

Hitchin map on even very stable upward flows

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We define even very stable Higgs bundles and study the Hitchin map restricted to their upward flows. In the GL_n case, we classify the type $(1, \dots, 1)$ examples, and find that they are governed by a root system formed by the roots of even height. We discuss how the spectrum of equivariant cohomology of real and quaternionic Grassmannians, $4n$ -spheres and the real Cayley plane appear to describe the Hitchin map on even cominuscule upward flows. The even upward flows in question are the same as upward flows in Higgs bundle moduli spaces for quasi-split inner real forms. The latter spaces have been pioneered by Oscar García-Prada and his collaborators.

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1. Introduction

This paper is a write-up of the second author's talk [12] at the conference “Moduli spaces and geometric structures” in honor of Oscar García-Prada on the occasion of his 60th birthday at ICMAT Madrid in September 2022.

In [14], motivated by mirror symmetry, the notion of very stable Higgs bundle was introduced. Let C be a smooth projective curve. Let \mathcal{M} denote the moduli space of rank n degree d semistable Higgs bundles (E, Φ) , where E is a rank n degree

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vector bundle and $\Phi \in H^0(C; \text{End}(E) \otimes K)$ is a Higgs field. There is a \mathbb{T} -action on \mathcal{M} by scaling the Higgs field, i.e. $\lambda \in \mathbb{T}$ acts by sending (E, Φ) to $(E, \lambda\Phi)$. A fixed point $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is called *very stable*, if the upward flow

$$W_{\mathcal{E}}^+ := \left\{ \mathcal{F} \in \mathcal{M} : \lim_{\lambda \rightarrow 0} \lambda \cdot \mathcal{F} = \mathcal{E} \right\} \subset \mathcal{M}$$

is closed. In Sec. 2, we recall the basic properties of very stable upward flows in general as well as for the moduli space of Higgs bundles \mathcal{M} .

One of the main results of [14] is the classification of very stable Higgs bundles $(E, \Phi) \in \mathcal{M}^{\mathbb{T}}$ of type $(1, \dots, 1)$. A fixed point is of type $(1, \dots, 1)$ when the vector bundle $E = L_0 \oplus \dots \oplus L_{n-1}$ is a direct sum of line bundles, and the Higgs field $\Phi|_{L_i} : L_i \rightarrow L_{i+1}K \subset EK$, which we denote by

$$b_i := \Phi|_{L_i} \in \text{Hom}(L_i, L_{i+1}K) \cong H^0(C; L_i^{-1}L_{i+1}K).$$

Then we have the following theorem.

Theorem 1.1 ([14, Theorem 4.16]). *The type $(1, \dots, 1)$ Higgs bundle $(E, \Phi) \in \mathcal{M}^{\mathbb{T}}$ is very stable if and only if the divisor $\text{div}(b_1) + \dots + \text{div}(b_{n-1})$ is reduced.*

We recall this classification in Theorem 2.1, and a reformulation of it in Remark 2.1 in terms of minuscule dominant weights of $\text{GL}(n, \mathbb{C})$.

Garcia-Prada and Ramanan in [9] study involutions on the moduli space of Higgs bundles. One important involution $\theta : \mathcal{M} \rightarrow \mathcal{M}$ is given by $\theta(E, \Phi) := (E, -\Phi)$. In [9], it is shown that the fixed points \mathcal{M}^{θ} correspond to $U(p, n-p)$ -Higgs bundles (including the case $p = 0$, where $U(0, n) := U(n)$). We recall these notions in Sec. 3.

In this paper, we will be interested in the so-called *even upward flows* $W_{\mathcal{E}}^{2+}$ for any $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}} \subset \mathcal{M}^{\theta}$ which are defined to be the upward flows \mathcal{E} in the semi-projective \mathcal{M}^{θ} , or equivalently, the intersection $W_{\mathcal{E}}^{2+} := W_{\mathcal{E}}^+ \cap \mathcal{M}^{\theta}$. Then we can define *even very stable* Higgs bundles $\mathcal{E} \in \mathcal{M}^{\mathbb{T}}$ for which the even upward flow $W_{\mathcal{E}}^{2+} \subset \mathcal{M}^{\theta}$ is closed. One of the main results of this paper is the following theorem.

Theorem 1.2. *The type $(1, \dots, 1)$ Higgs bundle $(E, \Phi) \in \mathcal{M}^{\mathbb{T}}$ is even very stable, if and only if the divisors $\text{div}(b_2) + \dots + \text{div}(b_{n-2})$ and $\text{div}(b_i) + \text{div}(b_{i+2k+1})$ for $1 \leq i \leq i + 2k + 1 \leq n - 1$ are all reduced.*

To clarify the meaning of this complicated looking set of divisors, we reformulate this theorem in Theorem 3.1 in terms of so-called *even minuscule* dominant weights using positive weights of even height.

As the Hitchin map restricted to very stable upward flows is finite flat and \mathbb{T} -equivariant between affine spaces, with positive \mathbb{T} -action of the same dimension it is susceptible of explicit description. In the type $(1, \dots, 1)$ very stable case, the second author found such an explicit description in [13] in terms of the spectrum of equivariant cohomology of the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$. We will recall this in Sec. 4.2.

Finally, in Sec. 4.3, we study the problem of modeling the Hitchin map on certain even very stable upward flows, in terms of the equivariant cohomology of

homogeneous spaces. We will find in Theorem 4.1 that for GL_{2n} the equivariant cohomology of quaternionic Grassmannians, for SO_{4n+2} the equivariant cohomology of the $4n$ -sphere and finally for E_6 the equivariant cohomology of the real Cayley plane should model the Hitchin map on some specific even very stable flows. The appearance of these symmetric spaces is interesting, partly because they are not of Hermitian type, and also because they are quotients of the Nadler group [22] of the quasi-split real form of Hodge type (see [9, Sec. 2.3] for the definition).

In this paper, we are concentrating on type $(1, \dots, 1)$ very stable and even very stable upward flows. By now there are many interesting results about other types of very stable or wobbly Higgs bundles see e.g. [20] for multiplicity algebras of type (2) very stable Higgs bundles, [6] for many wobbly Higgs bundles — both papers in this conference proceedings — and [25] for a classification of all type (n_1, n_2) very stable components.

2. Bialynicki–Birula Decomposition

In this section, we first recall the definition of a semi-projective variety and then collect the basics of the Bialynicki–Birula decomposition associated to such a variety.

Definition 2.1. Let X be a normal complex quasi-projective variety equipped with a $\mathbb{T} := \mathbb{C}^\times$ action. X is *semi-projective* if the fixed point locus $X^\mathbb{T}$ is projective, and for every $x \in X$ there is a $p \in X^\mathbb{T}$ such that $\lim_{\lambda \rightarrow 0} \lambda x = p$.

The latter is to be understood as the existence of a \mathbb{T} -equivariant morphism $f : \mathbb{A}^1 \rightarrow X$ such that $f(1) = x$ and $f(0) = p$. Semi-projective varieties are endowed with a stratification in affine subvarieties known as the *Bialynicki–Birula decomposition* [3], which we now recall. We refer to [14, Sec. 2] for further details.

Definition 2.2. Let X be a semi-projective variety and $\alpha \in X^\mathbb{T}$. The *upward flow from α* is defined to be

$$W_\alpha^+ := \left\{ x \in X : \lim_{\lambda \rightarrow 0} \lambda x = \alpha \right\}.$$

Similarly, the *downward flow from α* is

$$W_\alpha^- := \left\{ x \in X : \lim_{\lambda \rightarrow \infty} \lambda x = \alpha \right\}.$$

The *Bialynicki–Birula partition* is $X = \bigsqcup_{\alpha \in X^\mathbb{T}} W_\alpha^+$. The *core* of X is defined to be $\mathcal{C} := \bigsqcup_{\alpha \in X^\mathbb{T}} W_\alpha^-$.

Definition 2.3. For a connected component of the fixed locus, $F \in \pi_0(X^\mathbb{T})$, we define its *attractor* as $W_F^+ := \bigcup_{\alpha \in F} W_\alpha^+$, and its *repeller* as $W_F^- := \bigcup_{\alpha \in F} W_\alpha^-$. The *Bialynicki–Birula decomposition* is $X = \bigsqcup_{F \in \pi_0(X^\mathbb{T})} W_F^+$.

Given a smooth fixed point $\alpha \in X^{s\mathbb{T}}$, the \mathbb{T} -action on X induces a representation of \mathbb{T} on the tangent space $T_\alpha X$. We denote, for $k \in \mathbb{Z}$, the weight space $(T_\alpha X)_k \subset$

$T_\alpha X$ where $\lambda \in \mathbb{T}$ acts via multiplication by λ^k . This leads to a decomposition $T_\alpha X = \bigoplus_{k \in \mathbb{Z}} (T_\alpha X)_k$ in weight spaces. We denote $T_\alpha^+ X := \bigoplus_{k > 0} (T_\alpha X)_k$ the positive part and $T_\alpha^- X := \bigoplus_{k < 0} (T_\alpha X)_k$ the negative part. We have:

Proposition 2.1. *Given a smooth fixed point $\alpha \in X^{s\mathbb{T}}$, the upward flow W_α^+ (respectively, the downward flow W_α^-) is a locally closed \mathbb{T} -invariant subvariety of X which is isomorphic to $T_\alpha^+ X$ (respectively, $T_\alpha^- X$) as varieties with \mathbb{T} -action.*

The proof was originally given in [3] for smooth complete X . A proof for the general case is given in [14, Proposition 2.1].

Finally, suppose further that X^s is equipped with a symplectic form $\omega \in \Omega^2(X^s)$ such that, for $\lambda \in \mathbb{T}$, we have $\lambda^*(\omega) = \lambda\omega$. This supposition is motivated by the fact that the semi-projective variety we will be studying, the moduli space of semistable Higgs bundles, is endowed with such a form. Then, we have the following proposition.

Proposition 2.2. *For a smooth point $\alpha \in X^{s\mathbb{T}}$, the subspaces $T_\alpha^+ X$ and $T_\alpha^{\leq 0} X := (T_\alpha X)_0 \oplus T_\alpha^- X$ of $T_\alpha X$ are Lagrangian. Moreover, the subvarieties W_α^+ and $W_{F_\alpha}^-$ are also Lagrangian.*

The proof is given in [14, Proposition 2.10]. The main idea is that, for $v \in (T_\alpha X)_k$ and $w \in (T_\alpha X)_l$, we have

$$\lambda\omega(v, w) = \lambda^*(\omega)(v, w) = \omega(\lambda \cdot v, \lambda \cdot w) = \omega(\lambda^k v, \lambda^l w) = \lambda^{k+l}\omega(v, w),$$

so that $\omega(v, w)$ can only be nonzero in the situation $k + l = 1$, which does not happen if $k, l > 0$ or if $k, l \leq 0$.

Definition 2.4. We say that $\alpha \in X^{s\mathbb{T}}$ is *very stable* if $W_\alpha^+ \cap C = \{\alpha\}$.

This definition was introduced in [14, Definition 4.1], where it was proven [14, Lemma 4.4] that $\alpha \in X^{s\mathbb{T}}$ is very stable if and only if $W_\alpha^+ \subset X$ is closed.

2.1. Lagrangian upward flows in \mathcal{M}

In this section, we introduce Higgs bundles and show how the previous theory of Białynicki–Birula applies to the moduli space of semistable Higgs bundles. For this, we fix a smooth projective curve C over the complex numbers with genus $g \geq 2$ and canonical line bundle K .

Definition 2.5. A *Higgs bundle* is a pair (E, Φ) where E is a holomorphic vector bundle over C and $\Phi \in H^0(C, \text{End}(E) \otimes K)$.

Such an object can be defined in more generality for a real reductive Lie group G [7, Definition 3.1], giving G -Higgs bundles. The above definition is recovered by setting $G = GL_n(\mathbb{C})$ for $n = \text{rank } E$. Recall that a Higgs bundle (E, Φ) is *stable* if,

for every nonzero proper vector subbundle $F \subset E$ such that $\Phi(F) \subseteq F \otimes K$, we have

$$\mu(F) := \frac{\deg F}{\text{rank } F} < \mu(E) := \frac{\deg E}{\text{rank } E}$$

and it is *semistable* if for the same subbundles we have $\mu(F) \leq \mu(E)$. We denote by $\mathcal{M} := \mathcal{M}_n^d$ the moduli space of semistable Higgs bundles of fixed rank n and degree d . It was constructed via gauge theory in [18] and by algebraic geometric methods in [23, 27]. It is a normal [28] quasi-projective variety with a hyperkähler metric at its smooth points, which are the stable Higgs bundles. In particular, it has a symplectic structure $\omega \in \Omega^2(\mathcal{M}^s)$.

This space also carries a natural \mathbb{T} -action defined by $(E, \Phi) \mapsto (E, \lambda\Phi)$ which turns it into a semi-projective variety and such that $\lambda^*(\omega) = \lambda\omega$. Thus, the Bialynicki–Birula theory from the previous section applies. The fixed locus $\mathcal{M}^{\mathbb{T}}$ can be identified as follows. We have, for any $\lambda \in \mathbb{T}$, an isomorphism of vector bundles $f_\lambda \in \text{Aut}(E)$ such that

$$\Phi \circ f_\lambda = f_\lambda \circ (\lambda\Phi). \tag{2.1}$$

In other words, we have an action of \mathbb{T} on E which is linear on each fiber. Hence we can decompose $E = L_0 \oplus \dots \oplus L_k$ into weight spaces, where $f_\lambda|_{L_i} = \lambda^{w_i} \cdot \text{Id}_{L_i}$. The compatibility condition (2.1) shows that, if $v_i \in L_i$, then $f_\lambda(\lambda \cdot \Phi(v_i)) = \Phi(f_\lambda(v_i)) = \lambda^{w_i} \Phi(v_i)$, thus $f_\lambda(\Phi(v_i)) = \lambda^{w_i-1} \Phi(v_i)$. Hence, Φ maps the space for weight w_i into the space for weight $w_i - 1$. In particular, the weights can be chosen to be of the form $w_i = w_0 - i$ and the Higgs field has the property $\Phi(L_i) \subseteq L_{i+1} \otimes K$. From this, we can associate an invariant $(\text{rank } L_0, \dots, \text{rank } L_k)$ to the fixed point, known as the *type*.

The previous decomposition shows that, in fact, a Higgs bundle fixed by the \mathbb{T} -action is nilpotent, since $\Phi^{k+1} \equiv 0$. Another way of seeing this is via the *Hitchin map*:

$$h : \mathcal{M} \rightarrow \mathcal{A} := \bigoplus_{i=1}^n H^0(C, K^i)$$

defined by the coefficients $a_i \in H^0(C, K^i)$ of the characteristic polynomial $\det(\Phi - xI_n) = x^n + \sum_{j=1}^n a_j x^{n-j}$. This map is a proper, completely integrable Hamiltonian system [17, 23] whose fibers are Lagrangian at their smooth points, and the generic fibers are abelian varieties [26]. Moreover, by letting \mathbb{T} act on $H^0(C, K^i)$ with weight i , the Hitchin map is \mathbb{T} -equivariant. Thus, $\mathcal{M}^{\mathbb{T}} \subseteq h^{-1}(0)$ so that fixed points (E, Φ) have characteristic polynomial x^n and are nilpotent.

The upward and downward flows from the Bialynicki–Birula partition have been characterized in [14, Propositions 3.4 and 3.11]:

Proposition 2.3. *Let $\mathcal{E} = (E', \Phi') \in \mathcal{M}^{s\mathbb{T}}$ and $(E, \Phi) \in \mathcal{M}$. We have $(E', \Phi') \in W_{\mathcal{E}}^+$ if and only if there exists a filtration*

$$0 = E_0 \subset E_1 \subset \dots \subset E_k = E$$

such that $\Phi(E_i) \subseteq E_{i+1} \otimes K$ and the associated graded object verifies $(\text{Gr}E, \text{Gr}(\Phi)) \simeq (E', \Phi')$. The same is true replacing $W_{\mathcal{E}}^+$ with $W_{\mathcal{E}}^-$ and the ascending filtration with a descending filtration.

The downward flows have a concrete characterization via the Hitchin map. Since the action of \mathbb{T} on \mathcal{A} is by positive weights, the core is just $\mathcal{C}_{\mathcal{A}} = \{0\}$. Because h is \mathbb{T} -equivariant, this shows that $\mathcal{C} := \mathcal{C}_{\mathcal{M}} \subset h^{-1}(0)$. On the other hand, the properness of h shows that $h^{-1}(0)$ is projective and thus $\mathcal{C} = h^{-1}(0)$. This is typically called the *nilpotent cone* in this context. Note that \mathbb{T} -equivariance of h implies $\mathcal{M}^{\mathbb{T}} \subseteq \mathcal{C}$. From this, the notion of being very stable becomes:

Definition 2.6. A Higgs bundle $\mathcal{E} = (E, \Phi) \in \mathcal{M}^{s\mathbb{T}}$ is *very stable* if the only nilpotent Higgs bundle in $W_{\mathcal{E}}^+$ is \mathcal{E} itself. Otherwise, it is *wobbly*.

One interesting aspect of such objects is that the Hitchin map restricts nicely to the upward flow [14, Lemma 4.6]:

Proposition 2.4. *If $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is very stable then $h : W_{\mathcal{E}}^+ \rightarrow \mathcal{A}$ is finite, flat, surjective and generically tale.*

2.2. Examples of very stable Higgs bundles

In this section, we recall from [14] some examples of very stable Higgs bundles. First we consider the fixed point component of type (n) . The fixed points of this type are elements of the form $(E, 0)$ with E a semistable Higgs bundle. Thus, this component is just the moduli space of semistable rank n degree d vector bundles, \mathcal{N} . The upward flow for $\mathcal{E} = (E, 0)$ is given by $W_{\mathcal{E}}^+ = \{(E, \Phi) : \Phi \in H^0(C, \text{End}(E) \otimes K)\}$, so that \mathcal{E} is very stable if and only if the only nilpotent Higgs field $\Phi \in H^0(C, \text{End}(E) \otimes K)$ it admits is $\Phi \equiv 0$. This is the notion of very stable vector bundle introduced by Drinfeld and Laumon [21], for which they prove that very stable bundles form an open dense subset of the component.

Next, we shall focus on the type $(1, 1, \dots, 1)$ case. The starting example of very stable Higgs bundle in this component is the *canonical uniformising Higgs bundle*, $\mathcal{E}_0 = (E_0, \Phi_0)$, where

$$E_0 = \mathcal{O} \oplus K^{-1} \oplus \dots \oplus K^{1-n},$$

and, given $a = (a_1, \dots, a_n) \in \mathcal{A} = H^0(C, K) \oplus \dots \oplus H^0(C, K^n)$, the Higgs field

$$\Phi_a = \begin{pmatrix} 0 & 0 & \dots & 0 & a_n \\ 1 & 0 & \dots & 0 & a_{n-1} \\ 0 & 1 & \dots & 0 & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_1 \end{pmatrix}$$

is given by the companion matrix. The map $a \mapsto (E_0, \Phi_a)$ provides a section of the Hitchin map, known as the *Hitchin section* [19]. By means of Proposition 2.3, it

follows that $\{(E_0, \Phi_a) : a \in \mathcal{A}\} \subseteq W_{\mathcal{E}_0}^+$. Moreover, since both are affine spaces of equal dimension $\dim \mathcal{M}/2$, the upward flow is precisely the Hitchin section, hence \mathcal{E}_0 is very stable.

It is possible to completely classify very stable Higgs bundles of this type by starting with this example and performing Hecke transformations. First, we note that the data of a type $(1, 1, \dots, 1)$ fixed point is equivalent to the choice of a line bundle L_0 over C (that is, a divisor δ_0 up to principal divisor), as well as effective divisors $\delta_1, \dots, \delta_{n-1}$ on C . Indeed, such a fixed point (E, Φ) is of the form $E = L_0 \oplus \dots \oplus L_{n-1}$, where for all j we have $\text{rank } L_j = 1$, and $\Phi|_{L_{j-1}} = b_{j-1}$ for nonconstant maps $b_j : L_{j-1} \rightarrow L_j \otimes K$, $1 \leq j \leq n - 1$. Thus, L_0 is given, and δ_i is obtained as the zero locus of b_i with multiplicities. On the other hand, given $(\delta_0, \delta_1, \dots, \delta_{n-1})$ we construct E by setting $L_i := L_0 \otimes \mathcal{O}(\delta_1 + \dots + \delta_{i-1}) \otimes K^{-i}$ and $b_i := s_{\delta_i} \in \mathcal{O}(\delta_i) = \mathcal{O}(L_{j-1}^* \otimes L_j \otimes K)$ the canonical section. We shall denote the bundle corresponding to $\delta := (\delta_0, \dots, \delta_{n-1})$ by $\mathcal{E}_\delta = (E_\delta, \Phi_\delta)$.

Another convenient way of labeling these points is via choosing a dominant weight of $\text{GL}(n, \mathbb{C})$ at each point of C , that is, a map

$$\mu : C \rightarrow \Lambda^+(\text{GL}(n, \mathbb{C})) = \left\{ \sum_{i=1}^n a_i \omega_i : a \in \mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z} \right\},$$

where the ω_i are the fundamental weights. We require that the set $C \setminus \{\mu = 0\}$ is finite. We then define

$$\delta_\mu := \left(\sum_{c \in C} \langle \mu(c), \omega_n^\vee \rangle c, \left(\sum_{c \in C} \langle \mu(c), \omega_i^\vee \rangle c \right)_{i=1, \dots, n-1} \right),$$

and $\mathcal{E}_\mu := \mathcal{E}_{\delta_\mu}$. Conversely, we can retrieve the map μ from δ as

$$\mu_\delta(c) = \delta_0(c) \omega_n + \sum_{i=1}^{n-1} \delta_i(c) \omega_i \in \Lambda^+,$$

where $D(c)$ for a divisor D means the coefficient of c in D .

2.3. Hecke transformations

Now, we explain Hecke transformations for Higgs bundles. These play a key role since they allow to relate the upward flows of the different fixed points in the type $(1, 1, \dots, 1)$ component. In order to define Hecke transformations of (E, Φ) , we start by choosing a point $c \in C$ and a subspace $V \in \text{Gr}(k, E|_c)$ which is $\Phi|_c$ -invariant, that is, $\Phi|_c(V) \subset V \otimes K|_c$. The Hecke transformation $\mathcal{H}_V(E, \Phi) := (E', \Phi')$ is defined by diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & E' & \rightarrow & E & \rightarrow & E|_c/V & \rightarrow & 0, \\ & & \Phi' \downarrow & & \Phi \downarrow & & \overline{\Phi}_c \downarrow & & \\ 0 & \rightarrow & E' \otimes K & \rightarrow & E \otimes K & \rightarrow & E|_c/V \otimes K & \rightarrow & 0, \end{array}$$

where $E|_c/V$ is to be regarded as a skyscraper sheaf at c . More details of the construction of this diagram can be found in [14, Definition 4.10].

It is possible to reach any \mathcal{E}_μ from successive Hecke transformations that start at \mathcal{E}_0 . This is due to the following fundamental operation: starting with \mathcal{E}_0 and selecting the natural invariant subspace $V_k = (L_k \oplus \cdots \oplus L_{n-1})|_c$, the resulting Hecke transformation gives $\mathcal{E}_{\mu_{c,k}}$, where $\mu_{c,k}(c) = \omega_k$ and zero otherwise. This is explained in [14, Example 4.13]. For arbitrary μ it suffices to iterate the previous operation for every $c \in C$, at the ω_k indicated by $\mu(c)$. One of the main results of [14] is that the upward flows are also related by Hecke transformations, from which the following classification can be deduced:

Theorem 2.1 ([14, Theorem 4.16]). *A stable fixed point of type $(1, 1, \dots, 1)$, $(E_\delta, \Phi_\delta) \in \mathcal{M}^{s\mathbb{T}}$, is very stable if and only if the divisor $\delta_1 + \cdots + \delta_{n-1}$ is reduced.*

Remark 2.1. The previous statement can be rephrased as (E_μ, Φ_μ) being very stable if and only if for every $c \in C$, either $\mu(c) = a_n\omega_n$ or $\mu(c) = a_n\omega_n + \omega_k$ where $k \in \{1, \dots, n-1\}$ and $a_n \in \mathbb{Z}$. In other words, the point is very stable if and only if for every $c \in C$, the weight $\mu(c)$ is *minuscule*, that is, minimal with respect to the partial ordering in $\Lambda^+(\mathrm{GL}(n, \mathbb{C}))$ given by $\mu_1 \geq \mu_2 \iff \mu_1 - \mu_2 \in \Phi^+ = \{\sum_{\alpha \in \Delta^+} a_\alpha \alpha : a_\alpha \in \mathbb{Z}_{\geq 0}\}$, where Δ^+ denotes the set of positive roots.

3. Even Very Stable Upward Flows

In this section, we extend the results summarized above to the subspace $\mathcal{M}^\theta \subset \mathcal{M}$ of the moduli space defined by the fixed points of the subgroup $C_2 = \{1, -1\} \subseteq \mathbb{T}$, acting as the involution $\theta : (E, \Phi) \mapsto (E, -\Phi)$. Clearly, this subspace contains all the fixed points by the \mathbb{T} -action, so the previous concepts can be extended naturally. Moreover, by [9, Theorem 6.3], the space \mathcal{M}^θ contains the images of the maps $\mathcal{M}_{U(p,q)} \rightarrow \mathcal{M}$, given by extension of structure group, of the moduli spaces of $U(p, q)$ -Higgs bundles (for the different $U(p, q)$ with $p + q = n$, $p \leq q$) into the moduli space of $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles. The stable locus $\mathcal{M}^{\theta, s}$ is covered by these images, which are the Higgs bundles that we will consider.

We start by recalling the following definition from [4, Definition 3.3].

Definition 3.1. A $U(p, q)$ -Higgs bundle (E, Φ) is a holomorphic vector bundle E of the form $E = V \oplus W$, where V and W are vector bundles of ranks p and q , respectively, and $\Phi \in H^0(\mathrm{End}(E) \otimes K)$ is a section satisfying $\Phi(V) \subset W \otimes K$, $\Phi(W) \subset V \otimes K$.

We denote by $\mathcal{M}_{U(p,q)}^s$ the moduli space of stable $U(p, q)$ -Higgs bundles, where stability is defined as for $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles. As explained before, this space sits inside \mathcal{M} as the fixed point locus of the involution θ , that is, a stable Higgs bundle (E, Φ) is a $U(p, q)$ -Higgs bundle (for some p and q) if and only if $(E, \Phi) \simeq (E, -\Phi)$ [9, Theorem 6.3]. We will often use interchangeably the moduli space $\mathcal{M}_{U(p,q)}$ and its image inside \mathcal{M} .

Note that every fixed point $\mathcal{E} = (E, \Phi)$ of the \mathbb{T} -action is in particular fixed by -1 and hence a $U(p, q)$ -Higgs bundle for some p and q . This can be seen more explicitly by observing that, in the decomposition $E = E_0 \oplus \dots \oplus E_{k-1}$ with $\Phi(E_i) \subset E_{i+1} \otimes K$, the Higgs field Φ interchanges the summands with odd indices by those of even indices. In particular, type $(1, 1, \dots, 1)$ fixed points are Higgs bundles for the quasi-split group $U(p, p)$ or $U(p, p + 1)$.

Moreover, recall from Bialynicki–Birula theory in Sec. 2 that the upward flow $W_{\mathcal{E}}^+$ is an affine space isomorphic to the subspace of positive weights $T_{\mathcal{E}}^+ \mathcal{M} \subset T_{\mathcal{E}} \mathcal{M}$. It is easy to identify the subspace of $U(p, q)$ -Higgs bundles.

Remark 3.1. The $U(p, q)$ -Higgs bundles in $W_{\mathcal{E}}^+$, that is, $\mathcal{M}_{U(p,q)} \cap W_{\mathcal{E}}^+$, correspond via the isomorphism $W_{\mathcal{E}}^+ \simeq T_{\mathcal{E}}^+ \mathcal{M}$ to the vector subspace of positive, *even* weights $T_{\mathcal{E}}^{2+} \mathcal{M} \subset T_{\mathcal{E}} \mathcal{M}$.

This is because the even weights are precisely the vector subspace fixed by multiplication by -1 , and the previous isomorphism is \mathbb{T} -equivariant. In other words, we can view the locus of $U(p, q)$ -Higgs bundles at the upward flow of a fixed point as a subspace:

Definition 3.2. The *even upward flow* at \mathcal{E} is the subspace of $W_{\mathcal{E}}^+$ corresponding to $T_{\mathcal{E}}^{2+} \mathcal{M}$ and is denoted by $W_{\mathcal{E}}^{2+}$.

Note that this coincides with the standard upward flow when defined in \mathcal{M}^θ instead of \mathcal{M} . Hence, we also have the following natural definition of very stable points:

Definition 3.3. We say that \mathcal{E} is *even very stable* if $W_{\mathcal{E}}^{2+} \cap \mathcal{C} = \{\mathcal{E}\}$, where $\mathcal{C} = h^{-1}(0) \subseteq \mathcal{M}$ denotes the locus of nilpotent Higgs bundles. Otherwise, it is said to be *even wobbly*.

We remark, as one of the main interests for this study, that the subspaces of even weights $T_{\mathcal{E}}^2 \mathcal{M} \subset T_{\mathcal{E}} \mathcal{M}$ that we are considering are Lagrangian, since the symplectic form ω pairs the subspace of weight k with that of $1 - k$, as explained at the end of Sec. 2, so that the subspaces of even weights are paired with those of odd weights. In fact, the subvariety $\mathcal{M}^\theta \subset \mathcal{M}$ itself is Lagrangian, as explained in [9, Theorem 8.10].

Obviously, a very stable fixed point is also even very stable. However, a wobbly fixed point \mathcal{E} can either remain even wobbly or instead be even very stable, depending on whether the nontrivial intersection $(W_{\mathcal{E}}^+ \cap \mathcal{C}) \setminus \{\mathcal{E}\}$ happens at even weights or not. We will classify the even very stable Higgs bundles of type $(1, 1, \dots, 1)$, revealing that both situations already arise in this case. We follow the same notation of Sec. 2.2.

Proposition 3.1. *Let $\mathcal{E} = (E, \Phi)$ be a smooth fixed point of type $(1, 1, \dots, 1)$. Suppose that there is some point $c \in \mathcal{C}$ such that $b_i(c) = b_j(c) = 0$ for some i, j of different parity. Then \mathcal{E} is even wobbly.*

Proof. The proof follows the approach of constructing a curve via Hecke transformations followed in the proof of [14, Theorem 4.16]. We will see that, in this situation, the curve can be constructed along even weights. We have that $b_k(c) = b_{k+l}(c) = 0$ where $l > 0$ and odd. We start by performing a Hecke transformation at the Φ_c -invariant subspace $(L_0 \oplus \dots \oplus L_{k-1})|_c \subset E|_c$, yielding

$$E' = L_0 \oplus \dots \oplus L_{k-1} \oplus L_k(-c) \oplus \dots \oplus L_{n-1}(-c),$$

$$\Phi' = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ b_1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & b_2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{b_k}{s_c} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & b_{n-1} & 0 \end{pmatrix},$$

where $s_c \in H^0(\mathcal{O}(c))$ is the canonical section. This bundle is still stable [14, Lemma 4.17]. We have that $V_0 = (L_k(-c) \oplus \dots \oplus L_{n-1}(-c))|_c \subset E'|_c$ is the $(n - k)$ -dimensional Φ'_c -invariant subspace that transforms (E', Φ') back into $(E(-c), \Phi)$. Now, let $\{v_0, \dots, v_{n-1}\}$ be a basis of $E'|_c$, each v_j taken in the corresponding component. We define the following $(n - k)$ -dimensional subspace:

$$V := \langle v_{k-1} + v_{k+l}, v_k, v_{k+1}, \dots, v_{k+l-1}, v_{k+l+1}, v_{n-1} \rangle \subset E'|_c.$$

This is another Φ'_c -invariant subspace, since $\Phi'_c(v_{k-1} + v_{k+l}) \in \langle v_k, v_{k+l+1} \rangle$, $\Phi'_c(v_m) \in \langle v_{m+1} \rangle$ for $m \in \{k, \dots, k+l-2\} \cup \{k+l+1, \dots, n-1\}$, and $\Phi'_c(v_{k+l-1}) = 0$ (in this analysis we take $v_j = 0$ for any $j > n - 1$). Now, recall that there is an induced \mathbb{T} -action on $Gr_{n-k}(E'|_c)$ given by the \mathbb{T} -action on E' with weight $-i$ on the i th summand of E' . With this, given $\lambda \in \mathbb{T}$ we define $V_\lambda := \lambda V$, that is

$$V_\lambda = \langle \lambda^{l+1}v_{k-1} + v_{k+l}, v_k, v_{k+1}, \dots, v_{k+l-1}, v_{k+l+1}, v_{n-1} \rangle.$$

Note that this subspace is always $n - k$ -dimensional. This yields a curve within the connected subvariety $S_{n-k}(\Phi'|_c) \subset Gr_{n-k}(E'|_c)$ of vector subspaces of the fiber $E'|_c$ which are invariant by $\Phi'|_c$. As argued in the proof of [14, Theorem 4.16], this translates into a curve in the moduli space of Higgs bundles, defined by $\mathcal{H}_{V_\lambda}(E', \Phi') \simeq \lambda \mathcal{H}_V(E', \Phi')$. This curve connects $\mathcal{H}_{V_0}(E', \Phi') = (E(-c), \Phi)$ with a different fixed point, given by $\mathcal{H}_{V_\infty}(E', \Phi')$, where $V_\infty = \langle v_{k-1}, v_k, v_{k+1}, \dots, v_{k+l-1}, v_{k+l+1}, v_{n-1} \rangle$. This fixed point is also stable [14, Lemma 4.18].

Moreover, since l is odd, then $l + 1$ is even and we have that $V_\lambda = V_{-\lambda}$, hence $\lambda \mathcal{H}_V(E', \Phi') \simeq -\lambda \mathcal{H}_V(E', \Phi')$ so that this curve is fixed by the action of -1 and hence it lies in $W_{(E(-c), \Phi)}^{2+}$. This shows that $(E(-c), \Phi)$ is even wobbly thus (E, Φ) as well. □

There are still more examples of even wobbly stable Higgs bundles of type $(1, 1, \dots, 1)$, which are covered by the following proposition.

Proposition 3.2. *Let $\mathcal{E} = (E, \Phi)$ be a smooth fixed point of rank $n \geq 4$. Suppose that $b := b_{n-2} \circ \dots \circ b_2$ has a multiple zero at $c \in C$. Then \mathcal{E} is even wobbly.*

Proof. The proof is identical to that of Proposition 3.1, with the only difference being the construction of the invariant subspace $V \in S_{n-k}(\Phi'|_c)$.

Let $1 < i_1 < i_2 < \dots < i_m < n - 1$ be all the indices other than 1 and $n - 1$ such that $b_{i_k}(c) = 0$. We can assume that all of them are of the same parity since otherwise we apply the previous proposition. We start again with a Hecke transform at $(L_0 \oplus L_1 \cdots \oplus L_{i_m-1})|_c$. This yields (E', Φ') with the same form as in the proof of Proposition 3.1.

Now, we will construct a basis $\{v_0, v_1, \dots, v_{n-1}\}$ of $E'|_c$ as follows. We start with any nonzero $v_0 \in L'_0|_c$, then apply $\Phi'|_c$ until a zero vector is obtained. This will produce a string: $v_1 = \Phi'|_c(v_0)$, $v_2 = \Phi'|_c(v_1) \dots$ all the way up to v_{i_1-1} , if $b_1(c) \neq 0$, or just v_0 if $b_1(c) = 0$. The process then iterates: we pick v_{i_1} (or v_1 if $b_1(c) = 0$) nonzero in the corresponding summand and iterate, repeating until v_{n-1} . By construction, the basis is partitioned in strings inside of which each element maps to the next one and the last one maps to zero. Also, each v_i is in the corresponding summand $L'_i|_c$. Let $V_0 = \langle v_{i_m}, \dots, v_{n-1} \rangle$.

Now, recall that $i_m < n - 1$ and $i_1 \geq 2$. Note that if $m = 1$, it is possible that $i_1 = i_m$, meaning a multiple zero of b_{i_1} at c . In any case, we define the following $(n - i_m)$ -dimensional subspace:

$$V = \langle v_{i_m} + v_{i_1-2}, v_{i_m+1} + v_{i_1-1}, v_{i_m+2}, \dots, v_{i_m+i_1-1}, v_{i_m+i_1}, \dots, v_{n-1} \rangle.$$

It is $\Phi'|_c$ -invariant: each generator is taken to the next, and the last one to zero. Note that it is important that $i_m < n - 1$ since otherwise we get a 1-dimensional space that does not work, for $\langle v_{i_m} + v_{i_1-2} \rangle \mapsto \langle v_{i_1-1} \rangle$. Also notice that if $b_{n-1}(c) = 0$ then the second generator might map to zero instead of v_{i_m+2} , but this is not a problem for the invariance. We compute λV :

$$\lambda V = \langle v_{i_m} + \lambda^\nu v_{i_1-2}, v_{i_m+1} + \lambda^\nu v_{i_1-1}, v_{i_m+2}, \dots, v_{i_m+i_1-1}, v_{i_m+i_1}, \dots, v_{n-1} \rangle,$$

where $\nu := i_m - i_1 + 2$. Once again this gives a curve in $S_{n-k}(\Phi'|_c)$ connecting V_0 with a different V_∞ such that the Hecke transform is a new fixed point and, since $i_m - i_1 + 2$ is even, it follows that $\lambda V = -\lambda V$, as desired. \square

We now prove that these constitute all even wobbly cases.

Proposition 3.3. *Let $\mathcal{E}_\delta = (E, \Phi)$ be a smooth fixed point of type $(1, 1, \dots, 1)$. Suppose that for every $c \in C$ we have that $b_i(c) = b_j(c) = 0$ implies $i \equiv j \pmod{2}$, and $b_{n-2} \circ \dots \circ b_2$ has at most a single zero at c . Then, \mathcal{E} is even very stable.*

Proof. We argue by induction on $\deg(\delta_{n-1} + \dots + \delta_1)$, as we already know from Sec. 2.2 that the degree 0 case, which is the canonical uniformising Higgs bundle,

is even very stable. Suppose that $(E', \Phi') \in W_{\mathcal{E}}^+ \cap \mathcal{C}$ is a nilpotent element in the upward flow of \mathcal{E} , with the full filtration

$$0 = W_0 \subset W_1 \subset \dots \subset W_{n-1} \subset W_n = E'$$

given by Proposition 2.3. Take one of the points $c \in C$ with $\mu_{\delta}(c) \neq 0$, and suppose first that $b_{n-1}(c) = 0$. This means that $V' := (W_{n-1})|_c$ is $\Phi'|_c$ -invariant, and, as explained in [14, Sec. 4], the Hecke transformation $(E'_1, \Phi'_1) = \mathcal{H}_{V'}(E', \Phi')$ is stable, nilpotent and lies in the upward flow of $(E_1, \Phi_1) := \mathcal{H}_V(E, \Phi)$, where $V = (L_0 \oplus \dots \oplus L_{n-2})|_c \subseteq E|_c$. We have that (E_1, Φ_1) is another type $(1, 1, \dots, 1)$ point given by $\delta^1 = (\delta_0^1, \dots, \delta_{n-1}^1)$ with the only difference being that $\delta_{n-1}^1 = \delta_{n-1} - c$. Thus, by the induction hypothesis it follows that (E_1, Φ_1) is even very stable, which results in the following two options:

- If $(E'_1, \Phi'_1) \not\cong (E_1, \Phi_1)$, since the latter is even very stable, we have that (E'_1, Φ'_1) lies in an odd weight space of $T_{(E_1, \Phi_1)}\mathcal{M}$ and, by using the invariant subspaces $V'_1 \subseteq E'_1|_c$ and $V_1 \subseteq E_1|_c$ such that $\mathcal{H}_{V'_1}(E'_1, \Phi'_1) = (E', \Phi')$ and $\mathcal{H}_{V_1}(E_1, \Phi_1) = (E, \Phi)$ (up to twisting by a fixed line bundle), it follows that (E', Φ') also has odd weight in the upward flow for (E, Φ) and hence $(E', \Phi') \notin W_{\mathcal{E}_{\delta}}^{2+}$.
- If $(E'_1, \Phi'_1) \cong (E_1, \Phi_1)$, then the one-dimensional subspace $V'_1 \subseteq E'_1|_c$ such that $\mathcal{H}_{V'_1}(E'_1, \Phi'_1) = (E', \Phi')$ is one of the $\Phi_1|_c$ -invariant subspaces in the pair (E_1, Φ_1) , which we know well since it comes from a Hecke transformation of the starting (E, Φ) . Indeed, we have:

$$E_1 = L_0 \oplus \dots \oplus L_{n-2} \oplus L_{n-1}(-c),$$

$$\Phi_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ b_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & b_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & b_{n-2} & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{b_{n-1}}{s_c} & 0 \end{pmatrix}.$$

Let $V = \{v_0, \dots, v_{n-1}\}$ be a basis of $E_1|_c$. From the starting hypotheses about (E, Φ) , the only basis vectors that could vanish via $\Phi_1|_c$ are $\{v_0, v_{j-1}, v_{n-2}, v_{n-1}\}$ where j is the only index with $2 \leq j \leq n-2$ such that $b_j(c) = 0$, if it exists. Necessarily, $j \equiv n-1 \pmod{2}$. Also, if v_0 vanishes then $b_1(c) = 0$ and hence $n \equiv 0 \pmod{2}$. Finally, v_{n-2} vanishes if and only if b_{n-1} had a multiple zero at c .

Since Φ_1 is nilpotent, the desired invariant 1-dimensional subspace must be of the form $V'_1 = \langle v \rangle$ where $\Phi_1|_c(v) = 0$, that is, $v = \alpha v_0 + \beta v_{j-1} + \gamma v_{n-2} + \delta v_{n-1}$. Moreover,

$$\lambda \cdot V'_1 = \langle \alpha \lambda^{n-1} v_0 + \beta \lambda^{n-j} v_{j-1} + \gamma \lambda v_{n-2} + \delta v_{n-1} \rangle$$

must verify that $\lim_{\lambda \rightarrow 0} \lambda V'_1 = V_1 = \langle v_{n-1} \rangle$, which implies that $\delta \neq 0$. Hence, because of the parity of the exponents of λ appearing in the expression for $\lambda \cdot V'_1$, if $(E', \Phi') = \mathcal{H}_{V'_1}(E'_1, \Phi'_1)$ were in an even weight space, that is, if $\lambda \cdot V'_1 = -\lambda \cdot V'_1$, it would be necessary that $\alpha = \beta = \gamma = 0$, thus $V'_1 = V_1$, meaning $(E', \Phi') \simeq (E, \Phi)$.

This concludes the analysis for the $b_{n-1}(c) = 0$ case. Now, if $b_{n-1}(c) \neq 0$, we may also assume that $b_1(c) \neq 0$. This is because the involution $(E, \Phi) \mapsto (E^*, \Phi^t)$ of \mathcal{M} naturally bijects fixed points and upward flows, sending a point with $b_1(c) = 0$ to a point with $b_{n-1}(c) = 0$. Hence, the remaining case is when the only $b_j(c) = 0$ happens at a single j with $2 \leq j \leq n - 2$ and multiplicity one. This case is treated exactly as in the proof of [14, Theorem 4.16], which we now recall.

We have the $\Phi'|_c$ -invariant j -dimensional subspace $V' := (W_{j-1})|_c$ giving a nilpotent $(E'_1, \Phi'_1) = \mathcal{H}_{V'}(E', \Phi')$ in the upward flow of $(E_1, \Phi_1) = (E_1, \Phi_1) := \mathcal{H}_V(E, \Phi)$, where $V = (L_0 \oplus \dots \oplus L_{j-1})|_c \subseteq E|_c$. Exactly as before, if $(E'_1, \Phi'_1) \neq (E_1, \Phi_1)$, the induction hypothesis gives $(E', \Phi') \notin W_{\mathcal{E}_\delta}^{2+}$. Otherwise, notice that $\Phi'_1|_c$ is now a regular nilpotent, so the only $\Phi'_1|_c$ -invariant $(n - j)$ -dimensional subspace is $V'_1 = V_1 = (L_j \oplus \dots \oplus L_{n-1})|_c$. Hence (up to tensoring everything by a fixed line bundle) we have $(E', \Phi') = \mathcal{H}_{V_1}(E_1, \Phi_1) = (E, \Phi)$. \square

3.1. Even minuscule weights in $GL(n, \mathbb{C})$

We will now see how the conditions found before arise naturally in the context of the root system of the structure group $GL(n, \mathbb{C})$ of E . We have $\omega_1, \dots, \omega_{n-1}, \omega_n \in \Lambda^+$ the fundamental weights. Recall that the *height* of a root is the number of simple roots in its decomposition. In terms of the fundamental weights the positive roots of height $k \in \{1, \dots, n - 1\}$ can be indexed by $p \in \{1, \dots, n - k\}$ and given by

$$\alpha_{k,p} := -\omega_{p-1} + \omega_p + \omega_{p+k-1} - \omega_{p+k},$$

where ω_0 is understood as 0. In particular, the simple roots are the positive roots of height 1 of the form $\alpha_{1,p} = -\omega_{p-1} + 2\omega_p - \omega_{p+1}$ for $p = 1, \dots, n - 1$. The highest root $\alpha_{n-1,1} = \omega_1 + \omega_{n-1} - \omega_n$ has height $n - 1$.

Definition 3.4. For dominant weights λ, μ , we define the *even partial ordering* as

$$\lambda \geq_2 \mu \iff \lambda - \mu \in \Phi^{2+},$$

where

$$\Phi^{2+} = \left\{ \sum_{\substack{k \text{ even,} \\ 1 \leq p \leq n-k}} c_{k,p} \alpha_{k,p} : c_{k,p} \in \mathbb{Z}_{\geq 0} \right\}$$

is the set of positive linear combinations of even positive roots. Minimal elements for this ordering are called *even minuscule*.

We will say for a weight λ that its *i th coordinate/entry*, λ_i , is the coefficient of ω_i when λ is written in the basis $\{\omega_1, \dots, \omega_n\}$. The *position* of the i th coordinate

will be just i . Now, we will characterize the even minuscule weights, seeing that the conditions exactly match those for even very stable Higgs bundles. The characterization will be carried out solely via combinatorial arguments.

Proposition 3.4. *Let λ be a dominant weight such that at least one of these holds:*

- (1) *The weight λ has nonzero coordinates at two positions $1 \leq i < j \leq n - 1$ with $i \not\equiv j \pmod 2$.*
- (2) *The weight λ verifies $\lambda_2 + \dots + \lambda_{n-2} \geq 2$.*

Then λ is not even minuscule.

Proof. For the first situation, if λ has nonzero coordinates at i and j , where $j - i := k$ is an odd positive number, then consider $\mu := \lambda - \alpha_{i,k+1}$. Because of the nonzero coordinates at i and j it follows that μ is dominant. By construction, $\mu < \lambda$.

If $\lambda_2 + \dots + \lambda_{n-2} \geq 2$, choose indices i, j such that $2 \leq i \leq j \leq n - 2$ with either $i \neq j$ and $\lambda_i, \lambda_j \geq 1$, or $i = j$ and $\lambda_i \geq 2$. We can assume the positions are of the same parity, since otherwise we apply the previous. Hence write $k := j - i$ a nonnegative even number. Defining $\mu := \lambda - (\alpha_{k+2,i} + \alpha_{k+2,i-1})$ works as before, since $\alpha_{k+2,i} + \alpha_{k+2,i-1} = -\omega_{i-1} + \omega_i + \omega_{j+1} - \omega_{j+2} - \omega_{i-2} + \omega_{i-1} + \omega_j - \omega_{j+1} = -\omega_{i-2} + \omega_i + \omega_j - \omega_{j+2}$. □

Remark 3.2. Notice in the last part of the previous proof that $2 \leq i, j \leq n - 2$ is indeed required: for example, if $i = 1$ we cannot take $p = i - 1 = 0$. Similarly, we need for $\alpha_{k+2,i}$ to make sense that $i \leq n - (k - 2) = n - k + 2$ which is equivalent to $j \leq n - 2$.

As an example for the proof, take $n = 9$ and $\lambda = (0, 1, 0, 0, 0, 1, 0, 0, 0)$. Following the proof, we can consider $\alpha_{6,2} + \alpha_{6,1} = (-1, 1, 0, 0, 0, 0, 1, -1, 0) + (1, 0, 0, 0, 0, 1, -1, 0, 0) = (0, 1, 0, 0, 0, 1, 0, -1, 0)$, so that $\mu = (0, 0, 0, 0, 0, 0, 0, 1, 0)$ is lower than λ .

For the reciprocal, the combinatorial arguments will be easier if we stop considering ω_0 as zero and rather see it as an extra linearly independent vector. That is, we view the weight space $W := \langle \omega_1, \dots, \omega_{n-1}, \omega_n \rangle$ as a subspace of a new vector space $\widetilde{W} := \langle \omega_0, \omega_1, \dots, \omega_{n-1}, \omega_n \rangle$ which is a dimension higher. We have a projection $\pi : \widetilde{W} \rightarrow W$. We define $\widetilde{\alpha}_{k,p} := -\omega_{p-1} + \omega_p + \omega_{p+k-1} - \omega_{p+k}$ as before but in \widetilde{W} (i.e. we do not consider $\omega_0 = 0$ in that expression anymore), so that $\pi(\widetilde{\alpha}_{k,p}) = \alpha_{k,p}$. We also lift the even positive lattice as

$$\widetilde{\Phi}^{2+} = \left\{ \sum_{\substack{k \text{ even,} \\ 1 \leq p \leq n-k}} c_{k,p} \widetilde{\alpha}_{k,p} : c_{k,p} \in \mathbb{Z}_{\geq 0} \right\},$$

so that $\pi(\widetilde{\Phi}^{2+}) = \Phi^{2+}$.

For example, in this new vector space the simple positive roots have the following lifts: $\tilde{\alpha}_1 = (-1, 2, -1, 0, 0, \dots)$, $\tilde{\alpha}_2 = (0, -1, 2, -1, 0, 0, \dots)$, etc. As can be seen, this avoids the situation of the negative coordinate being truncated at the beginning. In this setting, we have the following facts about $\tilde{\Phi}^{2+}$, which will be the only ones we will need for our proof:

Lemma 3.1. *Take a nonzero $\tilde{x} = (x_0, x_1, \dots, x_{n-1}, x_n) \in \tilde{\Phi}^{2+}$, we have*

- (1) *The values x_0 and x_n are not positive.*
- (2) *The values $\sum_{j=0}^n x_j$, $\sum_{j \text{ even}} x_j$ and $\sum_{j \text{ odd}} x_j$ are all zero.*
- (3) *The value $\sum_{j=2}^{n-2} x_j$ is not negative.*
- (4) *The first and last nonzero coordinates of \tilde{x} are negative.*

Proof. Immediate, by induction on the number of positive even roots into which \tilde{x} decomposes. First, it is clear that any $\tilde{\alpha}_{k,p}$ for even k has all those properties. Second, it is easy to check that the sum of any two vectors with those properties keeps satisfying them. Hence the result follows. □

Proposition 3.5. *Let λ be a dominant weight that is not even minuscule. Then at least one of these hold:*

- (1) *The weight λ has nonzero coordinates at two positions $1 \leq i < j \leq n - 1$ with $i \not\equiv j \pmod{2}$.*
- (2) *The weight λ verifies $\lambda_2 + \dots + \lambda_{n-2} \geq 2$.*

Proof. Take $\mu < \lambda$. Denote $x := \lambda - \mu \in \Lambda^{2+}$. Since $\lambda = x + \mu$ and the coordinates of μ at positions between 1 and $n - 1$ are non-negative, it suffices to check that either x has positive coordinates at positions between 1 and $n - 1$ of different parity, or that the sum of the positive coordinates in positions $\{2, \dots, n - 2\}$ of x , which we shall denote S from now on, verifies $S \geq 2$.

In order to do this we can work with a lift $\tilde{x} \in \tilde{\Lambda}^{2+}$ of x by lifting each of the positive even roots in some decomposition. We will prove that \tilde{x} has at least one of the two desired properties, and, since $x_0, x_n \leq 0$ by Lemma 3.1, then $x = \pi(\tilde{x})$ will also have them and the proof will be complete.

Assume that \tilde{x} has every positive coordinate between 1 and $n - 1$ at even positions. We will show that $S \neq 0$ and $S \neq 1$, hence $S \geq 2$.

If $S = 0$ this means there are no positive entries in positions $\{2, \dots, n - 2\}$. By Lemma 3.1(3) there are no negative entries either. So, we have $\tilde{x} := (x_0, x_1, 0, \dots, 0, x_{n-1}, x_n)$. Now, if n is even then by Lemma 3.1(2), we have $x_0 + x_n = 0$ hence $x_0 = x_n = 0$, the latter by Lemma 3.1(1). But, since $x_1 = -x_{n-1}$ this contradicts Lemma 3.1(4) reaching a contradiction. If n is odd then $x_0 + x_{n-1} = 0$ and $x_1 + x_n = 0$. By assumption that positive entries are in even positions, we have $x_1 = x_n = 0$ and we reach the same contradiction.

If $S = 1$ there will be exactly one positive entry in positions $\{2, \dots, n - 2\}$ of value 1, say the i th one. By assumption i is even. By Lemma 3.1(3), from positions

2 to $n - 2$ there could either be no negative entries or one negative entry of value -1 at position j . Now, a case by case analysis follows.

- If n is odd, the case j odd is excluded because we would get, summing odd entries, that $x_1 + x_n - 1 = 0$ and we have $x_1, x_n \leq 0$, the first by assumption and the second by Lemma 3.1(1). So, we have $x_1 = x_n = 0$ and either j is even or there is no j at all. In either case we find, summing the even entries, that $x_0 + x_{n-1} \in \{0, -1\}$, from which it is impossible that both are negative, contradicting Lemma 3.1(4).
- If n is even, the case j odd is again excluded since, summing odd entries, we get $x_1 + x_{n-1} - 1 = 0$, but by assumption both x_1 and x_{n-1} are not positive. Hence $x_1 = x_{n-1} = 0$ and j is even or there is no j at all. In this case, we have to distinguish: if there is no j then $x_0 + x_n = -1$ so both cannot be negative at the same time, and since the only other nonzero entry is the 1 at the i th position, Lemma 3.1(4) is contradicted. If there is a j then $x_0 + x_n = 0$, so that $x_0 = x_n = 0$ and once again we have a contradiction with Lemma 3.1(4).

The only remaining task to complete the proof is to work out the case where all positive entries of \tilde{x} are at odd positions instead of even ones. There is a symmetry in $\tilde{\Phi}^{2+}$ via $\omega_k \mapsto \omega_{n-k}$ that will change odd positions and even positions if n is odd. So, the only remaining cases are for even n , which can be approached exactly the same as before:

- If $S = 0$ then the previous argument for even n did not use the positions of positive entries being even so it still works when they are odd.
- If $S = 1$ we now have that i is odd. If j were even, summing the even entries we get $x_0 + x_n - 1 = 0$ which is not possible ($x_0, x_n \leq 0$) so that j is either odd or does not exist. If it is odd, we sum the odd entries to get $x_1 + x_{n-1} = 0$, which yields a contradiction with Lemma 3.1(4). If it does not exist, we get $x_1 + x_{n-1} = -1$, from which they cannot both be negative and we reach the same contradiction.

□

Thus, we can rephrase the condition for even wobbliness as follows.

Theorem 3.1. *Let $\mu : C \rightarrow \Lambda^+$. Then $\mathcal{E}_\mu \in \mathcal{M}^{s\mathbb{T}}$ is even very stable if and only if for every $c \in C$, the dominant weight $\mu(c)$ is even minuscule.*

As an example, the fixed point in rank $n = 4$ associated to $\mu(c) = \omega_1 + \omega_3$ and $\mu(d) \neq 0$ for $d \neq c$ is even very stable, while being wobbly in the usual sense. In rank $n = 2$ every fixed point is even very stable, which can be deduced more easily by the lack of even weights in downward flows. This is because the only weights appearing in the upward flows in rank $n = 2$ are those appearing in the Hitchin base, which are 1 and 2. We then know that a negative weight k is paired to a nonnegative one, $1 - k$, via the symplectic form ω , hence the only possible negative weight is $k = -1$.

As a final remark, it is possible to work with higher order automorphisms defined by $(E, \Phi) \mapsto (E, \zeta_r \Phi)$, where ζ_r is a primitive r th root of unity, giving subspaces $\mathcal{M}^r \subset \mathcal{M}$ and the corresponding notions of r -very stable bundles. Most of the results presented here naturally generalize to that situation (by using r -minuscule weights), however in this case there is no longer an associated real group and the subspaces considered are not Lagrangian.

4. Hitchin Map on Upward Flows

One important motivation to consider very stable upward flows in [14] was the observation that the Hitchin map

$$h_{\mathcal{E}} := h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \rightarrow \mathcal{A}$$

restricted to them is proper. Furthermore, $W_{\mathcal{E}}^+ \cong T_{\mathcal{E}}^+ \mathcal{M}$ as \mathbb{T} -varieties. Thus, in the very stable case $h_{\mathcal{E}}$ is a proper, even finite flat [14, Lemma 4.6], \mathbb{T} -equivariant morphism between semi-projective affine spaces of the same dimension. As such it is susceptible for explicit description.

4.1. Equivariant cohomology of homogeneous spaces

We will describe the Hitchin map explicitly on some very stable upward flows in terms of equivariant cohomology. First, we recall some of the basic properties of equivariant cohomology see e.g. [1] for more details.

Let G be a connected complex affine (or compact Lie) group. Consider the classifying G -bundle $EG \rightarrow BG$, where EG is contractible. BG is called the classifying space, and its cohomology ring can be computed as follows:

$$H_G^* := H^*(BG; \mathbb{C}) \cong H^*(BT; \mathbb{C})^{W_G} \cong \mathbb{C}[\mathfrak{t}]^{W_G},$$

where $\mathfrak{t} = \text{Lie}(T)$ is the Lie algebra of a maximal torus $T \subset G$ and W_G is the Weyl group of G .

Let now G act on a variety (or manifold) X . We can form the Borel, or homotopy, quotient $X_G := (EG \times X)/G$, by the diagonal action of G , which is an X -bundle over BG . Its cohomology ring is what is taken to be its equivariant cohomology

$$H_G^*(X; \mathbb{C}) := H^*(X_G; \mathbb{C}).$$

As X_G is an X -bundle over BG we get a ring map $H_G^* \rightarrow H_G^*(X; \mathbb{C})$, making $H_G^*(X; \mathbb{C})$ an H_G^* -algebra.

Let now $T < H < G$ be a connected closed subgroup containing the maximal torus T of G . Because EG is contractible, we have $EG/H \sim EH/H$ so we can compute the G -equivariant cohomology of the homogeneous space G/H as follows:

$$H_G^*(G/H; \mathbb{C}) \cong H^*((EG \times G/H)/G; \mathbb{C}) \cong H^*(EG/H; \mathbb{C}) \cong H_H^*,$$

with the ring map $H_G^* \rightarrow H_H^*$ from the natural map $BH \rightarrow BG$ induced from the embedding $H \subset G$. This way we have a simple way to compute equivariant

cohomology of a homogeneous space explicitly as in the following diagram of graded algebras:

$$\begin{array}{ccc}
 H_G^*(G/H; \mathbb{C}) \cong \mathbb{C}[\mathfrak{t}]^{W_H} & & \\
 \uparrow & & \uparrow \\
 H_G^* & \cong & \mathbb{C}[\mathfrak{t}]^{W_G}.
 \end{array} \tag{4.1}$$

Note that the structure maps above are finite free, meaning that they define a finite free module. This is because the equivariant cohomology of equal rank homogeneous spaces are equivariantly formal [10, (1.2)], for example because they have no odd cohomology [11, Theorem VII, p. 467].

We will see below that on certain very stable upward flows the Hitchin map can be modeled by the spectrum of equivariant cohomology of appropriate homogeneous spaces. For this reason, we also take the spectrum of (4.1) and record the corresponding diagram:

$$\begin{array}{ccc}
 \text{Spec}(H_G^*(G/H; \mathbb{C})) \cong \mathfrak{t} // W_H & & \\
 \downarrow & & \downarrow \\
 \text{Spec}(H_G^*) & \cong & \mathfrak{t} // W_G.
 \end{array} \tag{4.2}$$

We note that half the grading on cohomology will induce the \mathbb{T} -action on $\mathfrak{t} // W_H$, which agrees with the \mathbb{T} -action induced from weight one action on \mathfrak{t} . The down arrows in the diagram then become \mathbb{T} -equivariant finite flat (ultimately because of equivariant formality), in particular proper, morphisms.

Because we will also consider how certain involutions act on the equivariant cohomology of homogeneous spaces, here we record the following lemma. To formulate it recall that for a unital commutative ring R with 2 invertible and with involution $\theta : R \rightarrow R$, the *coinvariant ring* is defined as

$$R_\theta := R / (r - \theta(r)),_{r \in R} \cong R / (r \in R : \theta(r) = -r).$$

While if A is a commutative R -algebra and $\theta : A \rightarrow A$ also acts on A , compatibly with the action on R , then A_θ is naturally an R_θ -algebra, called the *coinvariant algebra*. Their relevance is in forming the fixed point scheme of the affine R -scheme $\text{Spec}(A)$ under the involution $\text{Spec}(\theta) : \text{Spec}(A) \rightarrow \text{Spec}(A)$, which we also denote by θ . We get that the fixed point scheme

$$\text{Spec}(A)^\theta \cong \text{Spec}(A_\theta) \tag{4.3}$$

is an affine R_θ -scheme. Using arguments from [24] we have the following lemma.

Lemma 4.1. *Let G be a connected complex reductive group (or a connected compact Lie group), and $\tau : G \rightarrow G$ a complex algebraic (smooth) involution, with $\mathbb{T} \subset G$ a τ -stable torus. Let G_0^τ denote the identity component of the fixed point group G^τ . Assume that*

$$H_G^* \twoheadrightarrow H_{G_0^\tau}^* \tag{4.4}$$

is surjective. Let $\theta := \tau^* : H_G^* \rightarrow H_G^*$ denote the induced action on the cohomology of BG . Then we have that the coinvariant algebra

$$(H_G^*)_\theta \cong H_{G_0^\tau}^*.$$

Moreover, if $T \subset H \subset G$ is a τ -invariant closed connected subgroup, such that $H_H^* \twoheadrightarrow H_{H_0^\tau}^*$ is surjective, then we have the commutative diagram:

$$\begin{array}{ccccccc} H_G^*(G/H; \mathbb{C})_\theta & \cong & (H_H^*)_\theta & \cong & H_{H_0^\tau}^* & \cong & H_{G_0^\tau}^*(G_0^\tau/H_0^\tau; \mathbb{C}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ (H_G^*)_\theta & \cong & (H_G^*)_\theta & \cong & H_{G_0^\tau}^* & \cong & H_{G_0^\tau}^*. \end{array}$$

Proof. We know that $H_G^* \cong \mathbb{C}[\mathfrak{t}]^W$, the algebra of invariant polynomials on \mathfrak{t} by the Weyl group W of G . By [29, Lemma 6.1], we can choose algebra generators p_1, \dots, p_r in $\mathbb{C}[\mathfrak{t}]^W$ such $\tau(p_i) = \epsilon_i p_i$, where $\epsilon_i = \pm 1$. By [29, Lemma 6.5], the generators with $\epsilon_i = 1$ give the generators of $\mathbb{C}[\mathfrak{t}^\tau]^{W^\tau}$, where W^τ is the subgroup of W fixed by τ , itself a reflection group on \mathfrak{t}^τ . Thus the number of p_i 's with $\epsilon_i = 1$ is exactly the dimension of \mathfrak{t}^τ , which in turn agrees with the rank of G_0^τ .

Finally, all p_i 's with $\epsilon_i = -1$ restrict trivially to \mathfrak{t}^τ . Thus by the assumption (4.4) the restriction of those with $\epsilon_i = 1$ generate $\mathbb{C}[\mathfrak{t}^\tau]^{W_{G_0^\tau}} \cong H_{G_0^\tau}^*$, and as there are $\text{rank}(G_0^\tau)$ of them, they should restrict algebraically independent. Thus, the kernel of the surjection is generated by the anti-invariant generators, showing the claim $(H_G^*)_\theta \cong H_{G_0^\tau}^*$.

The second statement follows from the first and (4.1). □

Remark 4.1. The assumption of surjection (4.4) is quite restrictive. For simple G it only happens [24, after (4.1)] for the symmetric pairs

$$(G, G_0^\tau) \cong (\text{SL}_{2n+1}, \text{SO}_{2n+1}), (\text{SL}_{2n}, \text{Sp}_n), (\text{SO}_{2n}, \text{SO}_{2n-1}), \text{ and } (E_6, F_4).$$

All these examples will appear in Sec. 4.3.

4.2. Explicit Hitchin map on very stable upward flows

First, we recall from [13] how to describe $h_k := h_{\mathcal{E}_k}$ for the very stable upward flows $W_k^+ := W_{\mathcal{E}_k}^+$ explicitly, where $\mathcal{E}_k := \mathcal{E}_{\mu_c^k}$, $c \in C$ fixed and $\mu_c^k : C \rightarrow \Lambda^+$ is defined by $\mu_c^k(c) = \omega_k$ and 0 otherwise.

Let us define the evaluation map at $c \in C$

$$\text{ev}_c : \mathcal{A} \rightarrow \mathbb{C}^n$$

sending the characteristic polynomial $(a_1, \dots, a_n) \in \mathcal{A}$ to

$$(a_1(c), \dots, a_n(c)) \in K_c^1 \times \dots \times K_c^n \cong \mathbb{C}^n$$

after identifying the fiber $K \cong \mathbb{C}$. We can further identify

$$\mathbb{C}^n \cong \mathfrak{gl}_n // \text{GL}_n \cong \mathfrak{t}_n // S_n \cong \text{Spec}(H^*(\text{BGL}_n, \mathbb{C})),$$

where $\mathfrak{gl}_n = \text{Lie}(\text{GL}_n)$ and $\mathfrak{t}_n = \text{Lie}(\mathbb{T}_n)$ the Lie algebra of the maximal torus $\mathbb{T}_n \subset \text{GL}_n$, and the symmetric group S_n is the Weyl group of GL_n . Then with the notation $H_{\text{GL}_n}^* := H^*(B\text{GL}_n; \mathbb{C})$ for the cohomology ring of the classifying space $B\text{GL}_n$ we have the following pullback diagram from [13]:

$$\begin{CD} W_k^+ @>>> \text{Spec}(H_{\text{GL}_n}^{2*}(\text{Gr}_k(\mathbb{C}^n), \mathbb{C})), \\ @V h_k VV @VV \downarrow V \\ \mathcal{A} @> \text{ev}_c >> \text{Spec}(H_{\text{GL}_n}^{2*}), \end{CD}$$

where $\text{Gr}_k(\mathbb{C}^n)$ is the Grassmannian of k -dimensional subspaces in \mathbb{C}^n with the usual action of GL_n . Additionally, all maps are \mathbb{T} -equivariant with respect to the usual \mathbb{T} -action on $W_k^+ \subset \mathcal{M}$ and \mathcal{A} , and the one induced by the grading on $H_{\text{GL}_n}^{2*}(\text{Gr}_k(\mathbb{C}^n); \mathbb{C})$ and $H_{\text{GL}_n}^{2*}$. In other words, the Hitchin map h_k can be modeled by the equivariant cohomology of the Grassmannian $\text{Gr}_k(\mathbb{C}^n)$.

4.3. Explicit Hitchin map on even very stable upward flows

We can then ask what models the even Hitchin map $h_k^\theta : W_k^{2+} \rightarrow \mathcal{A}^\theta$. We can note that θ acts on \mathcal{M} as $-1 \in \mathbb{T}$ in the natural \mathbb{T} -action, thus we can induce an action of θ on $H_{\text{GL}_n}^{2*}(\text{Gr}_k(\mathbb{C}^n); \mathbb{C})$ and $H_{\text{GL}_n}^{2*}$ as $-1 \in \mathbb{T}$. Therefore, we have

$$\begin{CD} W_k^{2+} @>>> \text{Spec}(H_{\text{GL}_n}^{2*}(\text{Gr}_k(\mathbb{C}^n), \mathbb{C}))^\theta \\ @V h_k^\theta VV @VV \downarrow V \\ \mathcal{A}^\theta @> \text{ev}_c >> \text{Spec}(H_{\text{GL}_n}^{2*})^\theta \end{CD}, \tag{4.5}$$

and, as a result, the even Hitchin map h_k^θ can be modeled on the θ -fixed point scheme

$$\text{Spec}(H_{\text{GL}_n}^{2*}(\text{Gr}_k(\mathbb{C}^n), \mathbb{C}))^\theta.$$

For simplicity, we will start with GL_{2n} and an even number $0 < 2k < 2n$. To understand the fixed point scheme $\text{Spec}(H_{\text{GL}_n}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n}), \mathbb{C}))^\theta$ we recall a presentation of the equivariant cohomology of the Grassmannian. In practice it can be done by following through the restriction of invariant polynomials in (4.1).

Let $e_1, \dots, e_{2k}, f_1, \dots, f_{2n-2k}, c_1, \dots, c_{2n}$ be variables of degree given by their index. Then we have the following presentation of the graded ring $H_{\text{GL}_n}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n}), \mathbb{C}) \cong$

$$\frac{\mathbb{C}[e_1, \dots, e_{2k}, f_1, \dots, f_{2n-2k}, c_1, \dots, c_{2n}]}{((t^{2k} + \sum e_i t^{2k-i})(t^{2n-2k} + \sum f_i t^{2n-2k-i}) - (t^{2n} + \sum c_i t^{2n-i}))},$$

where the ideal is generated by the coefficients of the given polynomial in t . It is naturally an algebra over $H_{\text{GL}_{2n}}^* \cong \mathbb{C}[c_1, \dots, c_{2n}]$. The action of θ is easy to figure out in this presentation, namely, all elements of degree i will get multiplied by $(-1)^i$. To compute the fixed point scheme (4.3) we will have to determine the coinvariant algebra of this θ -action. To form the coinvariant algebra we add to the ideal the algebra elements which are acted upon by θ as -1 , that is we add the odd

degree generators to the ideal. This way we get the following presentation of the coinvariant algebra: $H_{\text{GL}_{2n}}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n}), \mathbb{C})_\theta \cong$

$$\frac{\mathbb{C}[e_2, \dots, e_{2k}, f_2, \dots, f_{2n-2k}, c_2, \dots, c_{2n}]}{((t^{2k} + \sum e_{2i}t^{2k-2i})(t^{2n-2k} + \sum f_{2i}t^{2n-2k-2i}) - (t^{2n} + \sum c_{2i}t^{2n-2i}))},$$

where all generators have even degree, and the ideal is generated by the coefficients of the indicated polynomial in t^2 . This is an algebra over $\mathbb{C}[c_2, \dots, c_{2n}]$ which we can and will identify with $H_{\text{Sp}(n)}^* := H^*(B\text{Sp}(n), \mathbb{C})$ the cohomology ring of the classifying space of the compact unitary symplectic group. In turn, we can identify $H_{\text{GL}_{2n}}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n}), \mathbb{C})_\theta$ with the equivariant cohomology $H_{\text{Sp}(n)}^*(\text{Gr}_k(\mathbb{H}^n), \mathbb{C})$ of the quaternionic Grassmannian $\text{Gr}_k(\mathbb{H}^n)$ of k -dimensional \mathbb{H} -subspaces of \mathbb{H}^n , which is a non-Hermitian compact homogeneous space isomorphic to $\text{Sp}(n)/\text{Sp}(k) \times \text{Sp}(n-k)$.

We can summarize our observation in the following diagram:

$$\begin{array}{ccc} \text{Spec}(H_{\text{GL}_{2n}}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n}), \mathbb{C})^\theta) & \cong & \text{Spec}(H_{\text{Sp}(n)}^{2*}(\text{Gr}_k(\mathbb{H}^n), \mathbb{C})), \\ \downarrow & & \downarrow \\ \text{Spec}(H_{\text{GL}_{2n}}^{2*})^\theta & \cong & \text{Spec}(H_{\text{Sp}(n)}^{2*}). \end{array} \tag{4.6}$$

Thus, in light of (4.5), the Hitchin map h_{2k}^θ on the even upward flow W_{2k}^{2+} can be modeled by the spectrum of the equivariant cohomology of the quaternionic Grassmannian $\text{Gr}_k(\mathbb{H}^n)$.

In fact, we can find similar coincidences of coinvariant algebras of the equivariant cohomology of cominuscule flag varieties in some other types. Cominuscule flag varieties correspond to maximal parabolic subgroups associated to minuscule coweights, or equivalently to simple roots, which occur with coefficient 1 in the highest root (see also [14, §8] for more context). For example, we can consider the action of θ on

$$H_{\text{SO}(4n+2)}^{2*}(\text{SO}(4n+2)/\text{SO}(2) \times \text{SO}(4n), \mathbb{C}) \tag{4.7}$$

given by $(-1)^{\text{deg}}$, where $\text{deg} = *$ is the degree of the grading on (4.7). The Hermitian symmetric space $\text{SO}(4n+2)/\text{SO}(2) \times \text{SO}(4n)$ is an even quadric, a cominuscule flag variety for the special orthogonal group SO_{4n+2} . The corresponding coinvariant algebra

$$H_{\text{SO}(4n+2)}^{2*}(\text{SO}(4n+2)/\text{SO}(2) \times \text{SO}(4n), \mathbb{C})_\theta \cong H_{\text{SO}(4n+1)}^{2*}(\text{SO}(4n+1)/\text{SO}(4n), \mathbb{C}),$$

can be identified with the $\text{SO}(4n+1)$ -equivariant cohomology ring of the sphere

$$S^{4n} \cong \text{SO}(4n+1)/\text{SO}(4n).$$

In fact we have the following diagram:

$$\begin{array}{ccc} H_{\text{SO}(4n+2)}^{2*}(\text{SO}(4n+2)/\text{SO}(2) \times \text{SO}(4n), \mathbb{C})_\theta & \cong & H_{\text{SO}(4n+1)}^{2*}(\text{SO}(4n+1)/\text{SO}(4n), \mathbb{C}), \\ \uparrow & & \uparrow \\ \left(H_{\text{SO}(4n+2)}^{2*} \right)_\theta & \cong & H_{\text{SO}(4n+1)}^{2*}. \end{array} \tag{4.8}$$

In other words, we can expect that the Hitchin map on even cominuscule upward flows in the SO_{4n+2} Higgs moduli space — equivalently the Hitchin map on cominuscule upward flows in the $SO(2n + 2, 2n)$ -Higgs moduli space — to be modeled by the spectrum of the equivariant cohomology of the sphere S^{4n} .

For our final example, we can consider the unique cominuscule flag variety for the exceptional E_6 . It is the complex Cayley plane $E_6/\text{Spin}(10) \times U(1)$ which is a compact Hermitian symmetric space. We can identify the coinvariant algebra of the $\theta = (-1)^{\text{deg}}$ action on its equivariant cohomology ring:

$$\begin{array}{ccc}
 H_{E_6}^{2*}(E_6/\text{Spin}(10) \times U(1), \mathbb{C})_{\theta} & \cong & H_{F_4}^{2*}(F_4/\text{Spin}(9), \mathbb{C}), \\
 \uparrow & & \uparrow \\
 (H_{E_6}^{2*})_{\theta} & \cong & H_{F_4}^{2*}.
 \end{array} \tag{4.9}$$

Thus, we expect to model the Hitchin map on even cominuscule upward flows in the E_6 Higgs moduli space — equivalently on the cominuscule upward flows in the $E_{6(2)}$ -Higgs moduli space — by the spectrum of equivariant cohomology of the real Cayley plane $F_4/\text{Spin}(9)$.

Mysteriously, in the above examples the symmetric spaces whose equivariant cohomology we found to give the θ -coinvariant algebra of the equivariant cohomology of the cominuscule flag variety are homogeneous spaces for the Nadler group [22, Table 1] of the corresponding quasi-split real form of Hodge type. That is

$$U(n, n)^{\vee} \cong \text{Sp}_n, \text{SO}(2n + 2, 2n)^{\vee} \cong \text{SO}_{4n+1}, \text{ and } E_{6(2)}^{\vee} \cong F_4.$$

Conjecturally, [2, §7] the Higgs bundle moduli space for the Nadler group should give the support of the mirror of the Lagrangian brane given by the Higgs moduli space attached to a real form. However these appearances of the Nadler group remain to be understood.

Even more surprising is that the Nadler groups in the above examples happen to be the fixed point subgroups of an involution of the ambient Langlands dual group. The corresponding anti-holomorphic involution can then be used to construct [5] an anti-holomorphic involution in the cominuscule flag varieties in the above examples, and show that the Lagrangian fixed point manifold is isomorphic to the corresponding non-Hermitian compact symmetric spaces we have found above. Thus by uniformly denoting this anti-holomorphic involution by τ we get that

$$\begin{aligned}
 \text{Gr}_{2k}(\mathbb{C}^{2n})^{\tau} &\cong \text{Gr}_k(\mathbb{H}^n), \\
 (\text{SO}(4n + 2)/\text{SO}(2) \times \text{SO}(4n))^{\tau} &\cong \text{SO}(4n + 1)/\text{SO}(4n)
 \end{aligned}$$

and

$$(E_6/\text{Spin}(10) \times U(1))^{\tau} \cong F_4/\text{Spin}(9).$$

Using Lemma 4.1, it can be shown — by observing that precisely the even degree generators of invariant polynomials survive for the τ -fixed groups — that in each

of these cases τ induces our

$$\tau^* = \theta = (-1)^{\text{deg}} \tag{4.10}$$

on equivariant cohomology, where again deg is half the degree of a homogeneous cohomology class. In turn, this observation and again Lemma 4.1 can be used to geometrically prove our final.

Theorem 4.1. *The diagrams in (4.6), (4.8), (4.9) commute and are induced by the involution τ .*

Remark 4.2. Because of (4.10) we can deduce, by the Lefschetz fixed point theorem, that the signatures of our Hermitian symmetric spaces $X = G/H$ agree

$$\text{sign}(X) = \text{tr}(\theta : H^*(X) \rightarrow H^*(X)) = \chi(X^\tau) \tag{4.11}$$

with the Euler characteristic of the corresponding non-Hermitian symmetric spaces X^τ . The quantity $\text{sign}(X)$ is relevant in our considerations as it agrees with the rank of the coinvariant algebra $H_G^*(X)_\theta$, which in turn should compute the multiplicity of the even Hitchin map on the corresponding even cominuscule upward flow.

It is interesting to note that a very similar approach to (4.11) was studied in [16, Remark (1) p. 337] to determine the signature of our Hermitian symmetric spaces as the Euler characteristic of X^σ , using the involution $\sigma : X \rightarrow X$ induced by the split real form — instead of our real form given by the Nadler group. The fixed point sets are different in the type A -case — certain real Grassmannians — from our quaternionic Grassmannians but the induced actions on the cohomology $\sigma^* = \tau^* = \theta$ agree, because the real forms σ and τ are inner to each other.

Remark 4.3. The involution σ however suggests a solution for modeling the even Hitchin system inside the equivariant cohomology of the last family of cominuscule flag varieties with non-zero signature. Namely, we can consider the anti-holomorphic action of σ on $\text{Gr}_{2k}(\mathbb{C}^{2n+1})$ with fixed point set $\text{Gr}_{2k}(\mathbb{C}^{2n+1})^\sigma \cong \text{Gr}_{2k}(\mathbb{R}^{2n+1})$. Then it appears that we have σ inducing

$$\begin{array}{ccc} H_{\text{SL}_{2n+1}}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n+1}), \mathbb{C})_\theta & \cong & H_{\text{SO}_{2n+1}}^{2*}(\text{Gr}_{2k}(\mathbb{R}^{2n+1}), \mathbb{C}), \\ \uparrow & & \uparrow \\ \left(H_{\text{SL}_{2n+1}}^{2*} \right)_\theta & \cong & H_{\text{SO}_{2n+1}}^{2*}. \end{array} \tag{4.12}$$

The subtlety of this case is that $\text{Gr}_{2k}(\mathbb{R}^{2n+1})$ is no longer simply connected, thus the usual computation of its equivariant cohomology (4.1) does not apply. One can proceed by first determining the equivariant cohomology ring of its universal double cover — the oriented Grassmannian — from (4.1), and then take invariants of the cover map. In fact, recently the equivariant cohomology of $\text{Gr}_{2k}(\mathbb{R}^{2n+1})$ was computed in [15, Theorem 5.23], and the result matches the coinvariant algebra $H_{\text{SL}_{2n+1}}^{2*}(\text{Gr}_{2k}(\mathbb{C}^{2n+1}), \mathbb{C})_\theta$. We note that in this example the group SO_{2n+1} is the

Langlands dual of the Nadler group Sp_n of our quasi-split real form $\mathrm{SU}(n, n + 1)$ of Hodge type. In particular, their classifying spaces have isomorphic cohomology

$$H_{\mathrm{SO}_{2n+1}}^* \cong \mathbb{C}[\mathfrak{so}_{2n+1}]^{\mathrm{W}_{\mathrm{SO}_{2n+1}}} \cong \mathbb{C}[\mathfrak{t}_{\mathrm{Sp}_n}]^{\mathrm{W}_{\mathrm{Sp}_n}} \cong H_{\mathrm{Sp}_n}^*, \tag{4.13}$$

because $\mathrm{W}_{\mathrm{SO}_{2n+1}} \cong \mathrm{W}_{\mathrm{Sp}_n}$, $\mathfrak{so}_{2n+1} \cong \mathfrak{t}_{\mathrm{Sp}_n}^*$ and the two representations in (4.13) can be identified by the Killing form.

Remark 4.4. The final example of symmetric pairs from Remark 4.1 we have not discussed yet is $(\mathrm{SO}_{4n}, \mathrm{SO}_{4n-1})$. In this case, τ will induce on the cohomology of our cominuscule flag variety an involution

$$\theta_\tau := \tau^* : H_{\mathrm{SO}_{4n}}^*(\mathrm{SO}_{4n}/\mathrm{SO}_2 \times \mathrm{SO}_{4n-2}; \mathbb{C}) \rightarrow H_{\mathrm{SO}_{4n}}^*(\mathrm{SO}_{4n}/\mathrm{SO}_2 \times \mathrm{SO}_{4n-2}; \mathbb{C}),$$

which is different from the usual Hodge type $(-1)^{\mathrm{deg}}$ unlike in the previous cases (4.10). In fact we expect that this θ_τ will be the involution corresponding to the (only) quasi-split real form $\mathrm{SO}(2n + 1, 2n - 1)$ which is not split or of Hodge type. Thus, we expect that the spectrum of the diagram we get from Lemma 4.1

$$\begin{array}{ccc} H_{\mathrm{SO}_{4n}}^{2*}(\mathrm{SO}_{4n}/\mathrm{SO}_2 \times \mathrm{SO}_{4n-2}; \mathbb{C})_{\theta_\tau} & \cong & H_{\mathrm{SO}_{4n-1}}^{2*}(\mathrm{SO}_{4n-1}/\mathrm{SO}_{4n-2}; \mathbb{C}), \\ \uparrow & & \uparrow \\ (H_{\mathrm{SO}_{4n}}^{2*})_{\theta_\tau} & \cong & H_{\mathrm{SO}_{4n-1}}^{2*} \end{array} \tag{4.14}$$

models the Hitchin map on a cominuscule upward flow in the $\mathrm{SO}(2n + 1, 2n - 1)$ -Higgs moduli space.

Remark 4.5. Oscar García-Prada has pointed out to us that the symmetric pairs in Remark 4.1 precisely correspond to the complexifications of the maximal split subgroups of the non-split quasi-split real forms in [8, Table 1. p. 2914]. They are used [8, Theorem 6.13(1)] to construct the Hitchin–Kostant–Rallis section in the quasi-split real form cases. This demystifies their appearance in our descriptions of the Hitchin maps on cominuscule upward flows in the quasi-split Higgs moduli spaces in the diagrams (4.6), (4.8), (4.9), (4.12) and (4.14).

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