seem to converge to the graph of some continuous functions. We will see, for example, the normal, Gumbel and Airy distributions emerging in the limit in our examples. We will conclude the paper with some proofs and heuristics towards establishing such facts.

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### 1. Semiprojective varieties

**1.1. Definition and examples.** — We start with the definition of a semiprojective variety, first considered in [Si2, Theorem 11.2].

**Definition 1.1.1.** — Let  $X$  be a complex quasi-projective algebraic variety with a  $\mathbb{C}^\times$ -action. We call  $X$  *semiprojective* when the following two assumptions hold:

1. The fixed point set  $X^{\mathbb{C}^\times}$  is proper.
2. For every  $x \in X$  the  $\lim_{\lambda \rightarrow 0} \lambda x$  exists as  $\lambda \in \mathbb{C}^\times$  tends to 0.

The second condition could be phrased more algebraically as follows: for every  $x \in X$  we have an equivariant map  $f : \mathbb{C} \rightarrow X$  such that  $f(1) = x$  and  $\mathbb{C}^\times$  acts on  $\mathbb{C}$  by multiplication.

First example is a projective variety with a trivial (or any)  $\mathbb{C}^\times$ -action. For a large class of non-projective examples one can take the total space of a vector bundle on a projective variety, which together with the canonical  $\mathbb{C}^\times$ -action will become semiprojective.

A good source of examples arise by taking GIT quotients of linear group actions of reductive groups on vector spaces. Examples include the semiprojective toric varieties of [HS] (even though the definition of semiprojectiveness is different there, but equivalent with ours) and quiver varieties studied by Reineke [Re1].

*1.1.1. Semiprojective hyperkähler varieties.* — In this survey we are interested in semiprojective hyperkähler varieties. Examples arise by taking the algebraic symplectic quotient of a complex symplectic vector space  $\mathbb{M}$  by a symplectic linear action of a reductive group  $\bar{\rho} : G \rightarrow \mathrm{Sp}(\mathbb{M})$ . In practice  $\mathbb{M} = \mathbb{V} \times \mathbb{V}^*$  and  $\bar{\rho}$  arises as the doubling of a representation  $\rho : G \rightarrow \mathrm{GL}(\mathbb{V})$ . If  $\mathfrak{g}$  denotes the Lie algebra of  $G$ , we have the derivative of  $\rho$  as  $\varrho : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathbb{V})$ . This gives us the moment map

$$\mu : \mathbb{M} \longrightarrow \mathfrak{g}^*, \tag{1.1.1}$$

at  $x \in \mathfrak{g}$  by the formula

$$\langle \mu(v, w), x \rangle = \langle \varrho(x)v, w \rangle. \tag{1.1.2}$$

By construction  $\mu$  is equivariant with respect to the coadjoint action of  $G$  on  $\mathfrak{g}^*$ . Taking a character  $\sigma \in \mathrm{Hom}(G, \mathbb{C}^\times)$  will yield the GIT quotient  $\mathcal{M}_\sigma^\rho := \mu^{-1}(0) //_\sigma G$  using the linearization induced by  $\sigma$ . Sometimes  $\sigma$  can be chosen generically so that  $\mathcal{M}_\sigma^\rho$  becomes non-singular (and by construction) quasi-projective. We assume this henceforth. By construction of the GIT quotient we have the proper affinization map

$$\mathcal{M}_\sigma^\rho \longrightarrow \mathcal{M}_0^\rho \tag{1.1.3}$$

to the affine GIT quotient  $\mathcal{M}_0^\rho = \mu^{-1}(0) // G$ .

The  $\mathbb{C}^\times$ -action on  $\mathbb{M}$  given by dilation will commute with the linear action of  $G$  on it so that the moment map (1.1.1) will be equivariant with respect to this and the weight 2 action of  $\mathbb{C}^\times$  on  $\mathfrak{g}^*$ . This will induce a  $\mathbb{C}^\times$ -action on  $\mathcal{M}_\sigma^\rho$ , such that on the affine GIT quotient  $\mathcal{M}_0^\rho$  it will have a single fixed point corresponding to the origin

in  $\mu^{-1}(0) \subset \mathbb{M}$ . This and the fact that the affinization map (1.1.3) is proper implies, that  $\mathcal{M}_\sigma^\rho$  is semiprojective.

An important special case is when  $Z(\mathrm{GL}(\mathbb{V})) \subset \mathrm{im}\rho$ . In this case we can take a square root of the  $\mathbb{C}^\times$  action above by acting only on  $\mathbb{V}^*$  by dilation and trivially on  $\mathbb{V}$ . This action will also commute with the action of  $G$  on  $\mathbb{M} = \mathbb{V} \times \mathbb{V}^*$  and will indeed reduce to a  $\mathbb{C}^\times$ -action on the quotient  $\mathcal{M}_\sigma^\rho$  whose square is the  $\mathbb{C}^\times$ -action we considered in the previous paragraph. In particular this new action also makes  $\mathcal{M}_\sigma^\rho$  semiprojective. In fact, it will have an additional property. Namely, the natural symplectic form  $\omega_{\mathbb{M}}$  on  $\mathbb{M}$  will be of homogeneity 1 with respect to the  $\mathbb{C}^\times$  action; in other words, it will satisfy

$$\lambda^*(\omega_{\mathbb{M}}) = \lambda\omega_{\mathbb{M}} \tag{1.1.4}$$

under this action. This property will be inherited by the quotient  $\mathcal{M}_\sigma^\rho$ . Following [Ha2] we make the following

**Definition 1.1.2.** — A semiprojective hyperkähler variety with a symplectic form of homogeneity one as in (1.1.4) is called *hyper-compact*.

When  $G$  is a torus,  $\mathcal{M}_\sigma^\rho$  are the toric hyperkähler varieties of [HS]; these always can be arranged to become hyper-compact. When the representation  $\rho$  arises from a quiver with a dimension vector  $\mathcal{M}_\sigma^\rho$  is a quiver variety as constructed by Nakajima in [Na2]. When the quiver has no edge loops, one can always arrange that  $\mathcal{M}_\sigma^\rho$  becomes hyper-compact. When the quiver is the tennis-racquet quiver, *i.e.*, two vertices connected with a single edge and with a loop on one of them, and the dimension vector is 1 in the simple vertex and  $n$  on the looped one, the Nakajima quiver variety becomes isomorphic with  $(\mathbb{C}^2)^{[n]}$  the Hilbert scheme of  $n$  points on  $\mathbb{C}^2$ . This semiprojective hyperkähler variety however is not hyper-compact as we will see later.

Finally, the following hyper-compact examples originally arose from an infinite dimensional analogue of the above construction. In [Hi1] Hitchin constructs the moduli space of semistable rank  $n$  degree  $d$  Higgs bundles on a Riemann surface as an infinite dimensional gauge theoretical quotient. A Higgs bundle is a pair  $(E, \phi)$  of a rank  $n$  degree  $d$  vector bundle  $E$  on the Riemann surface  $C$  and  $\phi \in H^0(C; \mathrm{End}(E) \otimes K_C)$ . Nitsure [Ni] constructed such moduli spaces  $\mathcal{M}_n^d$  in the algebraic geometric category, which are non-singular quasi-projective varieties when  $(n, d) = 1$ . There is a natural  $\mathbb{C}^\times$  action on  $\mathcal{M}_n^d$  given by scaling the Higgs field  $(E, \phi) \mapsto (E, \lambda\phi)$ . Hitchin [Hi1] when  $n = 2$  and Simpson [Si2, Corollary 10.3] in general showed that  $\mathcal{M}_n^d$  is semiprojective. A nice argument to see this is similar to that given for  $\mathcal{M}_\sigma^\rho$  above. Namely, the affinization

$$\chi : \mathcal{M}_n^d \longrightarrow \mathbf{A} \tag{1.1.5}$$

turns out to be the famous Hitchin map [Hi2], which by results of Hitchin [Hi1] when  $n = 2$  and Nitsure [Ni] for general  $n$  is a proper map. It is also  $\mathbb{C}^\times$ -equivariant which covers a  $\mathbb{C}^\times$ -action on the affine  $\mathbf{A}$  with a single fixed point. This implies that  $\mathcal{M}_n^d$  is indeed semiprojective.

**1.2. Bialynicki-Birula decomposition of semiprojective varieties.** — Much in this section is due to Simpson [Si2], Nakajima [Na2] and Atiyah-Bott [AB].

Let  $X$  be a non-singular semiprojective variety. Let  $X^{\mathbb{C}^\times} = \coprod_{i \in I} F_i$  be the decomposition of the fixed point set into connected components. Then  $I$  is finite and  $F_i$  are non-singular projective subvarieties of  $X$ . According to [Dol, Corollary 7.2] we can linearize the action of  $\mathbb{C}^\times$  on a very ample line bundle  $L$  on  $X$ . On each  $F_i$  then  $\mathbb{C}^\times$  will act on  $L$  through a homomorphism  $\mathbb{C}^\times \rightarrow \mathbb{C}^\times$  with weight  $\alpha_i \in \mathbb{Z}$  which we can assume, by suitably changing the linearisation, are always non-negative  $\alpha_i \in \mathbb{Z}_{\geq 0}$ . We introduce a partial ordering on  $I$  by setting

$$i < j \iff \alpha_i > \alpha_j. \tag{1.2.1}$$

Introduce  $U_i \subset X$  as the set of points  $x \in X$  for which  $\lim_{t \rightarrow 0} tx \in F_i$ . Similarly, as above, we can define  $D_i$  as the points  $x \in X$  for which  $\lim_{t \rightarrow \infty} tx \in F_i$ . These are locally closed subsets and Bialynicki-Birula [Bia, Theorem 4.1] proves that both  $U_i$  and  $D_i$  are subschemes of  $X$  which are isomorphic to certain affine bundles (so-called  $\mathbb{C}^\times$ -fibrations) over  $F_i$ .

It will be convenient to make the following

**Definition 1.2.1.** — The *core* of the semiprojective variety  $X$  is

$$\mathcal{C} := \cup_{i \in I} D_i \subset X$$

By assumption 2 of Definition 1.1.1 we get the Bialynicki-Birula decomposition  $X = \coprod_{i \in I} U_i$ . This decomposition satisfies that

$$\overline{U_i} \subset \cup_{j \geq i} U_j. \tag{1.2.2}$$

To see this we note that using the linearisation on the very ample line bundle  $L$  we can equivariantly embed  $X$  into some projective space  $\mathbb{P}^N$  with a linear  $\mathbb{C}^\times$  action. (1.2.2) follows from the corresponding statement for the linear action of  $\mathbb{C}^\times$  on  $\mathbb{P}^N$ , where it is clear.

It follows from the Hilbert-Mumford criterium for semistability that  $X^{ss} = X \setminus \coprod_{i \in I} D_i$  with respect to our linearisation. Thus we have a geometric quotient  $Z := X^{ss}/\mathbb{C}^\times$ , which is proper according to [Si2, Theorem 11.2] and is, in fact, an orbifold as there are no fixed points of  $\mathbb{C}^\times$  on  $X^{ss}$ . Using this construction for the semiprojective  $X \times \mathbb{C}$  where  $\mathbb{C}^\times$  acts *via* the diagonal action (with the standard multiplication action on the second factor) we get

$$\overline{X} := (X \times \mathbb{C})^{ss}/\mathbb{C}^\times, \tag{1.2.3}$$

which decomposes as  $\overline{X} = X \coprod Z$  corresponding to points in  $(X \times \mathbb{C})^{ss}$  with non-zero (respectively zero) second component. This thus yields an orbifold compactification of  $X$ , the algebraic analogue of Lerman’s symplectic cutting [Ler], as studied in [Ha1].

An immediate consequence of this compactification is the following:

**Corollary 1.2.2.** — *The core  $\mathcal{C}$  of a semiprojective variety  $X$  is proper.*

*Proof.* — The proper  $\overline{X}$  has two Bialynicki-Birula decompositions. One of them is

$$\overline{X} = D_\infty \cup \coprod_{i \in I} D_i$$

where

$$D_\infty = (X \setminus \mathcal{C}) \cup Z \subset \overline{X}.$$

Thus by property (1.2.2) the core  $\mathcal{C} = \coprod_{i \in I} D_i$  is closed in the proper  $\overline{X}$ . The claim follows.  $\square$

### 1.3. Cohomology of semiprojective varieties

*1.3.1. Generalities on cohomologies of complex algebraic varieties.* — We denote by  $H^*(X; \mathbb{Z})$  the integer and by  $H^*(X; \mathbb{Q})$  the rational singular cohomology of a CW complex  $X$ .  $H^*(X; \mathbb{Z})$  is a graded anti commutative ring; while  $H^*(X; \mathbb{Q})$  is a graded anticommutative  $\mathbb{Q}$ -algebra.

When  $X$  is a complex algebraic variety there is further structure on its rational cohomology. Motivated by the Frobenius action on the  $l$ -adic cohomology of a variety defined over an algebraic closure of a finite field Deligne in 1971 [De] introduced mixed Hodge structures on the cohomology of any complex algebraic variety  $X$ .

Here we only recall the notion of the weight filtration on rational cohomology. It is an increasing filtration:

$$W_0(H^k(X; \mathbb{Q})) = 0 \subset \dots \subset W_i(H^k(X; \mathbb{Q})) \subset \dots \subset W_{2k}(H^k(X; \mathbb{Q})) = H^k(X; \mathbb{Q})$$

by  $\mathbb{Q}$ -vector spaces  $W_i(H^k(X; \mathbb{Q}))$ . It has many nice properties. For example it is functorial,

$$W_{k-1}(H^k(X; \mathbb{Q})) = 0 \tag{1.3.1}$$

for a smooth  $X$ , and

$$W_k(H^k(X; \mathbb{Q})) = H^k(X; \mathbb{Q}) \tag{1.3.2}$$

for a projective variety  $X$ . We say that the weight filtration on  $H^*(X; \mathbb{Q})$  is *pure* when both (1.3.1) and (1.3.2) holds for every  $k$ . In particular a smooth projective variety always has pure weight filtration. We will see in Corollary 1.3.2 that semiprojective varieties also have pure weight filtration.

We denote by

$$H(X; q, t) = \sum_{i,k} \dim(\text{Gr}_i^W H^k(X; \mathbb{Q})) q^{i/2} t^k \in \mathbb{Z}[q^{1/2}, t]$$

the *mixed Hodge polynomial*. It has two important specializations. The polynomial

$$P(X; t) = H(1, t) = \sum_k \dim(H^k(X; \mathbb{Q})) t^k \in \mathbb{Z}[t]$$



is the *Poincaré polynomial* of  $X$ , while the specialization

$$\begin{aligned}
 E(X; q) &= q^{\dim X} H(X; 1/q, -1) \\
 &= \sum_{i,k} (-1)^k \dim(\mathrm{Gr}_i^W H^k(X; \mathbb{Q})) q^{\dim X - i/2} \in \mathbb{Z}[q^{1/2}]
 \end{aligned}
 \tag{1.3.3}$$

the *virtual weight polynomial*. In the case when the weight filtration is pure on  $H^*(X; \mathbb{Q})$  we have the relationships

$$H(X; q, t) = P(X; q^{1/2}t) = E(X; qt^2).
 \tag{1.3.4}$$

In the general case however there is no such relationships.

*1.3.2. The case of semiprojective varieties.* — Let  $X = \coprod_{i \in I} U_i$  the Bialynicki-Birula decomposition of a semiprojective variety, with index set  $I$  given a partial ordering as in (1.2.1). Following [AB, pp 537] let  $J \subset I$  such that

$$j \in I \text{ and } i < j \text{ implies } i \in J.
 \tag{1.3.5}$$

Then by  $U_J := \cup_{j \in J} U_j$  is open in  $X$  by (1.2.2). Let  $\lambda \in I \setminus J$  be minimal and let  $J^+ := J \cup \lambda$ , this also satisfies (1.3.5) so  $U_{J^+}$  is also open in  $X$  and  $U_\lambda$  is closed in  $U_{J^+}$ . Furthermore the open subvarieties  $U_J$  and  $U_{J^+}$  of  $X$  are both semiprojective with core

$$D_J := \coprod_{j \in J} D_j \subset U_J$$

and

$$D_{J^+} = \coprod_{j \in J^+} D_j = D_J \cup D_\lambda \subset U_{J^+}.$$

We now have the following commutative diagram:

$$\begin{array}{ccccccc}
 \longrightarrow & H^{j-k_\lambda}(U_\lambda; \mathbb{Z}) & \longrightarrow & H^j(U_{J^+}; \mathbb{Z}) & \longrightarrow & H^j(U_J; \mathbb{Z}) & \longrightarrow \\
 & \downarrow i_\lambda^* & & \downarrow i_J^* & & \downarrow i_{J^+}^* & \\
 \longrightarrow & H^{j-k_\lambda}(F_\lambda; \mathbb{Z}) & \longrightarrow & H^j(D_{J^+}; \mathbb{Z}) & \longrightarrow & H^j(D_J; \mathbb{Z}) & \longrightarrow
 \end{array}
 \tag{1.3.6}$$

Here the top row is the cohomology long-exact sequence of the pair  $(U_{J^+}, U_J)$  and

$$H^{j-k_\lambda}(U_\lambda, \mathbb{Z}) \cong H^j(U_{J^+}, U_J; \mathbb{Z})$$

is excision followed by the Thom isomorphism theorem, where  $k_\lambda = \mathrm{codim} U_\lambda$ . The bottom row is the cohomology long-exact sequence of the pair  $(D_{J^+}, D_J)$ , where again

$$H^{j-k_\lambda}(F_\lambda, \mathbb{Z}) \cong H^j(D_{J^+}, D_J; \mathbb{Z})$$

is the Thom isomorphism. Finally  $i_\lambda : F_\lambda \rightarrow D_\lambda$ ,  $i_J : D_J \rightarrow U_J$  and  $i_{J^+} : D_{J^+} \rightarrow U_{J^+}$  denote the corresponding imbeddings.

Clearly  $i_\lambda^*$  is an isomorphism. So if we know that  $i_J^*$  is an isomorphism, so will be  $i_{J^+}^*$  by the five lemma. If  $J_{min} = \{\lambda_{min}\}$  denotes a minimal element in  $I$ , then  $D_{J_{min}} \cong F_{\lambda_{min}}$  and so  $i_{J_{min}}^* : H^*(U_{J_{min}}; \mathbb{Z}) \cong H^*(D_{J_{min}}; \mathbb{Z})$ . Therefore by induction we get that

$$i_J^* : H^*(U_J; \mathbb{Z}) \cong H^*(D_J; \mathbb{Z})$$

is an isomorphism for all  $J$  satisfying (1.3.5). Thus in particular we have:

**Theorem 1.3.1.** — *The embedding  $i : \mathcal{C} \cong D_I \rightarrow X \cong U_I$  induces an isomorphism  $i^* : H^*(X; \mathbb{Z}) \cong H^*(\mathcal{C}; \mathbb{Z})$ .*

**Corollary 1.3.2.** — *A smooth semiprojective variety has pure cohomology.*

*Proof.* — As  $X$  is non-singular all the non-trivial weights in  $H^k(X; \mathbb{Q})$  are at least  $k$  by (1.3.1). By Theorem 1.3.1, Corollary 1.2.2 and (1.3.2) all the weights in  $H^k(X; \mathbb{Q})$  are at most  $k$ . The statement follows <sup>(1)</sup>. □

Interestingly our techniques can also be used to prove the purity of the cohomology of certain, typically affine, varieties which are deformations of semiprojective varieties as in the following corollary.

**Corollary 1.3.3.** — *Let  $X$  be a non-singular complex algebraic variety and  $f : X \rightarrow \mathbb{C}$  a smooth morphism, i.e., a surjective submersion. In addition, let  $X$  be semiprojective with a  $\mathbb{C}^\times$  action making  $f$  equivariant covering a linear action of  $\mathbb{C}^\times$  on  $\mathbb{C}$  with positive weight. Then the fibers  $X_c := f^{-1}(c)$  have isomorphic cohomology supporting pure mixed Hodge structures.*

*Proof.* — The proof can be found in [HLV1, Appendix B]. It proceeds by proving that the embedding of every fiber of  $f$  induces an isomorphism

$$H^*(X_c; \mathbb{Q}) \cong H^*(X; \mathbb{Q}), \tag{1.3.7}$$

which implies the statement in light of Corollary 1.3.2. This is clear for  $c = 0 \in \mathbb{C}$  as  $X_c$  is itself semiprojective and it shares the same core  $\mathcal{C} \subset X_0 \subset X$  with  $X$ . The proof of (1.3.7) for  $0 \neq c \in \mathbb{C}$  is more difficult and is using a version of the compactification technique as in (1.2.3) and Ehresmann’s theorem for proper smooth maps; in particular the proof is not algebraic. □

**Remark 1.3.4.** — In fact Simpson’s [Si2] main example for a semiprojective variety was  $\mathcal{M}_{\text{Hod}}$ , the moduli space of stable rank  $n$  degree 1  $\lambda$ -connections on the curve which comes with  $f : \mathcal{M}_{\text{Hod}} \rightarrow \mathbb{C}$  satisfying the conditions of Corollary 1.3.3. Here  $f^{-1}(0) \cong \mathcal{M}_{\text{Dol}} = \mathcal{M}_n^g$  is our moduli space of Higgs bundles while  $f^\lambda = \mathcal{M}_{\text{DR}}$  is the moduli space of certain holomorphic connections. The Corollary 1.3.3 then shows that  $H^*(\mathcal{M}_{\text{DR}}; \mathbb{Q}) \cong H^*(\mathcal{M}_{\text{Dol}}; \mathbb{Q})$  have isomorphic and pure cohomology. This argument was used in [HT, Theorem 6.2] and [Ha3, Theorem 2.2] in connection with topological mirror symmetry.

---

1. This argument is folklore yoga of weights; we learned it from Gérard Laumon.

**Remark 1.3.5.** — Another crucial use of this Corollary 1.3.3 is in our arithmetic harmonic analysis technique explained in §2. We will be able to compute the virtual weight polynomial  $E(X_\lambda; q)$  of an affine symplectic quotient, and to deduce that it gives the Poincaré polynomial we will put  $X_\lambda$  in a family  $f : X \rightarrow \mathbb{C}$  satisfying the conditions of Corollary 1.3.3.

The following result was discussed in [HS, Theorem 3.5] in the context of semiprojective toric varieties, and the proof was sketched in [Ha7].

**Corollary 1.3.6.** — *The core  $\mathcal{C}$  is a deformation retract of the smooth semiprojective variety  $X$ .*

*Proof.* — First we note that replacing cohomology with homology in the proof of Theorem 1.3.1 yields that that  $i_* : H_*(X; \mathbb{Z}) \cong H_*(\mathcal{C}; \mathbb{Z})$  induced by the embedding  $i : \mathcal{C} \rightarrow X$  is also an isomorphism. By the homology long exact sequence this is equivalent with

$$H_*(X, \mathcal{C}; \mathbb{Z}) = 0. \tag{1.3.8}$$

We also claim that  $i_* : \pi_1(X) \cong \pi_1(\mathcal{C})$  induces an isomorphism on the fundamental group (from whose notation we omitted the base-point for simplicity). This follows by induction similarly as in the proof of Theorem 1.3.1. First note by [Bia, Theorem 4.1] that  $U_{\lambda_{min}}$  retracts to  $F_{\lambda_{min}} \cong D_{\lambda_{min}}$  thus have isomorphic fundamental group. Then by induction we assume  $(i_J)_* : \pi_1(D_J) \cong \pi_1(U_J)$  is an isomorphism for an index set  $J \subset I$  satisfying (1.3.5). Take  $\lambda \in I \setminus J$  minimal and cover  $U_{J+} = U_J \cup U_\lambda$  with open sets  $U_J$  and a small tubular neighborhood  $U_\lambda^{tub}$  of  $U_\lambda$ , small in the sense that  $U_\lambda^{tub} \cup D_J = \emptyset$  it is disjoint from the proper  $D_J$  ( $D_J$  is the core of the semiprojective  $U_J$ ; thus proper by Theorem 1.2.2). This implies that  $F_\lambda^{tub} := U_\lambda^{tub} \cap D_{J+} \subset D_\lambda$  is a tubular neighborhood of  $F_\lambda$ . Then we have two commutative diagrams:

$$\begin{array}{ccc}
 \pi_1(U_\lambda^{tub} \cap U_J) & \longrightarrow & \pi_1(U_\lambda^{tub}) \\
 \uparrow \cong & & \uparrow \cong \\
 \pi_1(F_\lambda^{tub} \cap (U_J \cap D_{J+})) & \longrightarrow & \pi_1(D_\lambda^{tub}) \\
 \\ 
 \pi_1(U_\lambda^{tub} \cap U_J) & \longrightarrow & \pi_1(U_J) \\
 \uparrow \cong & & \uparrow \cong \\
 \pi_1(F_\lambda^{tub} \cap (U_J \cap D_{J+})) & \longrightarrow & \pi_1(D_J)
 \end{array} \tag{1.3.9}$$

where the maps are all induced by the embedding of the indicated spaces in each other. The four vertical arrows are all isomorphisms. The last one because of the induction hypothesis. The second one as both  $U_\lambda^{tub}$  and  $D_\lambda^{tub}$  retract to  $F_\lambda$ . Finally, the first and the third because these spaces all retract to  $F_\lambda^{tub} \setminus F_\lambda$ .

Using the diagrams (1.3.9) and the Seifert-van Kampen theorem applied to both the open covering

$$U_{J+} = U_{\lambda}^{tub} \cup U_J$$

and

$$D_{J+} = F_{\lambda}^{tub} \cup (U_J \cap D_{J+})$$

we see that

$$\pi_1(U_{J+}) \cong \pi_1(D_{J+}).$$

By induction we get the desired

$$\pi_1(X) \cong \pi_1(\mathcal{C}).$$

In particular, the homotopy long exact sequence of the pair  $(X, \mathcal{C})$  implies that  $\pi_1(X, \mathcal{C}) = 0$  as well as that  $\pi_2(X, \mathcal{C})$  is a quotient of  $\pi_2(X)$  and so abelian. From this and (1.3.8) the relative Hurewitz theorem [Whi, Theorem IV.7.3] implies  $\pi_k(X, \mathcal{C}) = 0$  for every  $k$ , thus

$$i_* : \pi_k(X) \xrightarrow{\cong} \pi_k(\mathcal{C})$$

is an isomorphism. Therefore  $X$  and  $\mathcal{C}$  are weakly homotopy equivalent, and as varieties they are CW complexes and so by Whitehead's theorem [Whi, Theorem V.3.5]  $i$  is a homotopy equivalence.  $\square$

**Theorem 1.3.7.** — *The Bialynicki-Birula decomposition  $X = \coprod_{i \in I} U_i$  of a semiprojective variety is perfect. In particular  $P(X; t) = \sum_{\lambda \in I} P(F_{\lambda}; t)t^{2k_{\lambda}}$ .*

*Proof.* — This follows from studying the top long-exact sequence of (1.3.6) considered with rational coefficients. Here we assume the same situation as there:

$$H^q(U_{J+}, U_J; \mathbb{Q}) \longrightarrow H^q(U_{J+}; \mathbb{Q}) \longrightarrow H^q(U_J; \mathbb{Q}) \longrightarrow H^{q+1}(U_{J+}, U_J; \mathbb{Q}). \quad (1.3.10)$$

This is a sequence of Mixed Hodge structures, and the weights are pure according to Corollary 1.3.2 in the cohomology of the semiprojective varieties  $U_J$  and  $U_{J+}$ , and in  $H^q(U_{J+}, U_J; \mathbb{Q})$  by the Thom isomorphism. Therefore the connecting morphism  $H^q(U_J; \mathbb{Q}) \cong W^q(H^q(U_J; \mathbb{Q})) \rightarrow H^{q+1}(U_{J+}, U_J; \mathbb{Q})$  must be trivial. Therefore the long exact sequence splits, the stratification is perfect, and the formula for Poincaré polynomials follow by induction.  $\square$

**1.4. Weak Hard Lefschetz.** — Fix a very ample line bundle  $L$  on a smooth semiprojective variety  $X$  and let  $\alpha = c_1(L) \in H^2(X; \mathbb{Q})$ . Then we have

**Theorem 1.4.1 (Weak Hard Lefschetz).** — *Let  $X$  be a semiprojective variety  $X$  with core  $\mathcal{C} = \cup_{\lambda \in I} D_{\lambda}$ . Assume  $\mathcal{C}$  is equidimensional of pure dimension  $k = \dim \mathcal{C}$ . Then the Hard Lefschetz map*

$$\begin{aligned} L^i : H^{k-i}(X, \mathbb{Q}) &\rightarrow H^{k+i}(X, \mathbb{Q}) \\ L^i(\beta) &= \beta \wedge \alpha^i \end{aligned} \quad (1.4.1)$$

is injective for  $0 \leq i < k$ .

*Proof.* — It follows from Corollaries 1.3.6 and 1.3.2 that the core  $\mathcal{C}$  has pure cohomology. Then the result follows from [BE, Theorem 2.2] as we have assumed  $\mathcal{C}$  is equidimensional. Their argument goes by first showing that the natural map  $H^*(\mathcal{C}; \mathbb{Q}) \rightarrow IH^*(\mathcal{C}; \mathbb{Q})$  is injective, and then concludes by using [BBD, Theorem 5.4.10] for the Hard Lefschetz theorem for  $IH^*(\mathcal{C}; \mathbb{Q})$ .  $\square$

**Remark 1.4.2.** — An immediate consequence of the injectivity of (1.4.1) for  $0 \leq i < k$  are the inequalities

$$b_i(X) \leq b_{i+2j}(X) \text{ for all } 0 \leq j \leq k - i \tag{1.4.2}$$

for the Betti numbers of the smooth semiprojective variety. As a consequence both sequences of odd and even Betti numbers grow until  $k$  and satisfy  $b_{k-i}(X) \leq b_{k+i}(X)$ .

**Remark 1.4.3.** — Possibly the analogous result to (1.4.1) holds when  $\mathcal{C}$  is not equidimensional and  $k$  is the smallest dimension of the irreducible components of  $\mathcal{C}$ . It was proved in the case of smooth semiprojective toric varieties in [HS]. There however it was used that the components of the core are smooth; but conceivably this can be avoided.

**Remark 1.4.4.** — Of course a general semiprojective toric variety could have a non-equidimensional core (as it corresponds to the complex of bounded faces of a non-compact convex polyhedron). However, we do not know of an example of a semiprojective hyperkähler variety whose core is not equidimensional.

When the semiprojective variety is hyper-compact (Definition 1.1.2) one finds that  $D_\lambda$  is Lagrangian. In other words,  $\dim D_\lambda = \frac{\dim X}{2}$  and hence  $k = \frac{\dim X}{2}$  as  $\text{codim} U_{\min} = 0$ . Examples include toric hyperkähler manifolds, Nakajima quiver varieties (from quivers without edge-loops) and the moduli space of Higgs bundles. The fact that the nilpotent cone, which agrees with the core of  $\mathcal{M}_n^g$ , is Lagrangian was first observed by Laumon [Lau]. Retrospectively, this can also be considered as a consequence of the completely integrability of the Hitchin system [Hi2]. In the hyper-compact case Theorem 1.4.1 appeared as [Ha2, Corollary 4.3].

However, when the quiver contains an edge loop the Nakajima quiver varieties are not hyper-compact. Examples include  $(\mathbb{C}^2)^{[n]}$  and more generally the ADHM spaces  $\mathcal{M}_{n,m}$ . Nevertheless, in these cases we know by [Br] and respectively [ELL] and [Ba] that the cores are irreducible and in particular equidimensional of dimension  $n - 1$  and  $mn - 1$  respectively.

We do not know if equidimensionality or even irreducibility of the core of Nakajima quiver varieties for quivers with edge loops holds in general.

**Remark 1.4.5.** — In the case of smooth projective toric varieties  $Y$ , the Hard Lefschetz theorem, together with the fact that  $H^2(Y)$  generates  $H^*(Y)$ , famously [St1] gives a complete characterization of possible Poincaré polynomials of smooth projective toric varieties, and in turn the face vectors of rational simple complex polytopes.

The above Weak Hard Lefschetz theorem was used in [HS] and [Ha2] to give new restrictions on the Poincaré polynomials of toric hyperkähler varieties and, in turn, on the face vectors of rational hyperplane arrangements. However a complete classification in this case has not even been conjectured.

**Remark 1.4.6.** — For the moduli space of Higgs bundles  $\mathcal{M}_n^g$  Theorem 1.4.1 is a consequence of the Relative Hard Lefschetz theorem [dCHM] using the argument of [HV, 4.2.8].

Thus it is interesting to ask the following:

**Question 1.4.7.** — *For semiprojective hyperkähler varieties is there a stronger form of the Weak Hard Lefschetz theorem or the inequalities (1.4.2)? In particular how do the Betti numbers of semiprojective hyperkähler varieties behave after  $k = \dim \mathcal{C}$ ?*

This question was the original motivation to look at the Betti numbers of examples of large semiprojective hyperkähler varieties to find how the Betti numbers behave after the critical dimension  $k = \dim \mathcal{C}$ .

It turns out that partly due to an arithmetic harmonic analysis technique to evaluate such Betti numbers we have now efficient formulas to compute Poincaré polynomials. This allows us to investigate numerically the shape of Betti numbers of large semiprojective hyperkähler manifolds in several examples. We explain this arithmetic technique and the resulting combinatorial formulas for the Poincaré polynomials in the next section.

## 2. Arithmetic harmonic analysis on symplectic quotients: the microscopic picture

In the previous section we collected results on the cohomology of a general semiprojective variety  $X$ . In this section we show that when  $X$  arises as symplectic quotient of a vector space, we can use “arithmetic harmonic analysis” to count points on  $X$  over a finite field, and in turn to compute Betti numbers. Counting points of varieties over finite fields is what we call microscopic approach to study Betti numbers of complex algebraic varieties.

**2.1. Katz’s theorem.** — From Katz’s [HV, Appendix] we recall the definition that a complex algebraic variety  $X$  is strongly-polynomial count. This means that there is polynomial  $\mathcal{P}_X(t) \in \mathbb{Z}[t]$  and a spreading out <sup>(2)</sup>  $\mathcal{X}$  over a finitely generated commutative unital ring  $R$  such that for all homomorphism  $\phi : R \rightarrow \mathbb{F}_q$  to a finite field  $\mathbb{F}_q$  (where  $q = p^r$  is a power of the prime  $p$ ) we have

$$\#\mathcal{X}_\phi(\mathbb{F}_q) = \mathcal{P}_X(q).$$

We then have the following theorem of Katz from [HV, Theorem 6.1, Appendix].

---

2. *I.e.*, a homomorphism  $R \rightarrow \mathbb{C}$  such that  $X = \mathcal{X} \otimes_R \mathbb{C}$ .

**Theorem 2.1.1 (Katz).** — *Assume that  $X/\mathbb{C}$  is strongly-polynomial count with counting polynomial  $\mathcal{P}_X \in \mathbb{Z}[t]$ . Then*

$$E(X; q) = \mathcal{P}_X(q).$$

This result gives the Betti numbers of a strongly polynomial count variety  $X$ , when additionally it has a pure cohomology. In that case (1.3.4) will compute the Poincaré polynomial from the virtual weight polynomial. This will be the case for many of our semiprojective varieties, where we will be able to use an effective technique to find the count polynomial  $\mathcal{P}_X(t)$ . This technique from [Ha5, Ha6] we explain in the next section.

**2.2. Arithmetic harmonic analysis.** — We work in the setup of §1.1.1 but change coefficients from  $\mathbb{C}$  to a finite field  $\mathbb{F}_q$ . For simplicity we denote with the same letters  $\rho, G, \mathfrak{g}, \mathcal{M}, \mathbb{V}, \mu$  the corresponding objects over  $\mathbb{F}_q$ . We define the function  $a_\rho : \mathfrak{g} \rightarrow \mathbb{N} \subset \mathbb{C}$  at  $X \in \mathfrak{g}$  as

$$a_\rho(X) := |\ker(\rho(X))|. \tag{2.2.1}$$

In particular  $a_\rho(X)$  is always a  $q$  power. Our main observation from [Ha5, Ha6] is the following:

**Proposition 2.2.1.** — *Let  $\xi \in \mathfrak{g}^*$  and fix a non-trivial additive character  $\Psi : \mathbb{F}_q \rightarrow \mathbb{C}^\times$ . The number of solutions of the equation  $\mu(v, w) = \xi$  over the finite field  $\mathbb{F}_q$  equals:*

$$\#\{(v, w) \in \mathbb{M} \mid \mu(v, w) = \xi\} = |\mathfrak{g}|^{-1} |\mathbb{V}| \sum_{X \in \mathfrak{g}} a_\rho(X) \Psi(\langle X, \xi \rangle) \tag{2.2.2}$$

Thus in order to count the  $\mathbb{F}_q$  points of  $\mu^{-1}(\xi)$  we only need to determine the function  $a_\rho$  as defined in (2.2.1) and compute its Fourier transform as in (2.2.2). In turn we assume that  $\xi \in (\mathfrak{g}^*)^G$  and we use this to count the  $\mathbb{F}_q$  points of the affine GIT quotients  $X := \mu^{-1}(\xi)//G$ , in cases when  $G$  acts freely on  $\mu^{-1}(\xi)$ , when the number of  $\mathbb{F}_q$  points on  $\mu^{-1}(\xi)//G$  is just  $\#\mu^{-1}(\xi)/|G|$ . In our cases considered below this quantity will turn out to be a polynomial in  $q$ , yielding by (2.1.1) a formula for the virtual weight polynomials of affine GIT quotient  $\mu^{-1}(\xi)//G$ .

Finally we can connect the affine GIT quotient to the GIT quotients with generic linearization as in §1.1.1 by considering  $X := \mu^{-1}(\overline{\mathbb{C}^\times \xi})//_\sigma G$  a non-singular semiprojective variety with a projection  $f : X \rightarrow \mathbb{C} \cong \overline{\mathbb{C}^\times \xi} \subset \mathfrak{g}^*$  with generic fiber  $X_{\lambda\xi} = f^{-1}(\lambda\xi) = \mu^{-1}(\lambda\xi)//_\sigma G = \mu^{-1}(\lambda\xi)//G$  the affine GIT quotient when  $\lambda \neq 0$  and  $X_0 = \mu^{-1}(0)//_\sigma G$  the GIT quotient with linearization  $\sigma$ . Now Corollary 1.3.3 can be applied to show that  $X_0$  and  $X_\xi$  have isomorphic pure cohomology, and so our computation by Fourier transform above gives the Poincaré polynomial of our semiprojective varieties, which arise as finite dimensional linear symplectic quotients.

**2.3. Betti numbers of semiprojective hyperkähler varieties**

*2.3.1. Toric hyperkähler varieties  $\mathcal{M}_{\mathcal{H}}$ .* — Let  $\mathcal{H} \subset \mathbb{Q}^n$  be a rational hyperplane arrangement. In this case the toric hyperkähler variety  $\mathcal{M}_{\mathcal{H}}$  arises as linear symplectic

quotient, with generic linearization, induced by a torus action  $\rho_{\mathcal{H}} : \mathbb{T} \rightarrow \mathrm{GL}(V)$  constructed from  $\mathcal{H}$  as in [HS, §6]. The variety  $\mathcal{M}_{\mathcal{H}}$  is an orbifold and is non-singular when  $\mathcal{H}$  is unimodular. In the unimodular case it was first constructed in [BD] by differential geometric means.

As explained in [Ha5] the above arithmetic harmonic analysis can be used to compute the Betti numbers of the semiprojective  $\mathcal{M}_{\mathcal{H}}$ ; we get

$$P(\mathcal{M}_{\mathcal{H}}; t) = \sum h_i(\mathcal{H})t^{2i}, \tag{2.3.1}$$

where the Betti numbers  $h_i(\mathcal{H})$  are the  $h$ -numbers of the hyperplane arrangement  $\mathcal{H}$ ; a combinatorial quantity. In the unimodular case (2.3.1) was first determined in [BD] and in the general case it was proved in [HS].

As explained in [HS, §8] one can construct the so-called cographic arrangement  $\mathcal{H}_Q$  from any graph  $Q$ . Then  $\mathcal{M}_{\mathcal{H}_Q}$  is just the Nakajima quiver variety  $\mathcal{M}_{\mathbf{1}}^Q$  of §2.3.3 below attached to  $Q$  and constant dimension vector  $\mathbf{1}$ . In this case the  $h$ -polynomial of (2.3.1) can be computed from the Tutte polynomial as follows:

$$P(\mathcal{M}_{\mathbf{1}}^Q; t) = t^{\dim \mathcal{M}_{\mathbf{1}}^Q} R_Q(1/t^2) = t^{\dim \mathcal{M}_{\mathbf{1}}^Q} T_Q(1, 1/t^2), \tag{2.3.2}$$

Here the *Tutte polynomial*  $T_Q$  of a graph  $Q$  is a two variable polynomial invariant, universal with respect to contraction-deletion of edges. It can be defined explicitly as follows

$$T_Q(x, y) := \sum_{A \subseteq E} (x - 1)^{k(A) - k(E)} (y - 1)^{k(A) + \#A - \#V}, \tag{2.3.3}$$

where  $k(A)$  denotes the number of connected components of the subgraph  $Q_A \subseteq Q$  with edge set  $A$  and the same set  $V = V(Q)$  of vertices as  $Q$ . Note that the exponent  $k(A) + \#A - \#V$  equals  $b_1(Q_A)$ , the first Betti number of  $Q_A$ .

We will only consider the *external activity polynomial*  $R_Q$  of  $Q$  obtained by specializing to  $x = 1$ . For  $Q$  connected, we have

$$R_Q(q) := T_Q(1, q) = \sum_{Q' \subseteq Q} (q - 1)^{b_1(Q')}, \tag{2.3.4}$$

where the sum is over all connected subgraphs  $Q' \subseteq Q$  with vertex set  $V$ . (This polynomial is related to the reliability polynomial of  $Q$  by a simple change of variables, hence the choice of name.) A remarkable theorem of Tutte guarantees that  $T_Q$ , and hence also  $R_Q$ , has non-negative (integer) coefficients.

For example, the Tutte polynomial of complete graphs  $K_n$  was computed in [Tu], cf. also [Ar, Theorem 4.3]. This implies the following generating function of the Poincaré polynomials  $P(\mathcal{M}_{\mathbf{1}}^{K_n}; t) = R_{K_n}(t^2)$  of Nakajima toric quiver varieties attached to the complete graphs  $K_n$

$$\sum_{n \geq 1} R_{K_n}(q) \frac{T^n}{n!} = (q - 1) \log \sum_{n \geq 0} q^{\binom{n}{2}} \frac{(T/(q - 1))^n}{n!} \tag{2.3.5}$$



2.3.2. *Twisted ADHM spaces  $\mathcal{M}_{n,m}$  and Hilbert scheme of points on the affine plane  $(\mathbb{C}^2)^{[n]}$ .* — Here  $G = GL(V)$ , where  $V$  is an  $n$ -dimensional  $\mathbb{K}$  vector space <sup>(3)</sup>. We need three types of basic representations of  $G$ . The adjoint representation  $\rho_{ad} : GL(V) \rightarrow GL(\mathfrak{gl}(V))$ , the defining representation  $\rho_{def} = Id : G \rightarrow GL(V)$  and the trivial representations  $\rho_{triv}^W = 1 : G \rightarrow GL(W)$ , where  $\dim_{\mathbb{K}}(W) = m$ . Fix  $m$  and  $n$ . Define  $\mathbb{V} = \mathfrak{gl}(V) \times \text{Hom}(V, W)$ ,  $\mathcal{M} = \mathbb{V} \times \mathbb{V}^*$  and  $\rho : G \rightarrow GL(\mathbb{V})$  by  $\rho = \rho_{ad} \times \rho_{def} \otimes \rho_{triv}^W$ . Then we take the central element  $\xi = Id_V \in \mathfrak{gl}(V)$  and define the twisted ADHM space as

$$\mathcal{M}_{n,m} = \mathcal{M} //_{\xi} G = \mu^{-1}(\xi) // G,$$

where

$$\mu(A, B, I, J) = [A, B] + IJ,$$

with  $A, B \in \mathfrak{gl}(V)$ ,  $I \in \text{Hom}(W, V)$  and  $J \in \text{Hom}(V, W)$ .

The space  $\mathcal{M}_{n,m}$  is empty when  $m = 0$  (the trace of a commutator is always zero), diffeomorphic with the Hilbert scheme of  $n$ -points on  $\mathbb{C}^2$ , when  $m = 1$ , and is the twisted version of the ADHM space [ADHM] of  $U(m)$  Yang-Mills instantons of charge  $n$  on  $\mathbb{R}^4$  (cf. [Na3]). As explained in [Ha5, Theorem 2] the arithmetic Fourier transform technique of §2 yields the following generating function for the Poincaré polynomials of  $\mathcal{M}_{n,m}$  (originally due to [NY, Corollary 3.10]):

$$\sum_{n=0}^{\infty} P(\mathcal{M}_{n,m}; t) T^n = \prod_{i=1}^{\infty} \prod_{b=1}^m \frac{1}{(1 - t^{2(m(i-1)+b-1)} T^i)}. \tag{2.3.6}$$

In particular when  $m = 1$  this gives for the generating function of Poincaré polynomials of Hilbert schemes of points on  $(\mathbb{C}^2)^{[n]}$

$$\sum_{n=0}^{\infty} P((\mathbb{C}^2)^{[n]}; t) T^n = \prod_{i=1}^{\infty} \frac{1}{(1 - t^{2(i-1)} T^i)}, \tag{2.3.7}$$

Göttsche’s formula from [Göt], which by Euler’s formula reduces to

$$b_{2i}((\mathbb{C}^2)^{[n]}) = \#\{\lambda \mid |\lambda| = n, l(\lambda) = i\} \tag{2.3.8}$$

where  $l(\lambda)$  is the number of parts in the partition  $\lambda$  of  $n$ ; this was the original computation of Ellingsrud-Stromme in [EIS].

2.3.3. *Nakajima quiver varieties  $\mathcal{M}_{\mathcal{V},\mathbf{w}}^Q$  and  $\mathcal{M}_{\mathcal{V}}$ .* — Here we recall the definition of the affine version of Nakajima’s quiver varieties [Na2]. Let  $Q = (\mathcal{V}, \mathcal{E})$  be a quiver, *i.e.*, an oriented graph on a finite set  $\mathcal{V} = \{1, \dots, n\}$  with  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  a finite set of oriented (perhaps multiple and loop) edges. To each vertex  $i$  of the graph we associate two finite dimensional  $\mathbb{K}$  vector spaces  $V_i$  and  $W_i$ . We call  $(\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{w}_1, \dots, \mathbf{w}_n) = (\mathbf{v}, \mathbf{w})$  the

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3. Here  $\mathbb{K} = \mathbb{C}$  when we study the complex semiprojective varieties and  $\mathbb{K} = \mathbb{F}_q$  when we do arithmetic harmonic analysis on them.

dimension vector, where  $\mathbf{v}_i = \dim(V_i)$  and  $\mathbf{w}_i = \dim(W_i)$ . To this data we associate the grand vector space:

$$\mathbb{V}_{\mathbf{v},\mathbf{w}} = \bigoplus_{(i,j) \in \mathcal{E}} \text{Hom}(V_i, V_j) \oplus \bigoplus_{i \in \mathcal{V}} \text{Hom}(V_i, W_i),$$

the group and its Lie algebra

$$G_{\mathbf{v}} = \bigtimes_{i \in \mathcal{V}} \text{GL}(V_i) \quad \mathfrak{g}_{\mathbf{v}} = \bigoplus_{i \in \mathcal{V}} \mathfrak{gl}(V_i),$$

and the natural representation

$$\rho_{\mathbf{v},\mathbf{w}} : G_{\mathbf{v}} \longrightarrow \text{GL}(\mathbb{V}_{\mathbf{v},\mathbf{w}}),$$

with derivative

$$\varrho_{\mathbf{v},\mathbf{w}} : \mathfrak{g}_{\mathbf{v}} \longrightarrow \mathfrak{gl}(\mathbb{V}_{\mathbf{v},\mathbf{w}}).$$

The action is from both left and right on the first term, and from the left on the second.

We now have  $G_{\mathbf{v}}$  acting on  $\mathbb{M}_{\mathbf{v},\mathbf{w}} = \mathbb{V}_{\mathbf{v},\mathbf{w}} \times \mathbb{V}_{\mathbf{v},\mathbf{w}}^*$  preserving the symplectic form with moment map  $\mu_{\mathbf{v},\mathbf{w}} : \mathbb{V}_{\mathbf{v},\mathbf{w}} \times \mathbb{V}_{\mathbf{v},\mathbf{w}}^* \rightarrow \mathfrak{g}_{\mathbf{v}}^*$  given by (1.1.1). We take now  $\xi_{\mathbf{v}} = (Id_{V_1}, \dots, Id_{V_n}) \in (\mathfrak{g}_{\mathbf{v}}^*)^{G_{\mathbf{v}}}$ , and define the affine Nakajima quiver variety [Na2] as

$$\mathcal{M}_{\mathbf{v},\mathbf{w}}^Q = \mu_{\mathbf{v},\mathbf{w}}^{-1}(\xi_{\mathbf{v}}) // G_{\mathbf{v}}.$$

As explained in [Ha5] and [Ha6] the arithmetic harmonic analysis technique of §2 translates to the formula (2.3.9) below. We first introduce some notation on partitions following [Mac]. We let  $\mathcal{P}(s)$  be the set of partitions of  $s \in \mathbb{Z}_{\geq 0}$ . For two partitions  $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathcal{P}(s)$  and  $\mu = (\mu_1, \dots, \mu_m) \in \mathcal{P}(s)$  we define  $n(\lambda, \mu) = \sum_{i,j} \min(\lambda_i, \mu_j)$ . Writing  $\lambda = (1^{m_1(\lambda)}, 2^{m_2(\lambda)}, \dots) \in \mathcal{P}(s)$  we let  $l(\lambda) = \sum m_i(\lambda) = l$  be the number of parts in  $\lambda$ . For any  $\lambda \in \mathcal{P}(s)$  we have  $n(\lambda, (1^s)) = sl(\lambda)$ .

**Theorem 2.3.1.** — *Let  $Q = (\mathcal{V}, \mathcal{E})$  be a quiver, with  $\mathcal{V} = \{1, \dots, n\}$  and  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ , with possibly multiple edges and loops. Fix a dimension vector  $\mathbf{w} \in \mathbb{Z}_{\geq 0}^{\mathcal{V}}$ . The Poincaré polynomials  $P(\mathcal{M}_{\mathbf{v},\mathbf{w}}^Q)$  of the corresponding Nakajima quiver varieties are given by the generating function:*

$$\sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{\mathcal{V}}} P_t(\mathcal{M}_{\mathbf{v},\mathbf{w}}^Q) t^{-d(\mathbf{v},\mathbf{w})} T^{\mathbf{v}} = \frac{\sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda^1 \in \mathcal{P}(\mathbf{v}_1)} \cdots \sum_{\lambda^n \in \mathcal{P}(\mathbf{v}_n)} \frac{(\prod_{(i,j) \in \mathcal{E}} t^{-2n(\lambda^i, \lambda^j)}) (\prod_{i \in \mathcal{V}} t^{-2n(\lambda^i, (1^{\mathbf{w}_i})}))}{\prod_{i \in \mathcal{V}} (t^{-2n(\lambda^i, \lambda^i)} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1-t^{2j}))}}{\sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^{\mathcal{V}}} T^{\mathbf{v}} \sum_{\lambda^1 \in \mathcal{P}(\mathbf{v}_1)} \cdots \sum_{\lambda^n \in \mathcal{P}(\mathbf{v}_n)} \frac{\prod_{(i,j) \in \mathcal{E}} t^{-2n(\lambda^i, \lambda^j)}}{\prod_{i \in \mathcal{V}} (t^{-2n(\lambda^i, \lambda^i)} \prod_k \prod_{j=1}^{m_k(\lambda^i)} (1-t^{2j}))}}, \tag{2.3.9}$$

where  $d(\mathbf{v}, \mathbf{w}) = 2 \sum_{(i,j) \in \mathcal{E}} \mathbf{v}_i \mathbf{v}_j + 2 \sum_{i \in \mathcal{V}} \mathbf{v}_i (\mathbf{w}_i - \mathbf{v}_i)$  is the dimension of  $\mathcal{M}_{\mathbf{v},\mathbf{w}}^Q$  and  $T^{\mathbf{v}} = \prod_{i \in \mathcal{V}} T_i^{\mathbf{v}_i}$ .

**Example 2.3.2.** — We will look at the case of  $Q = A_1$  a single vertex with no edges. In this case the semiprojective quiver variety  $\mathcal{M}_{n,m}^{A_1}$  is isomorphic [Na1, Theorem 7.3] with the cotangent bundle  $T^*\text{Gr}(n, m)$  of the Grassmanian of  $m$  planes in  $\mathbb{C}^n$ . In this case we can directly count points on  $\text{Gr}(n, m)$  over finite fields and we get the following well known formula for its Poincaré polynomial:

$$P(T^*\text{Gr}(n, m); t) = \left[ \begin{matrix} n \\ k \end{matrix} \right]_{t^2} = \prod_{i=1}^k \frac{1 - t^{2(n+1-i)}}{1 - t^{2i}} \tag{2.3.10}$$

The combination of (2.3.10) and (2.3.9) gives a curious  $q$ -binomial type of theorem.

**Example 2.3.3.** — When the quiver is the Jordan quiver, *i.e.*, one loop on a single vertex, then  $\mathcal{M}_{\mathbf{v}, \mathbf{w}}^Q = \mathcal{M}_{n,m}$  the twisted Yang-Mills moduli spaces from §2.3.2. The formula (2.3.9) then reduces to (2.3.6).

We also consider Nakajima quiver varieties  $\mathcal{M}_{\mathbf{v}}^Q$  attached to a single dimension vector  $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$  on the same quiver  $Q$ . We construct

$$\mathbb{V}_{\mathbf{v}} := \bigoplus_{(i,j) \in \mathcal{E}} \text{Hom}(V_i, V_j),$$

which will also carry a natural representation

$$\rho_{\mathbf{v}, \mathbf{w}} : G_{\mathbf{v}} \longrightarrow \text{GL}(\mathbb{V}_{\mathbf{v}, \mathbf{w}}).$$

In the framework of (1.1.1) this gives rise to the symplectic vector space  $\mathbb{M}_{\mathbf{v}} := \mathbb{V}_{\mathbf{v}} \times \mathbb{V}_{\mathbf{v}}^*$  and the moment map  $\mu_{\mathbf{v}} : \mathbb{M}_{\mathbf{v}} \rightarrow \mathfrak{g}_{\mathbf{v}}^*$  leading to the quotient  $\mathcal{M}_{\mathbf{v}}^Q := \mu^{-1}(0) //_{\xi} G_{\mathbf{v}}$  where  $\xi \in \text{Hom}(G_{\mathbf{v}}, \mathbb{C}^{\times})$  is a character of  $G_{\mathbf{v}}$ . When  $\mathbf{v}$  is indivisible (*i.e.*, the equation  $\mathbf{v} = k\mathbf{v}'$  for an integer  $k$  and dimension vector  $\mathbf{v}'$  implies  $k = 1$ ) it is known that  $\xi$  can be chosen so that  $\mathcal{M}_{\mathbf{v}}^Q$  is smooth semiprojective hyperkähler variety. For  $\mathbf{v}$  indivisible it is proved in [CBvdB] that

$$P(\mathcal{M}_{\mathbf{v}}^Q; t) = t^{d_{\mathbf{v}}} A_Q(\mathbf{v}; t^{-2}), \tag{2.3.11}$$

where  $A_Q(\mathbf{v}; q) \in \mathbb{Z}[q]$  is the *Kac polynomial* [Kac], which counts absolutely indecomposable representations of the quiver  $Q$  and  $d_{\mathbf{v}} = \dim \mathcal{M}_{\mathbf{v}}^Q$ . A generating function formula was obtained for  $A_Q(\mathbf{v}; q)$  by Hua [Hua], and it takes the following combinatorial form:

$$\sum_{\mathbf{v} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} A_{\mathbf{v}}(q) T^{\mathbf{v}} = (q-1) \text{Log} \left( \sum_{\pi = (\pi^1, \dots, \pi^r) \in \mathcal{P}^r} \frac{\prod_{i \rightarrow j \in \Omega} q^{\langle \pi^i, \pi^j \rangle}}{\prod_{i \in I} q^{\langle \pi^i, \pi^i \rangle} \prod_k \prod_{j=1}^{m_k(\pi^i)} (1 - q^{-j})} T^{|\pi|} \right), \tag{2.3.12}$$

where  $\text{Log}$  denotes the plethystic logarithm, as explained in [HLV1, §2.3.3]. The formula (2.3.11) was proved by the arithmetic harmonic analysis technique of §2 in [HLV2] for all quivers, with a cohomological interpretation of  $A_Q(\mathbf{v}, q)$  in the case when  $\mathbf{v}$  is a divisible dimension vector, settling Kac’s conjecture [Kac] that the  $A$ -polynomial  $\mathbf{A}_Q(\mathbf{v}, q) \in \mathbb{Z}_{\geq 0}[q]$  has non-negative coefficients.

2.3.4. *Moduli of Higgs bundles  $\mathcal{M}_n^g$ .* — Denote by  $\mathcal{M}_n^g$  the moduli spaces of rank  $n$  degree 1 stable Higgs bundles on a smooth projective curve of genus  $g$ . The construction of the moduli space can be done by algebraic geometric techniques using GIT quotients as in [Ni] or by gauge theoretical means using an infinite dimensional hyperkähler quotient construction as was done in the original paper [Hi1]. This latter construction is not algebraic, and so it is unclear how our arithmetic harmonic analysis of §1.1.1 would extend to this case. The cohomology of  $\mathcal{M}_n^g$  is the most interesting (due to various connections to a variety of subjects) and the least understood. There are various results on its Betti numbers available [Hi1, Go, GHS] but we only have a conjectured formula. First we introduce rational functions  $H_n(z, w) \in \mathbb{Q}(z, w)$  by the generating function:

$$\sum_{n=0}^{\infty} H_n(z, w) T^n = (1-z)(1-w) \text{Log} \left( \sum_{\lambda \in \mathcal{P}} \prod \frac{(z^{2l+1} - w^{2a+1})^{2g}}{(z^{2l+2} - w^{2a})(z^{2l} - w^{2a+2})} T^{|\lambda|} \right) \tag{2.3.13}$$

Then we have the following conjecture [HV, Conjecture 4.2.1]

$$P(\mathcal{M}_n^g; t) = t^{d_n} H_n(1, -1/t), \tag{2.3.14}$$

where already part of the conjecture is that  $H_n(w, z) \in \mathbb{Z}[w, z]$  is a polynomial in  $w, z$ .

**Remark 2.3.4.** — This conjecture was obtained *via* a more elaborate version of the arithmetic harmonic analysis technique of §1.1.1. Namely a non-abelian version of the arithmetic harmonic analysis allows us [HV] to count points on certain  $\text{GL}_n$ -character varieties; and the conjecture (2.3.13) is a non-trivial extension of that result, and the non-abelian Hodge theorem of [Si1] which shows that this  $\text{GL}_n$ -character variety is canonically diffeomorphic with  $\mathcal{M}_n^g$  thus shares its cohomology.

**Remark 2.3.5.** — As  $\mathcal{M}_n^g$  is semiprojective by [Si2, Corollary 10.3] its cohomology is pure by Corollary 1.3.2. Therefore counting the  $\mathbb{F}_q$  rational points of  $\mathcal{M}_n^g$  would lead to its Betti numbers. However we do not know how to extend our arithmetic harmonic analysis of §1.1.1 to this case. There are recent works of Chaudouard and Laumon [CL, Ch] where a different kind of harmonic analysis is used to count  $\#\mathcal{M}_n^g(\mathbb{F}_q)$  but so far only the  $n = 3$  case is complete where those results confirm the conjecture (2.3.13).

**Remark 2.3.6.** — One last observation is that the similarity of (2.3.12) and (2.3.13) is not accidental. In fact it was proved in [HV, Theorem 4.4.1] that  $H_n(0, \sqrt{q}) = A_{S_g}((n), q)$  where  $S_g$  is the  $g$  loop quiver on one vertex. In particular a certain subring of  $H^*(\mathcal{M}^g; \mathbb{Q})$  is conjectured to have graded dimensions with Poincaré polynomial  $A_{S_g}((n), q)$ . This and more general versions of such conjectures [HLV3, Conjecture 1.3.2] show that the cohomology of Nakajima quiver varieties for comet-shaped quivers should be isomorphic with subrings of the cohomology of certain Higgs moduli spaces. This maybe relevant when we compare the large scale asymptotics of the Betti numbers of these varieties, as will be done in the remaining of this paper.

### 3. Visual distribution of Betti numbers: the big picture

Motivated by Question 1.4.7 in this section we will be studying pictures of the distribution of Betti numbers of our semiprojective hyperkähler manifolds. The reason we can look at very large examples are the combinatorially tractable formulas in the previous §2.3.

**3.1. Toric quiver variety  $\mathcal{M}_1^{K_{40}}$ .** — Using formula (2.3.5) one can efficiently compute the Betti numbers of the toric quiver variety  $\mathcal{M}_1^{K_{40}}$  for the complete graph  $K_{40}$  on 40 vertices. This is a hyper-compact semiprojective hyperkähler variety of real dimension 2964. The top non-trivial Betti number therefore is the middle one  $b_{1482} \approx 2 \times 10^{46}$ . The sequence of Betti numbers is *unimodal*<sup>(4)</sup> and the largest Betti number is  $b_{1288} \approx 8 \times 10^{58}$ . In Figure 1 we plotted only the non-trivial even Betti numbers<sup>(5)</sup>.

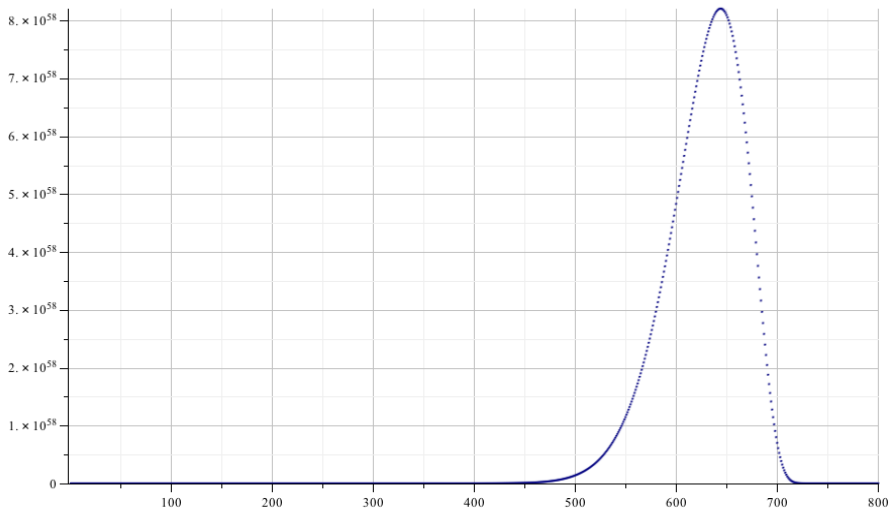


FIGURE 1. Distribution of even Betti numbers of the toric quiver variety  $\mathcal{M}_1^{K_{40}}$

**3.2. Hilbert scheme  $(\mathbb{C}^2)^{[500]}$ .** — We can efficiently compute Betti numbers of Hilbert schemes of  $n$  points on  $\mathbb{C}^2$  for large  $n$  using (2.3.7). When  $n = 500$ , Figure 2 shows the distribution of even Betti numbers. We have  $\dim_{\mathbb{R}}(\mathbb{C}^2)^{[500]} = 2000$ . The Hilbert scheme  $(\mathbb{C}^2)^{[500]}$  is a semiprojective hyperkähler manifold, but not hyper-compact, and the top non-trivial Betti number is  $b_{998} = 1$ . Again the sequence of Betti numbers is unimodal. The maximal Betti number is  $b_{896} \approx 5.5 \times 10^{19}$ .

4. Meaning it has single local maximum.

5. This means that one needs to double the value on the  $x$ -axis to get the correct degree for the Betti number.

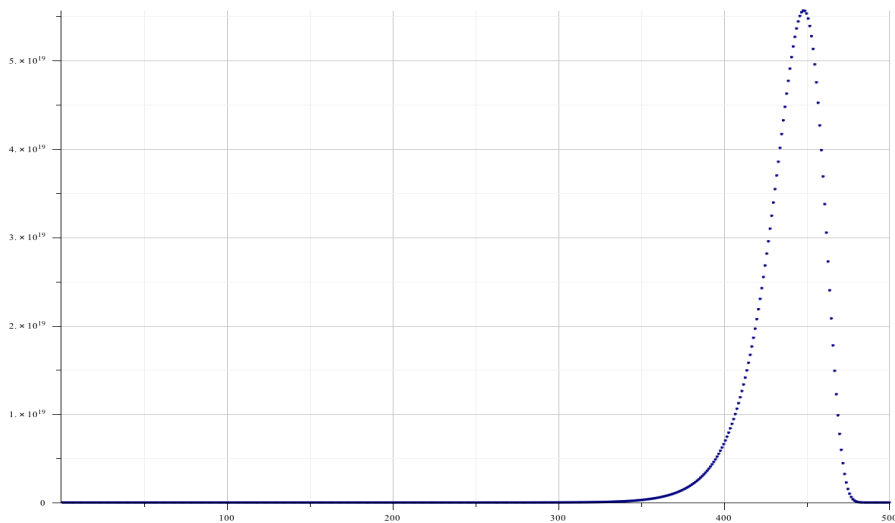


FIGURE 2. Distribution of even Betti numbers of the Hilbert scheme  $(\mathbb{C}^2)^{[500]}$  of 500 points on  $\mathbb{C}^2$

**3.3. Twisted ADHM space  $\mathcal{M}_{40,20}^{\hat{A}_0}$ .** — The Nakajima quiver variety  $\mathcal{M}_{m,n}^{\hat{A}_0}$  attached to the Jordan quiver  $\hat{A}_0$  and dimension vectors  $\mathbf{v} = (m)$  and  $\mathbf{w} = (n)$  is a semiprojective hyperkähler variety, which is not hyper-compact. When  $m = 40$  and

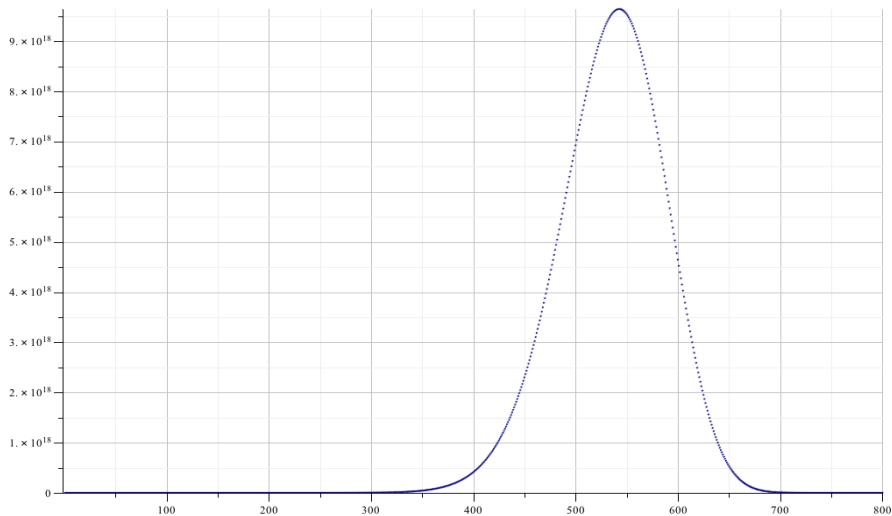


FIGURE 3. Distribution of even Betti numbers of ADHM space  $\mathcal{M}_{40,20}^{\hat{A}_0}$

$n = 20$ , Figure 3 shows the distribution of even Betti numbers. The top non-zero Betti number is  $b_{1598} = 1$ . There are only even Betti numbers and they form a unimodal sequence. The maximal Betti number is  $b_{1086} \approx 9.6 \times 10^{17}$ .

**3.4. Cotangent bundle of Grassmannian  $\mathcal{M}_{30,100}^{A_1} \cong T^*\text{Gr}(100, 30)$ .** — As was discussed earlier in §2.3.3 the Nakajima quiver variety  $\mathcal{M}_{30,100}^{A_1} \cong T^*\text{Gr}(100, 30)$  for the trivial  $A_1$  quiver with dimension vectors  $\mathbf{v} = (30)$  and  $\mathbf{w} = (100)$  is the cotangent bundle to the Grassmannian of 30 dimensional subspaces in  $\mathbb{C}^{100}$ . This is a semiprojective hyperkähler manifold which is hyper-compact. Of course in this case the core is the zero section of the cotangent bundle, thus it is the smooth projective Grassmannian  $\text{Gr}(100, 30)$ . It only has even cohomology and satisfies Hard Lefschetz. In particular the sequence of even Betti numbers is unimodal and symmetric. The top non-zero Betti number is  $b_{4200} = 1$  while the maximal one is  $b_{2100} \approx 8.7 \times 10^{22}$ .

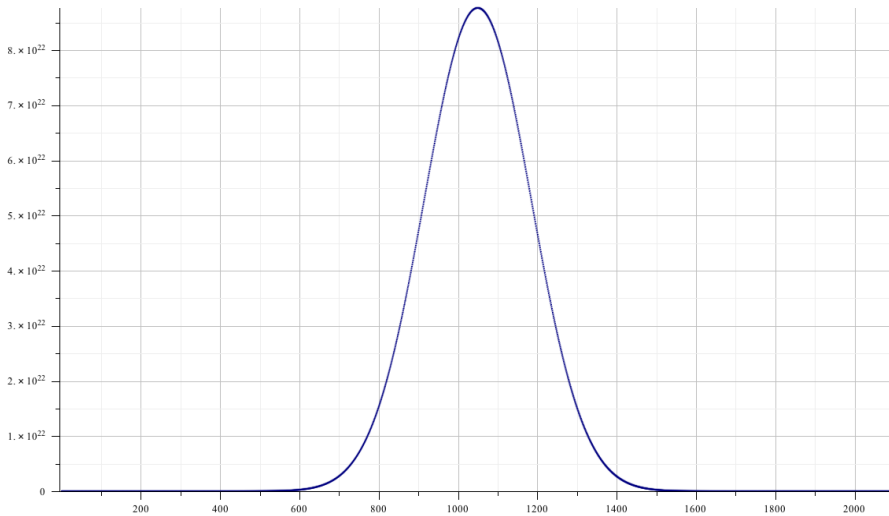


FIGURE 4. Distribution of even Betti numbers of cotangent bundle to Grassmannian  $T^*\text{Gr}(100, 30)$

**3.5. A quiver variety  $\mathcal{M}_{(15,7)}^Q$ .** — We include a smooth quiver variety of type  $\mathcal{M}_Q^Q$  where  $Q$  is the graph on two vertices, with 10 loops on the first vertex, and a connecting edge to the second vertex, and furthermore  $\mathbf{v} = (15, 7)$ . This is a smooth (because  $(15, 7)$  is indivisible) semiprojective hyperkähler variety, which is not hypercompact, due to the presence of loops on the first vertex. We have  $\dim_{\mathbb{R}} \mathcal{M}_{(15,7)}^Q = 8328$  and top non-trivial Betti number  $b_{3862} = 1$ . Again, there are only even Betti numbers and their sequence is unimodal, with maximal Betti number  $b_{3036} \approx 2.1 \times 10^{22}$ .

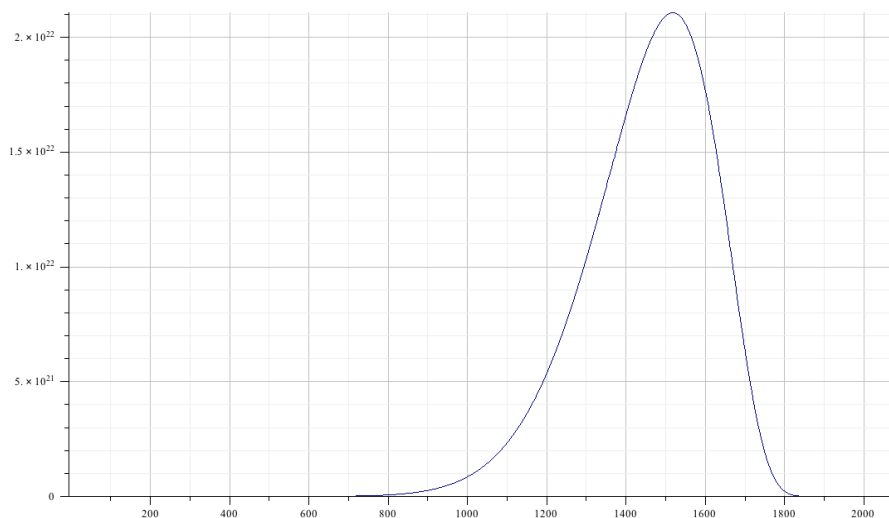


FIGURE 5. Distribution of even Betti numbers of the quiver variety  $\mathcal{M}_{(15,7)}^Q$

**3.6. Cotangent bundle of Jacobian  $\mathcal{M}_1^{100} \cong T^* \text{Jac}(C_{100})$ .** — When  $n = 1$  the moduli space of rank 1 degree 1 Higgs bundles is isomorphic with  $\mathcal{M}_1^g \cong T^* \text{Jac}(C_g)$  the cotangent bundle to the Jacobian of the curve  $C_g$  of genus  $g$ . Of course this is also a semiprojective hyperkähler manifold, which is hyper-compact. Just like in the Grassmannian case above, the Jacobian  $\text{Jac}(C_g)$  as the zero section of its cotangent bundle is the core of the semiprojective variety, that is the core is a smooth projective variety. The Poincaré polynomial then is just

$$P(\mathcal{M}_1^g; t) = (1 + t)^{2g}.$$

Figure 6 shows the distribution of all non-trivial Betti numbers when  $g = 100$ . The top non-trivial one is  $b_{200} = 1$ . The sequence of Betti numbers is unimodal, with maximal value  $b_{100} \approx 8.7 \times 10^{22}$ .

**3.7. Moduli space of Higgs bundles  $\mathcal{M}_8^2$ .** — The moduli space of rank  $n$  degree 1 stable Higgs bundles on a smooth projective curve of genus  $g$  is smooth semiprojective, hyper-compact hyperkähler manifold. We can use (2.3.13) and (2.3.14) to compute the conjectured Betti numbers of  $\mathcal{M}_n^g$  for small values of  $n$  and  $g$ . In fact this formula is the most computationally demanding, and we could only evaluate the  $g = 2$  and  $n = 8$  case. Part of the reason of the computational difficulty is because the calculation goes through evaluating the two variable polynomial  $H_8(q, t)$  from (2.3.13) which already has 11786 terms. At any rate in this particular case  $\dim_{\mathbb{R}} \mathcal{M}_8^2 = 252$ , thus the top non-trivial Betti number is  $b_{126}$  which equals 12300. An important difference between



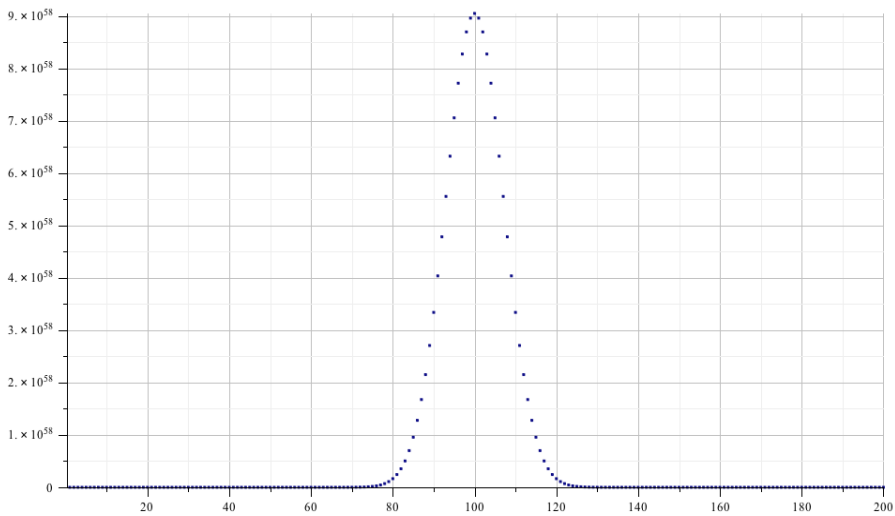


FIGURE 6. Distribution of all Betti numbers of cotangent bundle to Jacobian  $T^*\text{Jac}(C_{200})$

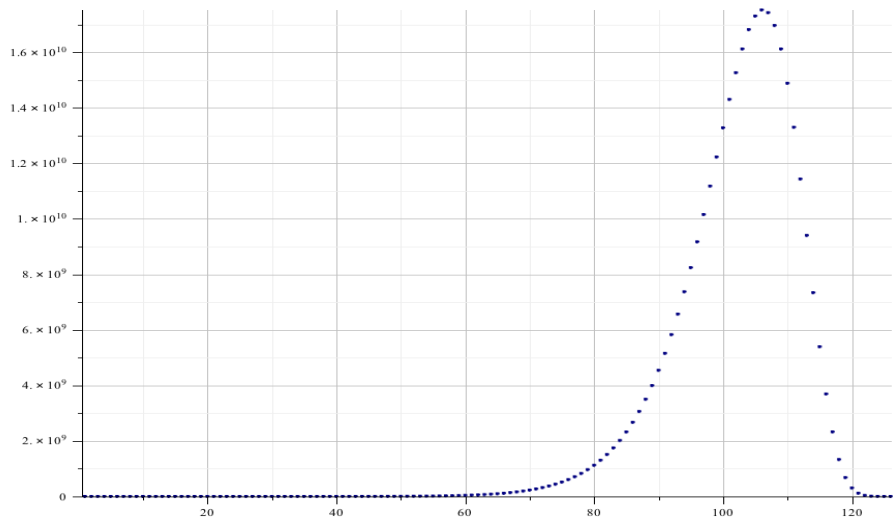


FIGURE 7. Distribution of all conjectured Betti numbers of moduli of rank 8 Higgs bundles  $\mathcal{M}_8^2$

this case and the previous ones, is that  $\mathcal{M}_8^2$  has non-trivial odd Betti numbers. The full sequence is not unimodal but both sequences of odd and even Betti numbers are unimodal. The maximal Betti number is  $b_{106} \approx 1.7 \times 10^{10}$ .

#### 4. Asymptotic shape of Betti numbers: the macroscopic picture

In the previous section we have plotted the distributions of Betti numbers of some large examples of our semiprojective hyperkähler varieties. Originally we were motivated by studying potential extensions of the Weak Hard Lefschetz Theorem 1.4.1. Surprisingly, the plots in the previous section behave in a rather peculiar manner. First, we note that the sequence of even Betti numbers is always unimodal. Second, and more puzzling, is the apparent asymptotic behavior of the distribution of Betti numbers: the plots above seem to suggest the existence of a certain continuous limiting distribution<sup>(6)</sup>. Furthermore it seems that the distributions on Figures 6, 4 and 3 are the same while the remaining ones on Figures 1, 2, 5 and 7 also look similar.

In this section we will prove some rigorous results about such limiting distributions, in particular we will determine this distribution in the case of Figures 6, 4, 2 and 1 and offer conjectures in the remaining cases.

First we discuss what we mean by a limiting distribution of Betti numbers of a family of varieties.

**4.1. Generalities.** — In this section we consider sequences of varieties  $X_0, X_1, \dots$  whose Betti numbers  $b_i(X_n)$  approach a limiting distribution as  $n \rightarrow \infty$ . For simplicity, we will typically assume that the all varieties  $X$  under consideration satisfy  $b_{2i+1}(X) = 0$  and let

$$E(X, q) := \sum_{i=0}^{\dim X} b_{2i}(X) q^{\dim X - i}.$$

If  $X$  is a polynomial count variety with pure mixed Hodge structure then by Theorem 2.1.1  $E(X, q) = \mathcal{P}(q)$ , where  $\mathcal{P}$  is a polynomial such that  $\#X(\mathbb{F}_q) = \mathcal{P}(q)$  for generic  $q$ .

To a Laurent polynomial  $E(q) = \sum_i e_i q^i$  with non-negative real coefficients we associate the discrete measure  $d\mu_E$  on  $[-\infty, \infty]$  such that

$$\int_{-\infty}^{\infty} \phi(x) d\mu_E := \sum_i \phi(i) e_i.$$

**Definition 4.1.1.** — Given a measure  $\mu$  on  $[-\infty, \infty]$  its *moments* are the real numbers

$$M_k := \int_{-\infty}^{\infty} x^k d\mu$$

and its *factorial moments* are the real numbers

$$m_k := \int_{-\infty}^{\infty} \binom{x}{k} d\mu.$$

---

6. Even a  $C^\infty$  one: one can also plot the higher discrete derivatives of the distributions above and still get some continuous looking distributions.

Clearly these two kinds of moments are linearly related and since the leading term in  $\binom{x}{k}$  is  $x^k/k!$ , typically the asymptotic behavior of  $m_k$  and  $M_k/k!$  for a sequence of measures is the same. It depends on the situation which set of moments is easier to deal with.

For a measure  $d\mu$  on  $[-\infty, \infty]$  we define the generating function of moments

$$M_\mu(t) := \sum_{k \geq 0} M_k \frac{t^k}{k!}, \quad m_\mu(\eta) := \sum_{k \geq 0} m_k \eta^k. \tag{4.1.1}$$

If  $d\mu_E$  is a discrete measure associated to the Laurent polynomial  $E = \sum_i e_i q^i$  then

$$M_{\mu_E}(t) = E(e^t), \quad m_{\mu_E}(\eta) = E(1 + \eta). \tag{4.1.2}$$

If  $d\mu_1, d\mu_2$  are two measures then

$$M_{\mu_1}(t)M_{\mu_2}(t) = M_\mu(t), \quad m_{\mu_1}(t)m_{\mu_2}(t) = m_\mu(t),$$

where  $d\mu := d\mu_1 * d\mu_2$  (additive convolution in  $\mathbb{R}$ ). If  $d\mu_1, d\mu_2$  have density functions  $\omega_1, \omega_2$  respectively then  $d\mu$  has density function

$$\omega_1 * \omega_2(x) := \int_{-\infty}^{\infty} \omega_1(y)\omega_2(x - y) dy.$$

If  $d\mu$  is a measure on  $[-\infty, \infty]$  then for any real number  $a$  we have

$$e^{-at}M_\mu(t) = \sum_{k \geq 0} M_k^{(a)} \frac{t^k}{k!}, \tag{4.1.3}$$

where  $M_k^{(a)}$  denotes the  $k$ -th moment of the translated measure  $d\mu(x + a)$ .

Let  $d\mu_n$  for  $n = 1, 2, \dots$  and  $d\mu$  be measures on  $[-\infty, \infty]$ . We say that  $d\mu_n$  converges *in distribution* to  $d\mu$  as  $n \rightarrow \infty$  if

$$\lim_{n \rightarrow \infty} \Phi_n(x) = \Phi(x)$$

at all points  $x$  of continuity of  $\Phi$ , where  $\Phi_n$  and  $\Phi$  are the cumulative density function associated to  $d\mu_n$  and  $d\mu$  respectively. *I.e.*,

$$\Phi_n(x) = \int_{-\infty}^x d\mu_n, \quad \Phi(x) = \int_{-\infty}^x d\mu.$$

Typically a proof of such convergence goes by proving that the appropriately scaled sequence of moments of  $d\mu_n$  converges to those of  $d\mu$  by means of the following

**Theorem 4.1.2.** — *Suppose that  $d\mu$  is determined by its moments, that  $d\mu_n$  has moments of all orders, and that  $\lim_{n \rightarrow \infty} M_k(d\mu_n) = M_k(d\mu)$  for  $k = 0, 1, \dots$ . Then  $d\mu_n$  converges in distribution to  $d\mu$ .*

*Proof.* — This is Theorem 30.2 in [Bil]. □

The kind of statement we look for is the following.

**Definition 4.1.3.** — The sequence of varieties  $X_n$  has *limiting Betti distribution*  $d\mu$  if up to rescaling and translation  $d\mu_n$ , the measure associated to  $E(X_n, q)$ , converges to  $d\mu$  in distribution. More precisely, if there exist real constants  $\alpha_n, \beta_n, \gamma_n$  with  $\alpha_n, \gamma_n > 0$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{\gamma_n} \Phi_n(\alpha_n x + \beta_n) = \Phi(x)$$

at all points  $x$  of continuity of  $\Phi$ , where  $\Phi_n$  and  $\Phi$  are the cumulative density function associated to  $E(X_n, q)$  and  $d\mu$  respectively. *I.e.*,

$$\Phi_n(x) = \int_{-\infty}^x d\mu_n, \quad \Phi(x) = \int_{-\infty}^x d\mu.$$

**Remark 4.1.4.** — Note that the limiting distribution  $d\mu$  itself is only determined up to rescaling and translation. This will suffice for the kind of qualitative analysis that we are interested in.

**4.2. Large Tori and Grassmannians.** — When  $X_n = \mathcal{M}_1^n = T^*\text{Jac}(C_n)$  it has the topology of a  $2n$ -dimensional split torus. Then

$$b_i(X_n) = \binom{2n}{i}.$$

It follows from the Central Limit theorem that the sequence  $X_0, X_1, \dots$  has Gaussian limiting Betti distribution. This we could observe in Figure 6 for  $n = 100$ .

Now fix a positive integer  $r$  and consider the Grassmanian variety  $X_n := G_r^{r+n}$  parametrizing  $r$ -dimensional planes in an ambient space of dimension  $r + n$ . It is well known that the number of points of the Grassmanian over a finite field is given by a  $q$ -binomial number. Explicitly,

$$E_n(q) := \#G_r^{r+n}(\mathbb{F}_q) = \begin{bmatrix} n+r \\ r \end{bmatrix} = \frac{\prod_{j=1}^r (q^{n+j} - 1)}{\prod_{j=1}^r (q^j - 1)}. \tag{4.2.1}$$

Consider the  $j$ -th factor of this product and assume that  $n = jm$  for some integer  $m$ . Then

$$\frac{q^{n+j} - 1}{q^j - 1} = q^{n/2} \frac{q^{(n+j)/2} - q^{-(n+j)/2}}{q^{j/2} - q^{-j/2}} = q^{n/2} \sum_{i=0}^m q^{j(i-m/2)}. \tag{4.2.2}$$

If we now replace  $q$  by  $q^{1/n}$  and ignore the power of  $q$  prefactor we obtain

$$\sum_{i=0}^m q^{i/m-1/2}.$$

As  $m$  approaches infinity the associated density function converges to  $\chi^{(1)} := \chi_{[-1/2, 1/2]}$ , the characteristic function of the interval  $[-1/2, 1/2]$ .

Therefore, if  $n$  is divisible by all  $j = 1, 2, \dots, r$  then  $E_n(q)$  has associated density function that scaled appropriately approximates the  $r$ -th iterated convolution of  $\chi^{(1)}$ :

$$\chi^{(r)} := \underbrace{\chi^{(1)} * \dots * \chi^{(1)}}_r.$$

Consequently, we should expect  $X_n$  to have limiting Betti distribution  $\chi^{(r)}$ . To prove this for the full sequence  $X_n$  (and not just for the subsequence of  $n$ 's in the previous argument) we consider the moment generating function  $E(e^t)$  (see (4.1.1)). By (4.2.2) we have

$$e^{-rt/2} E_n(e^{t/n}) = \prod_{j=1}^r \frac{e^{\frac{1}{2}(1+j/n)t} - e^{-\frac{1}{2}(1+j/n)t}}{e^{\frac{1}{2}(j/n)t} - e^{-\frac{1}{2}(j/n)t}}.$$

Taking the limit as  $n \rightarrow \infty$  we obtain

$$\lim_{n \rightarrow \infty} \frac{r!}{n^r} e^{-rt/2} E_n(e^{t/n}) = \left( \frac{e^t - e^{-t}}{t} \right)^r. \tag{4.2.3}$$

The function  $(e^{t/2} - e^{-t/2})/t$  is precisely the moment generating function for  $\chi^{(1)}$ . Hence (4.2.3) shows that indeed the  $X_n$  have limiting Betti distribution  $\chi^{(r)}$  by Theorem 4.1.2.

The density functions  $\chi^{(r)}$  have a long history. In approximation theory they are called *central B-splines* (see [Butz] for details). The support of  $\chi^{(r)}$  is the interval  $[-r/2, r/2]$ , it is a  $C^{n-2}$  function and a polynomial of degree  $n - 1$  in each subinterval  $[m - r/2, m + 1 - r/2]$  for  $m = 0, 1, \dots, r - 1$ . By the central limit theorem the distribution  $\chi^{(r)}$  approaches a Gaussian distribution as  $r \rightarrow \infty$ . More precisely [Butz, (4.7)],

$$\lim_{r \rightarrow \infty} \sqrt{r/6} \chi^{(r)} \left( \sqrt{r/6} x \right) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

**4.3. Large Hilbert schemes of points on  $\mathbb{C}^2$ .** — Consider the sequence of varieties  $X^{[n]}$  the Hilbert scheme of  $n$  points on  $X = \mathbb{C}^2$ . It follows from (2.3.8) that if  $d\mu_n$  denotes the discrete measure associated to  $X^{[n]}$  then

$$\Phi_n(x) = \int_{-\infty}^x d\mu_n = \#\{\lambda \mid |\lambda| = n, l(\lambda) \leq x\}. \tag{4.3.1}$$

By a theorem of Erdős and Lehner [ErL, Theorem 1.1]

$$\lim_{n \rightarrow \infty} \frac{1}{p(n)} \Phi_n(\alpha_n x + \beta_n) = e^{-c^{-1} e^{-cx}}, \tag{4.3.2}$$

where  $p(n)$  is the total number of partitions of  $n$ ,  $c := \pi/\sqrt{6}$  and

$$\alpha_n := \sqrt{n}, \quad \beta_n := 2c^{-1} \sqrt{n} \log n.$$

It follows from this result that the sequence  $X^{[n]}$  has limiting Betti distribution given by an instance of the *Gumbel distribution*. These appear as universal distributions when considering the maximum of samples (rather than the average as in the central limit theorem). Such *extreme value distributions* are of relevant in the prediction extreme natural phenomena like earthquakes, floods, etc. In our concrete case the density function is  $\omega(x) := \exp(-c^{-1}e^{-cx} - cx)$  (the derivative of the right hand side of (4.3.2)) whose graph is given in Figure 8. This should be compared, after scaling and reflection in the  $y$ -axis, to Figure 2.

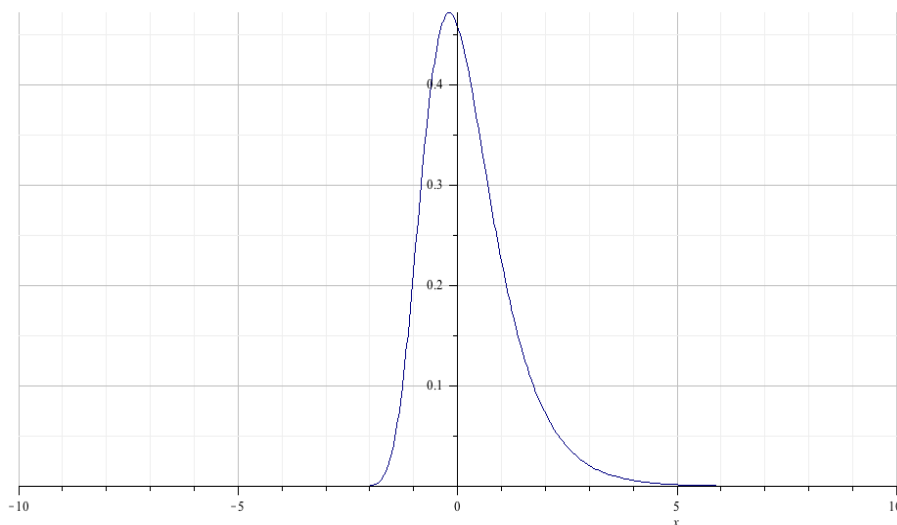


FIGURE 8. Gumbel distribution  $\exp(-c^{-1}e^{-cx} - cx)$

It is far from clear *a priori* why such a distribution would appear as a limiting Betti distribution of the Hilbert scheme of points on  $\mathbb{C}^2$ . It would be interesting to see what other limiting Betti distributions occur for the sequence  $S^{[n]}$  for an arbitrary smooth surface  $S$ .

On the other hand, it is not hard to convince ourselves of the relevance of extreme value distributions for our problem given (4.3.1). Indeed,  $l(\lambda) = \lambda'_1$ , where  $\lambda'$  is the partition dual to  $\lambda$ . In other words, the length of a partition equals the largest part of its dual.

**4.4. Large toric hyperkähler varieties.** — Take  $X_n = \mathcal{M}_1^{K_n}$  the hyperkähler toric quiver variety associated to the complete graph  $K_n$  on  $n$ -vertices. As mentioned in § 2.3 we have that  $E(X_n, q)$  equals a polynomial invariant, the external activity polynomial of the graph  $K_n$ .

We claim that  $X_n$  has a limiting Betti distribution known as the *Airy distribution* (for another instance of this phenomenon see [Re2]). This distribution appears in several different combinatorial and physical problems and has its origin as the distribution of the area of a Brownian excursion. Its density function is rather complicated to describe explicitly (its graph is given in Figure 9). In particular, its relation to the classical Airy function, from where the distribution gets its name, is not that straightforward to state. We will work instead with the moments which luckily determine the Airy distribution uniquely (see [Ta2, Thm. 3]).

There is a sizable literature on the Airy distribution; we will use the survey [Jan] as our main reference and point to the interested reader to the works cited there for more details. But we should warn the reader that there exists a different but related distribution called in the literature the map-Airy distribution.

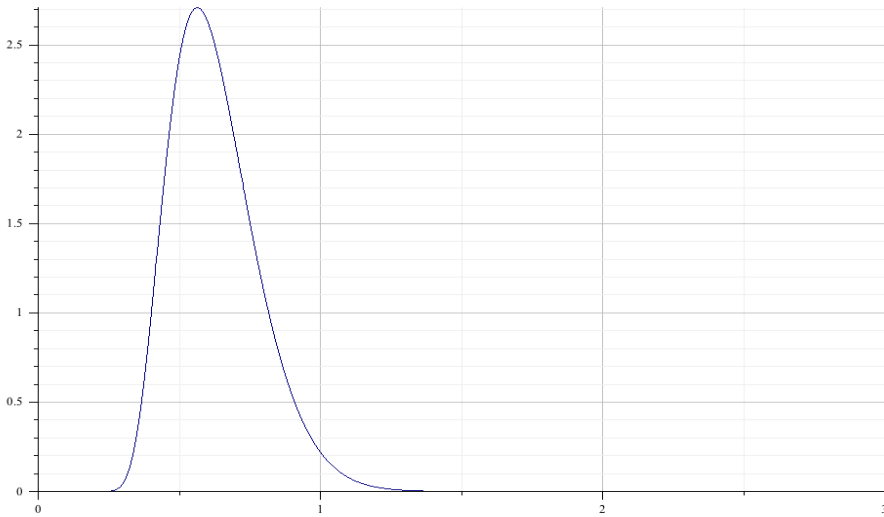


FIGURE 9. Airy distribution density function

We start by defining the rational constants  $c_k$  by means of the following expansion

$$\sum_{k \geq 1} c_k T^k := \log \sum_{n \geq 0} \frac{(1/6)_n (5/6)_n}{n!} \left(\frac{3}{2} T\right)^n, \tag{4.4.1}$$

where  $(a)_n := a(a+1) \cdots (a+n-1)$  is the Pochhammer symbol. The first few values are

$k$	1	2	3	4	5
$c_k$	5/24	5/16	1105/1152	565/128	82825/3072

We call  $c_k$  the *Wright constants*. There is a large number of different normalizations of these constants in the literature; see [Jan] for a comprehensive comparison between these. It is not hard to show that we have

$$c_k = \sum_Q \frac{1}{\text{Aut}(Q)},$$

where  $Q$  runs over all connected trivalent graphs on  $k$  (unlabeled) vertices and  $\text{Aut}(Q)$  denotes its group of automorphisms.

Now we define the constants  $\rho_k$  by [Jan, (34)]

$$\rho_{-1} := 1, \quad \rho_0 := \frac{\sqrt{2\pi}}{4}, \quad \rho_k := \frac{\sqrt{\pi}}{2^{\frac{1}{2}(3k-1)}\Gamma(\frac{3}{2}k)}c_k, \quad k \geq 1. \tag{4.4.2}$$

Then the  $k$ -th moment  $M_k$  of the Airy distribution is  $k!\rho_{k-1}$  [Jan, (36)]; concretely,

$$M_0 = 1, \quad M_1 = \frac{\sqrt{2\pi}}{4} = 0.626657068\dots \quad M_2 = \frac{5}{12} = 0.416666666\dots, \quad \text{etc.}$$

(more numerical values are listed in [Ta2, Table 4]).

To connect back to the external activity polynomial of  $K_n$  note that by (2.3.4)

$$R_{K_n}(q) = \sum_{k \geq 0} C_{n,n+k-1}(q-1)^k, \tag{4.4.3}$$

where  $C_{n,m}$  denotes the number of connected graph on  $n$  labeled vertices with  $m$  edges. It follows from (4.1.2) that

$$m_{\mu_{E_n}}(\eta) = R_{K_n}(1+\eta) = \sum_{k \geq 0} C_{n,n+k-1}\eta^k.$$

Hence the  $k$ -th factorial moment  $m_{n,k}$  of  $\mu_{E_n}$  is precisely  $C_{n,n+k-1}$ .

By a standard result  $C_{n,n-1}$ , the number of trees on  $n$  labeled vertices, is  $n^{n-2}$ . Wright proved [Jan, (20)] that for fixed  $k \geq 0$  we have

$$C_{n,n+k-1} \sim \rho_{k-1}n^{n-2+\frac{3}{2}k}, \quad n \rightarrow \infty. \tag{4.4.4}$$

Therefore,

$$\frac{m_{n,k}}{m_{n,0}} \sim \rho_{k-1}n^{\frac{3}{2}k}, \quad n \rightarrow \infty. \tag{4.4.5}$$

Our claim on the limiting Betti distribution of  $X_n$  now follows from Theorem 4.1.2.

Consider now the varieties  $X_{m,n}$  associated to the complete bipartite graph  $K_{m,n}$ . (Details of the proof of the assertions below can be found in the appendix §6.) By [HS]  $E_{m,n}(q) := E(X_{m,n}, q)$  is the external activity polynomial of  $K_{m,n}$ . Denote by  $M_k^{m,n}$  the  $k$ -th moment of  $\mu_{E_{m,n}}$ . For fixed  $k$  and  $n$ , the quantity  $M_k^{m,n}/n^m$  is a polynomial in  $m$  of degree  $n+k-1$  and

$$M_k^{m,n} \sim \alpha_{n,k}n^{m-n-k+1}m^{n+k-1}, \quad m \rightarrow \infty. \tag{4.4.6}$$



for some constants  $\alpha_{n,k}$ . We have

$$\beta_{n,k} := \binom{n+k-1}{k} \alpha_{n,k} = n \sum_{|\lambda|=n} (-1)^{l(\lambda)-1} \frac{(l(\lambda)-1)!}{\prod_{i \geq 1} m_i! i^{m_i}} n(\lambda')^{n+k-1}, \tag{4.4.7}$$

where the sum is over all partitions  $\lambda$  of  $n$ ,  $m_i := m_i(\lambda)$  is the multiplicity of  $i$  in  $\lambda$  and

$$n(\lambda') := \sum_{i \geq 1} \binom{i}{2} m_i.$$

Here is a table with the first few values of  $\beta_{n,k}$ , which are non-negative integers.

$n \backslash k$	0	1	2	3	4	5
1	1	0	0	0	0	0
2	1	1	1	1	1	1
3	3	12	39	120	363	1092
4	16	156	1120	7260	45136	275436
5	125	2360	30925	353500	3795225	39474960

Note that  $\beta_{n,0} = \alpha_{n,0} = n^{n-2}$ . We have the following generating function identity

$$\sum_{k \geq 0} \alpha_{n,k} \frac{t^k}{k!} = \left( \frac{e^t - 1}{t} \right)^{n-1} R_{K_n}(e^t), \tag{4.4.8}$$

where  $R_{K_n}$  is the external activity polynomial of the complete graph  $K_n$ . It follows that  $\alpha_{n,k}$  is the  $k$ -th moment of

$$\tilde{\omega}_n := \underbrace{\chi_{[0,1]} dx * \cdots * \chi_{[0,1]} dx}_{n-1} * d\mu_n,$$

where  $\chi_{[0,1]}$  is the characteristic function of the interval  $[0, 1]$  and  $d\mu_n$  is the measure  $d\mu_{E_n}$  for  $E_n := E(X_n, q)$  and  $X_n$  the variety of 4) associated to  $K_n$ . We deduce that the sequence of varieties  $X_{m,n}$  for fixed  $n$  has limiting Betti distribution  $\tilde{\omega}_n$  as  $m \rightarrow \infty$ .

As in the case of the Grassmanian the density function  $\tilde{\omega}_n$  is a spline of degree  $n - 2$ . For example, for  $n = 3$  we find that

$$\frac{\alpha_{3,k}}{\alpha_{3,0}} = \frac{3^{k+1} - 1}{(k+2)(k+1)}$$

are the moments of the density function

$$\begin{cases} 0 & x < 0 \\ \frac{2}{3}x & 0 \leq x < 1 \\ -\frac{1}{3}x + 1 & 1 \leq x < 3 \\ 0 & 3 \leq x \end{cases}.$$

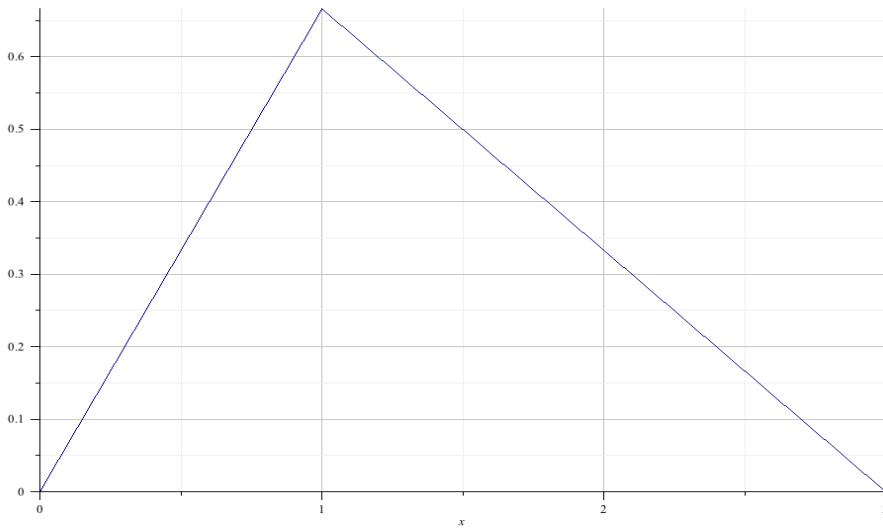


FIGURE 10. Spline approximation to the scaled Airy distribution

This is, up to scaling, a continuous piecewise linear approximation to the density function of the Airy distribution (its graph is given in Figure 10).

On the other hand the coefficients of the external activity polynomial for the graph  $K_{3,100}$  are shown in Figure 11.

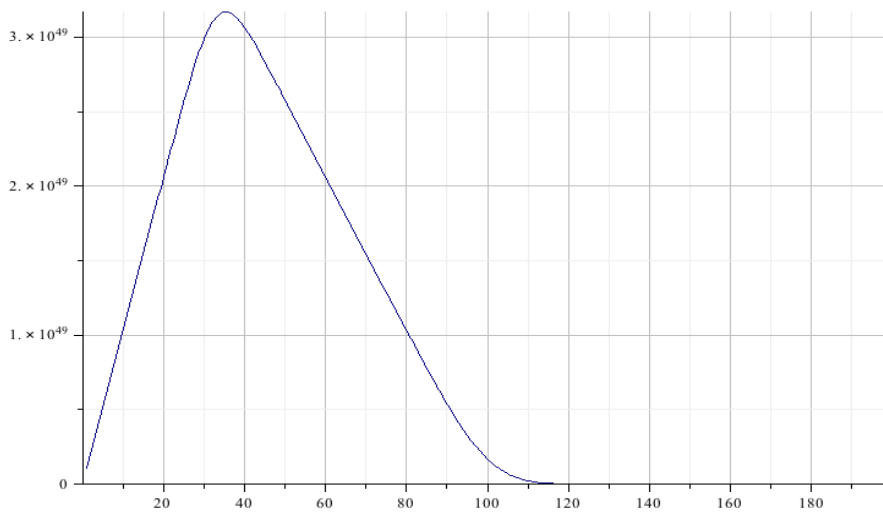


FIGURE 11. External activity polynomial of  $K_{3,100}$

The first factor on the right hand side of (4.4.8) has the expansion

$$\left(\frac{e^t - 1}{t}\right)^{n-1} =: \sum_{k \geq 0} \gamma_{n,k} t^k, \quad \gamma_{n,k} := \frac{(n-1)!}{(k+n-1)!} \left\{ \begin{matrix} n+k-1 \\ n-1 \end{matrix} \right\}$$

where  $\left\{ \begin{matrix} a \\ b \end{matrix} \right\}$  denotes the Stirling numbers of the second kind. We have for fixed  $k > 0$

$$\gamma_{n,k} \sim \frac{n^k}{2^k k!}, \quad n \rightarrow \infty.$$

It follows that coefficientwise

$$\lim_{n \rightarrow \infty} \left(\frac{e^{t/n^{3/2}} - 1}{t/n^{3/2}}\right)^{n-1} = 1.$$

Hence combined with (4.4.3) and (4.4.4) we deduce from (4.4.8) that

$$\lim_{n \rightarrow \infty} \frac{\alpha_{n,k}}{n^{3/2k} \alpha_{n,0}} = M_k$$

the  $k$ -th moment of the Airy distribution. Therefore, appropriately scaled, the distributions  $\tilde{\omega}_n$  converge to the Airy distribution as  $n \rightarrow \infty$ . We actually expect that the double sequence  $X_{m,n}$  (rather than the iterated limit  $\lim_n \lim_m$  we considered) should have the Airy distribution as its limiting Betti distribution.

**4.5. Large quiver varieties: heuristics.** — Let  $Q = (\mathcal{V}, \mathcal{E})$  be a quiver (see 2.3.3). Given a dimension vector  $\mathbf{v} = (v_1, \dots, v_n)$ , let  $A_{\mathbf{v}}(q)$  be the Kac polynomial for  $Q, \mathbf{v}$ .

Recall (2.3.11) that for  $\mathbf{v}$  indivisible  $A_{\mathbf{v}}(q)$  is the reverse of the Poincaré polynomial of a certain smooth quiver variety  $\mathcal{M}_{\mathbf{v}}^Q v^Q$ . We may consider the collection of all such Nakajima quiver varieties associated to indivisible dimension vectors  $\mathbf{v}$  for a fixed quiver  $Q$ . We expect that when  $\mathbf{v}$  tends to infinity generically the corresponding varieties  $\mathcal{M}_{\mathbf{v}}^Q$  have the Airy distribution as limiting Betti distribution independently of the quiver  $Q$ .

In this section we present some heuristics in support of this expected *universality* property of the Airy distribution. Concretely, we have Hua’s formula (2.3.12) for  $A_{\mathbf{v}}(q)$ , which though somewhat unwieldy, has a structure quite similar to (2.3.5) for computing the external activity polynomial of all complete graphs. We have already shown in § 4.4 that in this case we have the Airy distribution as the limit. The key fact is the asymptotic calculation of the moments (4.4.5), which in turn boils down to (4.4.4).

Flajolet and his collaborators [F1] use a saddle point analysis to prove (4.4.4). In a future publication we hope to apply to (2.3.12) this saddle point analysis to prove the expected universality of the Airy distribution for quiver varieties. We outline below the key ingredients of the saddle point approach following closely [F1], to which we refer the reader for details, and end with a brief discussion on how it could be applied to the case of quiver varieties.

Using the standard Gaussian integral

$$e^{-\frac{1}{2}y^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy - \frac{1}{2}x^2} dx$$

we rewrite the sum on the right hand side of (2.3.5) with  $q = 1 + \eta = e^{-t}$  and  $t > 0$  as

$$F(\eta, T) := \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^{-1}x^2} \sum_{n \geq 0} e^{(ix + \frac{1}{2}t)^n} \frac{(T/\eta)^n}{n!} dx = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{\phi(x, \eta, T)} dx,$$

where

$$\phi(x, \eta, T) := -\frac{1}{2}t^{-1}x^2 + e^{ix + \frac{1}{2}t}(T/\eta).$$

Note that by (2.3.5) and (4.4.3)

$$\eta \log F(\eta, T) = C_0(T) + C_1(T)\eta + \dots, \tag{4.5.1}$$

where

$$C_k(T) := \sum_{n \geq 0} C_{n, n+k-1} \frac{T^n}{n!}$$

is the exponential generating function for connected graphs on  $n$  vertices with a fixed  $n + k - 1$  first Betti number.

In order to study the asymptotics of  $C_{n, n+k-1}$  and prove (4.4.4), we compute the asymptotic behavior of  $F(\eta, T)$  as  $\eta$  approaches zero by using the saddle point method. For this purpose we compute the critical points of  $\phi_{-1}(x)$ , where

$$\phi(x, \eta, T) = \phi_{-1}(x, T)\eta^{-1} + \phi_0(x, T) + \phi_1(x, T)\eta + \dots.$$

We have

$$\phi_{-1}(x, T) = \frac{1}{2}x^2 + e^{ix}T$$

and the critical points are the solutions  $x = x(T)$  of the equation  $e^{ix}T = ix$ . In a neighborhood of  $T = 0$  we can solve this equation with  $ix = w = w(T)$  where

$$w(T) = \sum_{n \geq 1} n^{n-1} \frac{T^n}{n!},$$

is the *tree function* satisfying what we call the *saddle point equation*

$$Te^w = w. \tag{4.5.2}$$

In terms of the *saddle point parameter*  $w$  the expansion coefficients  $C_k(T)$  of (4.5.1) have the following form

$$C_0 = w - \frac{1}{2}w^2, \tag{4.5.3}$$

$$C_1 = -\frac{1}{2} \log(1 - w) + \frac{1}{2}w + \frac{1}{4}w^2 \tag{4.5.4}$$

$$C_k = \frac{E_{k-1}(w)}{(1 - w)^{3(k-1)}}, \quad k > 1 \tag{4.5.5}$$

for certain polynomials  $E_k$ . It follows from [KP, Lemma 2] that the asymptotics of  $C_{n,n+k-1}$  for large  $n$  and  $k > 1$  fixed is of the form

$$C_{n,n+k-1} \sim E_{k-1}(1) \frac{\sqrt{2\pi} n^{n+\frac{3}{2}k-2}}{2^{\frac{3}{2}(k-1)} \Gamma(\frac{3}{2}(k-1))}, \tag{4.5.6}$$

if  $E_{k-1}(1)$  does not vanish. As we will now see, in fact,

$$E_k(1) = c_k, \tag{4.5.7}$$

the Wright constants defined in (4.4.2). Equivalently, as  $w$  approaches 1

$$C_k \sim c_k(1-w)^{-3(k-1)} + \dots, \quad k > 1.$$

Hence, (4.5.6) is the same as (4.4.4).

We make the change of variables  $x = y - iw$  in the integral and get

$$\phi_{-1}(x, T) = -\frac{1}{2}w^2 + w + \psi(y, w),$$

where  $\psi(y, w) = \frac{1}{2}(1-w)y^2 + (e^{iy} - 1 - iy + \frac{1}{2}y^2)w$ . Therefore,

$$F(\eta, T) = e^{\eta^{-1}(-\frac{1}{2}w^2+w)} \frac{1}{\sqrt{2\pi t}} \int_{\mathcal{C}} e^{\eta^{-1}\psi(y,w)} h(y, \eta, w) dy \tag{4.5.8}$$

for a certain function  $h(y, \eta, w)$  holomorphic in  $\eta$  and an appropriate contour  $\mathcal{C}$ .

To prove (4.5.7) and get (4.5.6) one studies the behavior of  $C_k$  as  $w$  approaches 1. Note however that the critical point  $y = 0$  of

$$\psi(y, w) = \frac{1}{2}(1-w)y^2 - \frac{i}{6}wy^3 + O(y^4)$$

becomes degenerate for  $w = 1$ . It is well known that in this situation the standard saddle point method has to be modified to incorporate higher order terms. We have here the simplest case of this phenomenon known as the *coalescing of saddle points* [CFU].

We homogenize by letting  $y_*, w_*, \eta_*$  be new variables defined by

$$w = 1 - w_*, \quad y = w_*y_*, \quad \eta = w_*^3\eta_*.$$

Then

$$\eta^{-1}\psi(y, w) = \eta_*^{-1}y_*^2(1 - \frac{i}{3}y_* + w_*y_*r(y_*, w_*))$$

for some power series  $r(y_*, w_*)$  in  $y_*$  with coefficients polynomial in  $w_*$ . Now we can let  $w_*$  approach zero and check that up to a tractable factor the integral in (4.5.8) can be replaced by

$$\frac{1}{\sqrt{2\pi\eta_*}} \int_{-\infty}^{\infty} e^{\frac{1}{2}\eta_*^{-1}(y_*^2 - \frac{i}{3}y_*^3)} dy_*.$$

Since the asymptotic expansion of this integral is precisely

$$\sum_{n \geq 0} \frac{(1/6)_n (5/6)_n}{n!} \left(\frac{3}{2}\eta_*\right)^n$$

(4.5.7) follows.

To summarize, we prove that we have Betti limiting distribution the Airy distribution by computing the limiting moments. To do this, we

1. Express the generating function of all moments as an integral.
2. Find the critical points of the dominant exponential factor of the integrand, which satisfy saddle point equations.
3. Show that in terms of the saddle point parameters the generating function  $C_k$  of the  $k$ -th moment (for  $k > 1$ ) becomes a rational function whose leading term involves the Wright constants  $c_k$ , due to a coalescing of saddle points.
4. Deduce that the limiting moments are those of the Airy distribution.

As mentioned, we expect that these same steps can be applied to Hua's formula (2.3.12) to study the limiting distribution of quiver varieties for a fixed but arbitrary quiver as mentioned earlier. The first two steps are fairly routine. In general the saddle point equations however will determine an algebraic variety of higher dimension (equal to the number of nodes in the quiver). Carrying through the last two steps then becomes more of a challenge but at least the generic behaviour, when the dimension vector components increase to infinity independently, should be as above. It is conceivable that the non-generic behavior of the asymptotics could involve higher order singularities and hence distributions other than the Airy distribution. We will revisit this issue in a later publication.

## 5. Results and speculations on the asymptotics of discrete distributions

In the previous section we proved and gave heuristics for some asymptotical results on the distribution of Betti numbers of certain families of semiprojective hyperkähler varieties. Not surprisingly, we found in §4.2 that the Gaussian distribution appears in several examples. The classical binomial distribution, given for us as the Betti numbers of tori, is the most well known example of such asymptotic behavior. We also found in §4.2 that the Betti numbers of certain families of Grassmannians also have Gaussian limiting distributions. The same behavior was already observed by Takács [Ta1] in his studies of coefficients of  $q$ -binomial coefficients. More recent work of Stanley and Zanello [SZ] gives new results on asymptotics of these quantities, as well as studies unimodality properties of various sequences; not unlike our sequences of (even) Betti numbers of semiprojective hyperkähler varieties.

In fact, in our computed examples we observed that the sequence of even (similarly odd) Betti numbers form a unimodal sequence. This result follows from the Hard Lefschetz theorem for a smooth projective variety; or a semiprojective variety with core which is smooth projective. However it is unclear why this property may hold in larger generality. Clearly the even Betti numbers of smooth affine varieties will not necessarily be unimodal, as the case of  $SL(5, \mathbb{C})$  shows. In fact starting with Stanley's studies [St2] several combinatorial sequences have been conjectured and some proved to be unimodal. In particular recently Huh in [Huh] proved that the  $h$ -vector of a

representable matroid is log-concave and thus unimodal, in particular proving a long-standing conjecture of Colbourn on log-concavity of the external activity polynomial. This implies that the Betti numbers of toric hyperkähler varieties (which are the h-vectors of rationally representable matroids) form a unimodal sequence.

However, some combinatorial counterexamples are relevant for us too. For example, in our geometric language, Stanton [Sta] found examples of Poincaré polynomials of closures of Schubert cells in Grassmannians, which are not unimodal. It could be relevant for us as the closure of the Schubert cell is an equidimensional, in fact irreducible, proper variety with a paving, and thus has pure cohomology. Thus potentially it could be the core of a semiprojective hyperkähler variety.

The sequence of graphs, such as the complete graphs  $K_n$ , we studied in §4.4 is convergent in the sense of Lovász-Szegedy [LSz]. The continuous limit for their extremal activity polynomial we found there could possibly be related to some invariants of the limiting objects. This also raises the possibility of existence of a limiting object to our sequences of hyperkähler manifolds, whose “Poincaré series” in the appropriate sense would agree with our limiting distribution.

The Airy distribution in §4.4, governing the limit of Betti numbers of the toric quiver varieties attached to complete graphs and possibly sequences of more general quiver varieties as discussed in §4.5 was earlier noticed to be the limiting distribution of Betti numbers of certain non-commutative Hilbert schemes by Reineke in [Re2, Theorem 6.2]. In fact, our heuristics in §4.5 were motivated by an effort to systematize a proof of such results.

It is worthwhile noticing that the graphs of the Gumbel (Figure 8) and Airy (Figure 9) distributions seem very similar to the naked eye. In fact, they are really different (for example, they have different decay rate at the tails). However, looking at the distribution of Betti numbers in the case of the toric quiver variety  $\mathcal{M}_1^{K_{40}}$  (Figure 1) and the Hilbert scheme  $(\mathbb{C}^2)^{[500]}$  (Figure 2) one might easily believe they are approaching a common limit.

We also mention two recent appearances of the Gaussian distribution as a limit of series of discrete distributions. First in [EEL] it is conjectured that sequences of ranks of certain syzygies of a smooth projective variety also have Gaussian distribution in the limit. More directly relevant for us is the recent work of Morrison [Mo]. It was proved there that the sequence of discrete distributions given by the motivic DT invariants of  $(\mathbb{C}^3)^{[n]}$  is also normally distributed in the limit. In fact the generating function [BBS] of such motivic DT-invariants is similar, at least in the limit  $m \rightarrow \infty$ , to generating functions of the twisted ADHM spaces  $\mathcal{M}_{n,m}$  we discussed in §2.3.2. Thus it is conceivable, that in an appropriate limit the Betti numbers of  $\mathcal{M}_{n,m}$  will also be distributed normally; Figure 3 seems to support this possibility.

Very recently Bringmann and Dousse in [BD] have proved a long-standing conjecture of Dyson on the limiting shape of the crank generating function, which is important in the number theory of integer partitions. [BD] deduce this conjecture

from an asymptotic study of certain type of integer partitions, which will in particular imply, that the Poincaré polynomial  $P_t(S^{[n]})$  of the Hilbert scheme of  $n$ -points of a simply-connected projective smooth complex surface  $S$  has limiting distribution  $\frac{1}{\cosh^2(x)}$ , the so-called *logistic* distribution. Note that when  $S$  is a  $K3$  surface  $S^{[n]}$  is a compact hyperkähler manifold. The surprise here is that  $S^{[n]}$ , being smooth and projective, satisfies Poincaré duality, while the limiting distribution is not Gaussian. On the other hand the appearance of the logistic distribution is less surprising in light of the fact that it is the distribution of the difference of two independent random variables, which are distributed according to the Gumbel distribution. This latter distribution we found earlier to govern the limit of the Poincaré polynomials of Hilbert scheme of  $n$  points on  $\mathbb{C}^2$ .

Finally we mention a few potential connections to ideas from mathematical physics. The paper [BM] gives an asymptotic study of certain number theoretical quantities which are related to Betti numbers of Hilbert schemes of surfaces. This study was partly motivated by counting BPS states in black hole physics. Interestingly, very similar analysis to the saddle point method in §4.5 has been used in [DGLZ] to study the asymptotic properties of coloured  $SU(2)$  Jones polynomials. Also the large  $N$  or t’Hooft limit of various  $U(N)$  gauge theories, studied extensively by string theorists, involves asymptotic studies not unlike ours. Maybe in these contexts we will find an explanation of the continuous looking limit distribution (Figure 7) of the Betti numbers of the moduli space of rank  $n$  Higgs bundles  $\mathcal{M}_n^g$  as  $n \rightarrow \infty$ .

### 6. Appendix

Here we give more details of the arguments in § 4.4. Let  $R_{m,n}$  be the external activity polynomial of the complete bipartite graph  $K_{m,n}$ . The generating function identity analogous to (2.3.5) is

$$\sum_{m,n \geq 1} R_{m,n}(q) \frac{x^m}{m!} \frac{y^n}{n!} = (q-1) \log F(x/(q-1), y/(q-1)), \tag{6.0.9}$$

where

$$F(x, y) := \sum_{m,n \geq 0} q^{mn} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Note that for  $i = 0, 1, \dots$

$$\left. \frac{\partial^i F}{\partial y^i} \right|_{y=0} = e^{q^i x}$$

Combining this with Fa di Bruno’s formula [Mac, p. 31] we obtain

$$R_{m,n}(q) = \frac{1}{(q-1)^{n-1}} \sum_{|\lambda|=n} c_\lambda r_\lambda(q)^m, \tag{6.0.10}$$



where  $m_i$  is the multiplicity of  $i$  in  $\lambda$ ,

$$r_\lambda(q) := \sum_{i \geq 1} [i] m_i, \quad [r] := 1 + q + \dots + q^{r-1}$$

and

$$c_\lambda := \frac{(-1)^{l(\lambda)-1} (l(\lambda) - 1)! n!}{\prod_{i \geq 1} m_i! i^{m_i}}.$$

Consider

$$C_n(q, T) := \sum_{m \geq 0} R_{m,n}(q) T^m.$$

By (6.0.10) this power series is the Taylor expansion of the rational function

$$C_n(q, T) = \frac{1}{(q-1)^{n-1}} \sum_{|\lambda|=n} \frac{c_\lambda}{1 - r_\lambda(q)T}. \tag{6.0.11}$$

It follows that

$$C_{n,k}(T) := \left( q \frac{\partial}{\partial q} \right)^k C_n(q, T) \Big|_{q=1} = \sum_{m \geq 0} M_k^{m,n} T^m \tag{6.0.12}$$

is a rational function of  $T$  with the denominator dividing a power of  $1 - nT$  (since  $r_\lambda(1) = |\lambda|$  for all partitions  $\lambda$ ). In particular, for fixed  $n$  and  $k$ ,  $M_k^{m,n}/n^m$  is a polynomial in  $m$ , say  $f_{n,k}(m)$ . Note that

$$C_n(e^t, T) = \sum_{k \geq 0} C_{n,k}(T) \frac{t^k}{k!}. \tag{6.0.13}$$

From (6.0.11) we obtain upon replacing  $t$  by  $n(1 - nT)t$

$$n^n (1 - nT)^n C_n(q, T) \Big|_{T=1/n} = n \left( \frac{n(1 - nT)}{q-1} \right)^{n-1} \Big|_{T=1/n} \cdot \sum_{|\lambda|=n} c_\lambda \frac{1 - nT}{1 - r_\lambda(q)T} \Big|_{T=1/n}.$$

The first factor on the right hand side equals  $n/t^{n-1}$ ; the term corresponding to  $\lambda$  in the second factor equals  $c_\lambda / (1 - n(\lambda')t)$ , where

$$q \frac{d}{dq} r_\lambda(q) \Big|_{q=1} = n(\lambda') := \sum_{i \geq 1} \binom{i}{2} m_i.$$

Therefore, in combination with (6.0.13) we find

$$\frac{n^{n+k-1}}{k!} (1 - nT)^{n+k} C_{n,k}(T) \Big|_{T=1/n} = \sum_{|\lambda|=n} c_\lambda n(\lambda')^{n+k-1}. \tag{6.0.14}$$

On the other hand, for any positive integer  $r$

$$\frac{1}{(1 - nT)^r} = \sum_{m \geq 0} n^m \binom{r+m}{r} T^m.$$

Hence  $f_{n,k}(m)$  is at most of degree  $n+k-1$ . Denote by  $\delta_{n,k}$  its coefficient of degree  $n+k-1$ . Then

$$\delta_{n,k} = \frac{k!}{(n+k-1)!n^{n+k-1}} \sum_{|\lambda|=n} c_\lambda n(\lambda')^{n+k-1}$$

Again by Fra di Bruno's formula the sum equals

$$\left( q \frac{d}{dq} \right)^{n+k-1} G_n(q) \Big|_{q=1},$$

where  $G_n$  is given by the expansion

$$\log \sum_{n \geq 0} q^{\binom{n}{2}} \frac{x^n}{n!} = \sum_{n \geq 1} G_n(q) \frac{x^n}{n!}.$$

Note that by (2.3.5)  $G_n(q) = (q-1)^{n-1} R_n(q)$ , where  $R_n$  is the external activity polynomial of the complete graph  $K_n$ .

It follows that

$$\sum_{k \geq 0} \delta_{n,k} n^{n+k-1} \frac{t^k}{k!} = \frac{G_n(e^t)}{t^{n-1}} = \left( \frac{e^t - 1}{t} \right)^{n-1} R_n(e^t).$$

In particular, since both factors on the right hand side have positive coefficients in their Taylor expansion in  $t$  we conclude that  $\delta_{n,k}$  is also positive. Hence  $f_{n,k}$  has degree  $n+k-1$  and  $\delta_{n,k}$  is its leading coefficient. We also see that  $\delta_{0,n} = R_n(1)/n^{n-1} = n^{-1}$  by Cayley's theorem.

Finally, it follows that

$$M_k^{m,n} \sim \delta_{n,k} m^{n+k-1}, \quad m \rightarrow \infty$$

proving (4.4.6) and (4.4.7) with  $\alpha_{n,k} = \delta_{n,k} n^{n+k-1}$ .

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