Higgs bundle tournaments

based on joint project with Mirko Mauri

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AN AMERICAN TOURNAMENT TREATED BY THE CALCULUS OF SYMMETRIC FUNCTIONS.

By Major P. A. MacMahon.

Part I.

1. In a tournament of n players, where each player plays every other player, there are $\frac{1}{2}n(n-1)$ games. Since each game may be won or lost there are $2^{\frac{1}{2}n(n-1)}$ events and I propose to analyse them by means of the powerful calculus of symmetric functions. The final result of the play is that the players are arranged in a definite order, each with a certain number of games to his credit. These numbers constitute a partition of the number $\frac{1}{2}n(n-1)$, and we may ask how many of the $2^{\frac{1}{2}n(n-1)}$ events will yield a given partition of $\frac{1}{2}n(n-1)$ when the players are or are not in an assigned order.

2. Consider the symmetric function

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_{n-1}+\alpha_n)$$

of the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

It involves $\frac{1}{2}n(n-1)$ factors, and the terms, after carrying out the multiplication, are grouped together in monomial symmetric functions.

Score sequences of tournaments

- *tournament*:= orientation of complete graph on $[n] := \{1, .., n\}$
- *score* of a vertex:= its outdegree
- *score vector*:= vector of scores
- *score sequence*:= non-decreasing sequence of the scores
- *transitive* tournament:= $a \to b$ & $b \to c \Rightarrow a \to c$
  $\Leftrightarrow$ its score sequence is $(0, 1, \ldots, n-2, n-1)$
- *strong* tournament:= $[n] = A \coprod B$, $A \to B \Rightarrow A = \emptyset$ or $B = \emptyset$
  $\Leftrightarrow$ there exists a Hamiltonian circuit
- for any tournament there is unique $[n] = \coprod A_i$ such that
  $A_1 \to A_2 \to \cdots \to A_k$ & restricted tournament on A_i is strong,
  e.g. $A_1 = V$ for strong, $|A_i| = 1$ for transitive tournaments

Theorem (Landau 1953)

$(s_1 \leq \cdots \leq s_n)$ is the score sequence of a strong (tournament) $\Leftrightarrow$ $\sum_{i=1}^{n} s_i = \binom{n}{2}$ and $\sum_{i=1}^{k} s_i \underset{(=)}{>} \binom{k}{2}$ for $k < n$

Semi-stable type (1, .., 1)-Higgs bundles

- C complex smooth projective curve $D \in S^d(C)$ eff. divisor
  (E, Φ) *Higgs bundle* , rank n vector bundle E and
  $\Phi \in H^0(\mathrm{End}(E) \otimes K(D))$ *Higgs field*
  (semi)-stable: $\forall$ proper Φ-invariant $F \subset E \Rightarrow \frac{\deg F}{\mathrm{rank} F} \underset{(-)}{<} \frac{\deg E}{\mathrm{rank} E}$
- $E = L_1 \oplus \ldots \oplus L_n$ and $0 \neq \Phi(L_i) \subset L_{i+1}K(D) \Rightarrow$
  $(E, \Phi) \cong (E, \lambda\Phi)$ *type* $(1, .., 1)$*-Higgs bundle*
  (semi)-stable $\Leftrightarrow$ for $\ell_i := \deg(L_i)$ and $1 < k < n$
  $\frac{\ell_k + \cdots + \ell_n}{n-k+1} \underset{(-)}{<} \frac{\ell_1 + \cdots + \ell_n}{n}$
- when $C = \mathbb{P}^1$ and $|D| = 3$ then $\deg(K(D)) = 1$ and
  $\deg(E) = \ell_1 + \cdots + \ell_n = 0$ choosing $s_i = i - 1 - l_i \rightsquigarrow$

Theorem

$\{$*score sequences* $(s_1, \ldots, s_n)$ *of strong (tournaments) on* $[n]\} \leftrightarrow$
$\{$*degree sequences* $(\ell_1, \ldots, \ell_n)$ *of degree* 0 *rank* n
(semi)-stable type $(1, .., 1)$*-Higgs bundles*$\}$

- e.g. transitive tourn. $(0, 1, \ldots, n-1) \leftrightarrow (0, ..., 0)$ trivial bundle
- $(\deg(E), n) = 1 \rightsquigarrow$ similar combinatorics by
  (Villegas 2011, 2023, Reineke 2012, Rayan 2018)

Generating score sequences as Weyl character formula

- MacMahon's gf for
  $n(s) := \#\{\text{tournaments of score sequence } s\}$
  $\prod_{1 \le i < j \le n}(x_i + x_j) = \sum_{s=(s_1 \le \cdots \le s_n)} n(s) \sum_{\substack{s'=(s'_1,\ldots,s'_n) \\ \{s'\}=\{s\}}} x^{s'}$
- $\Lambda := \mathbb{Z}^n / \langle e_1 + \cdots + e_n \rangle$ weight lattice $\circlearrowleft W := S_n$ Weyl group
  $R := \{\pm(e_i - e_j)\}_{1 \le i < j \le n} = R_+ \coprod R_- \subset \Lambda$ type A_{n-1} root system
  $\omega_j = \sum_{i=1}^{j} e_i \in \Lambda$ fundamental weight
  $\Lambda^+ := \oplus_{i=1}^{n-1} \mathbb{N}\omega_i \subset \Lambda$ dominant weights $\cong \Lambda / W$
  $\rho := (\sum_{\alpha \in R_+} \alpha)/2 = \sum_{i=1}^{n-1} \omega_i \in \Lambda^+$
  half-sum of positive roots
- Weyl character formula for $\mathfrak{sl}_n \to \mathrm{End}(V^\rho)$ of highest weight ρ
  $\sum_{\lambda \in \Lambda} \dim(V^\rho_\lambda) x^\lambda = \frac{\sum_{w \in W} \det(w) x^{w(\rho+\rho)}}{\sum_{w \in W} \det(w) x^{w(\rho)}} = \frac{\prod_{\alpha \in R_+} (x^\alpha - x^{-\alpha})}{\prod_{\alpha \in R_+} (x^{\alpha/2} - x^{-\alpha/2})} =$
  $\prod_{\alpha \in R_+} (x^{\alpha/2} + x^{-\alpha/2}) = \prod_{1 \le i < j \le n}(x_i + x_j)$
- $\rightsquigarrow$ { weights in V^ρ} $\leftrightarrow$ {score vectors}
  { dominant weights in V^ρ} $\leftrightarrow$ {score sequences}
  {monomial basis in V^ρ} $\leftrightarrow \{\Delta \subset R : |\Delta \cap \{\alpha, -\alpha\}| = 1 \forall \alpha \in R\} \leftrightarrow$
  $\updownarrow$
  {tournaments on $[n]$}

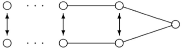
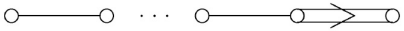
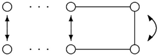
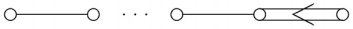
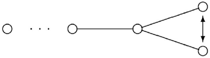
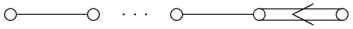
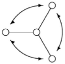
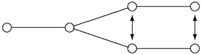
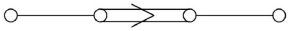
Bottom Lagrangian

- $\mathbb{M} := \mathbb{M}^{ss,0}_{\mathrm{PGL}_n} \ni (E,\Phi); \Phi \in H^0(C; \mathrm{End}_0(E) \otimes K)$, $\deg(E) = 0$
- $$\begin{array}{cccc} h: & \mathbb{M} & \to & \mathbb{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E,\Phi) & \mapsto & \det(x - \Phi) \end{array}$$ Hitchin map
- $\mathcal{E}_{triv} := (\mathcal{O}^n, 0)$ trivial Higgs bundle, *bottom Lagrangian*:
  $B := \{(\mathcal{O}^n, \Phi) : \Phi \in H^0(C; \mathrm{End}_0(\mathcal{O}^n) \otimes K_C)\} \subset \mathbb{M}$
  singular Lagrangian subvariety
  $B \cong \mathrm{End}_0(\mathbb{C}^n) \otimes H^0(K_C) // \mathrm{PGL}_n$
- $g(C) = 2$, $h^0(K_0) = 2 \rightsquigarrow \pi : C \to \mathbb{P}^1$ hyperelliptic $\rightsquigarrow$
  $\pi^*(\mathcal{O}(1)) = K_C \rightsquigarrow \overline{B} \cong \mathbb{M}^0_{\mathbb{P}^1, \mathcal{O}(1)}$ moduli of semi-stable (E,Φ)
  $\deg(E) = 0$, $\mathrm{rank}(E) = n$, $\Phi \in H^0(\mathbb{P}^1; \mathrm{End}_0(E)(1))$
- components of $\overline{B} \cap h^{-1}(0) \cong$ nilpotent cone in $\mathbb{M}^0_{\mathbb{P}^1, \mathcal{O}(1)} \leftrightarrow$
  components of stable type $(1, ..., 1)$ $\mathcal{O}(1)$-Higgs bundles on $\mathbb{P}^1$
  $\leftrightarrow$ score sequences of strong tournaments on $[n]$
  $\leftrightarrow$ dominant weights in V^ρ
- Expectations:
  1. geometry of $h|_{\overline{B}}$ should reflect $\mathrm{SL}_n \to \mathrm{GL}(V^\rho)$
  2. geometry of $h_{\mathrm{G}^\vee}|_{\overline{B}_{\mathrm{G}^\vee}}$ should reflect $\mathrm{G} \to \mathrm{GL}(V^\rho)$
  3. should be compatible with folding $\sigma : \mathrm{G} \to \mathrm{G}$

Root system tournaments

- $R \subset \Lambda \cong \mathbb{Z}^r$ finite root system in weight lattice, Weyl group W, $R_+ \subset R$ set of positive roots
  1. $|R_+ \cap \{\alpha, -\alpha\}| = 1 \forall \alpha \in R$
  2. $\alpha, \beta \in R_+$ and $\alpha + \beta \in R \Rightarrow \alpha + \beta \in R_+$
- *R-tournament*: $T \subset R$ s.t. $|T \cap \{\alpha, -\alpha\}| = 1 \forall \alpha \in R$
- example: positive roots $R_+ \subset R$ *transitive* R-tournament
- notion used by (Calderbank–Hanlon, 1986) to give a combinatorial proof for the Weyl denominator identity $\prod_{\alpha \in R_+}(x^{\alpha/2} - x^{-\alpha/2}) = \sum_{w \in W} \det(w) x^{w(\rho)}$ in types B, C, D after (Gessel, 1979) in type A
- Weyl character formula for $\mathfrak{g}_R \to \mathrm{End}(V^\rho)$, $\rho = (\sum_{\alpha \in R_+})/2$ $\sum_{\lambda \in \Lambda} \dim(V^\rho_\lambda) x^\lambda = \frac{\sum_{w \in W} \det(w) x^{w(\rho+\rho)}}{\sum_{w \in W} \det(w) x^{w(\rho)}} = \frac{\prod_{\alpha \in R_+}(x^\alpha - x^{-\alpha})}{\prod_{\alpha \in R_+}(x^{\alpha/2} - x^{-\alpha/2})} = \prod_{\alpha \in R_+}(x^{\alpha/2} + x^{-\alpha/2})$
- $\{R$-tournaments$\} \leftrightarrow \{$monomial basis in $V^\rho\}$
- *R-score vector*:= weight in V^ρ
  R-score sequence:= dominant weight in V^ρ

- $R \subset \Lambda \circlearrowleft W$ finite root system, weight lattice, Weyl group
- Dynkin diagram automorphism $\rightsquigarrow \sigma : \Lambda \to \Lambda$ s.t. $\sigma(R) = R$
- folding procedure $\rightsquigarrow$ root system $R^\sigma := R/\sigma \subset \Lambda^\sigma$ (except A_{2n})
- defining property: $\mathfrak{g}^\vee_{R^\sigma} \cong (\mathfrak{g}^\vee_R)^\sigma \subset \mathfrak{g}^\vee_R$
-

R	σ		R^σ	$\mathfrak{g}^\vee_{R^\sigma} \subset \mathfrak{g}^\vee_R$
A_{2n-1}		B_n		$\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$
A_{2n}		C_n		$\mathfrak{so}_{2n+1} \subset \mathfrak{sl}_{2n+1}$
D_n		C_{n-1}		$\mathfrak{so}_{2n-1} \subset \mathfrak{so}_{2n}$
D_4		G_2		$\mathfrak{g}_2 \subset \mathfrak{so}_8$
E_6		F_4		$\mathfrak{f}_4 \subset \mathfrak{e}_6$

Tournament folding $A_{2n-1} \rightsquigarrow B_n$

- $\sigma : [2n] := \{1, \ldots, 2n\} \to [2n]$ by $\sigma(i) = 2n - i$
- orbits $[2n]/\sigma$:= *married couples*
- tournament on $[2n]$ is *marriage balanced*:
  $a \to b \Rightarrow \sigma(b) \to \sigma(a)$
- score vector of m.b. tournament is $(s_1, \ldots, s_{2n})$
  satisfies $s_i + s_{\sigma_i} = 2n - 1$
- score sequence of m.b. tournament: $(s'_1 \leq \cdots \leq s'_n \leq n - 1)$

Theorem

$\{R^\sigma = B_n - \textit{tournaments}\} \leftrightarrow$
$\{\textit{marriage balanced tournaments on } [2n]\}$ *with generating function:*
$\prod_{i=1}^{n}(y_i + x_i) \prod_{1 \leq i < j \leq n}(y_i y_j + x_i x_j)(x_i y_j + x_j y_i)|_{y_i = 1} = \sum_{\lambda \in \Lambda} \dim V^\rho_\lambda x^\lambda$
$\{\textit{score vectors of m.b. tournaments}\} \leftrightarrow \{\textit{weights in } V^\rho\}$
$\{\textit{score sequences of m.b. tournaments}\} \leftrightarrow \{\textit{dominant weights in } V^\rho\}$

- $[2n] \circlearrowleft W = S_n \ltimes (\mathbb{Z}/2)^n \subset S_{2n}$ preserving couples
- games between spouses correspond to short roots of B_n

Tournament folding $D_n \rightsquigarrow C_{n-1}$

- $\sigma : [2n] := \{1, \ldots, 2n\} \to [2n]$ by $\sigma(i) = 2n - i$
- *marriage balanced couple*(=:m.b.c.) tournament on $[2n]$ everyone plays everyone except their spouses and $a \to b \Rightarrow \sigma(b) \to \sigma(a)$
- $[2n] \circlearrowleft W = S_n \ltimes (\mathbb{Z}/2)^{n-1} \subset S_{2n}$ preserving couples flipping even
- g.f. for score vectors of m.b.c. tournaments
  $\prod_{1 \le i < j \le n} (y_i y_j + x_i x_j)(x_i y_j + x_j y_i)|_{y_i=1} \overset{D_n\text{-Weyl}}{=} \sum_{\lambda \in \Lambda} \dim V_\lambda^\rho x^\lambda$
- D_n Dynkin auto: $\tau := (n, n+1) : [2n] \to [2n]$, $[2n]/\tau = [2n-1]$
- $\sigma : [2n-1] \to [2n-1]$ by $\sigma(i) = 2n - 1 - i$
  $[2n-1]/\sigma$ has $n - 1$ couples and one singleton $\{n\}$
- *marriage balanced single*(=:m.b.s.) tournament on $[2n-1]$ everyone plays everyone except their spouses, single n plays twice with everyone with same result, $a \to b \Rightarrow \sigma(b) \to \sigma(a)$
- g.f. for score vectors of m.b.s. tournaments
  $\prod_{i=1}^{i-1} (y_i^2 + x_i^2) \prod_{1 \le i < j < n} (y_i y_j + x_i x_j)(x_i y_j + x_j y_i)|_{y_i=1} \overset{C_{n-1}\text{-Weyl}}{=} \sum_{\lambda \in \Lambda} \dim V_\lambda^\rho x^\lambda$
- compatible with tournament folding $A_{2n-2} \rightsquigarrow C_{n-1}$

Skeletons of big algebras

- $\mu \in \Lambda^+(\mathrm{G}) \rightsquigarrow \mathcal{B}^\mu := \mathcal{B}^\mu(\mathfrak{g}) \subset (S(\mathfrak{g}) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}}$ *big algebra* commutative, graded, cyclic, $H^{2*}_{\mathrm{G}} = \mathbb{C}[\mathfrak{g}]^{\mathrm{G}}$-algebra $\sigma : \mathrm{G} \to \mathrm{G}$ Dynkin automorphism, $\mu \in \Lambda^+(\mathrm{G}_\sigma) \rightsquigarrow$ folding: $\mathcal{B}^\mu(\mathfrak{g})_\sigma \cong \mathcal{B}^\mu(\mathfrak{g}_\sigma)$ (Hausel, Zveryk 2023)
- base change for principal $\mathrm{SL}_2 \to \mathrm{G}$ $\mathcal{B}^\mu_{\mathrm{SL}_2} := \mathcal{B}^\mu \otimes_{H^{2*}_{\mathrm{SL}_n}} H^{2*}_{\mathrm{SL}_2}$ *principal* big algebras
- $\mathrm{Spec}(\mathcal{B}^\mu_{\mathrm{SL}_2}) \to \mathrm{Spec}(H^{2*}_{\mathrm{SL}_2}) \cong \mathbb{A}^1$ *big skeletons*
- weight of $\mathbb{C}^\times \circlearrowright \mathbb{A}^1 = 2 \Rightarrow$ components of the skeletons either $\begin{matrix} \mathbb{A}^1 & \to & \mathbb{A}^1 \\ x & \mapsto & x^2 \end{matrix}$ *parabolas* or $\begin{matrix} \mathbb{A}^1 & \to & \mathbb{A}^1 \\ x & \mapsto & x \end{matrix}$ *spikes*
- $-1 \in \mathbb{C}^\times$ fixes base $\mathbb{A}^1$ & spikes, swaps two sides of parabolas $\Rightarrow \#$spikes = $\mathrm{tr}(-1|\mathrm{V}^\mu) = \mathrm{D}^\mu(-1)$ $D^\mu(q) = \prod_{\alpha \in \Delta^+} \frac{(1-q^{(\rho+\mu,\alpha)})}{(1-q^{(\rho,\alpha)})}$ Dynkin polynomial / quantum dim.
- *principal basis* : $B^\mu \subset \mathbb{P}(V^\mu)$ common eigenvectors of $\mathcal{B}^\mu$ $-1 \circlearrowright B^\mu$ determines the big skeleton

Problems for Higgs bundle tournaments

- $\mu \in \Lambda^+(\mathrm{SL}_n)$ (Berenstein–Zelevinsky 1996) labeled Lusztig's dual canonical basis in V^μ with semi-standard Young tableaux $SST(\lambda, \leq n)$ & identified Lusztig's involution η as evacuation
- (Stembridge, 1996) showed $D^\mu(-1)$ are the η-invariants "$q = -1$ phenomenon"
- (Sundquist, 1992) (Yutsis, 1980) sets up bijection $SST((1, \ldots, n), \mu) \leftrightarrow$ tournaments with score sequence μ
- Problem 0: Can we label principal basis of V^ρ with tournaments?
- Problem 1: What does evacuation correspond on tournaments?
- Problem 2: What is the charge of an SST correspond to a tournament? Tournament analogue of Kostka-Foulkes $K^\rho_\mu(q)$?
- Problem 3: Can Sundquist–Yutsis be generalised for other types B,C,D,G? Tournament analogue of Lusztig's $m^\rho_\mu(q)$?