# On the center of classical family algebras

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## Lie algebras

A **Lie algebra over** $\mathbb{C}$ is a $\mathbb{C}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, satisfying

- antisymmetricity: $[x, x] = 0$ for all $x \in \mathfrak{g}$;
- Jacobi identity: $[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$.

For example, if $A$ is an associative algebra over $\mathbb{C}$, then it can be checked that the bilinear map defined by $[a, b] = ab - ba$ for $a, b \in A$ makes $A$ a Lie algebra over $\mathbb{C}$. We will always endow algebras with this Lie bracket.
We define maps of Lie algebras $\pi : \mathfrak{g} \to \mathfrak{h}$ to be linear maps such that $\pi([X, Y]) = [\pi(X), \pi(Y)]$ for all $X, Y \in \mathfrak{g}$.
A **representation** of $\mathfrak{g}$ is a $\mathbb{C}$-vector space $V$ with a Lie algebra map $\pi : \mathfrak{g} \to \operatorname{End} V$. We can as well say that $\mathfrak{g}$ **acts on** $V$, or that $V$ is a $\mathfrak{g}$-module. We define the subspace $V^{\mathfrak{g}}$ of $\mathfrak{g}$**-invariants of** $V$ as the set of all elements in $V$ which are sent to 0 by the action of $\mathfrak{g}$.

## Semisimple finite dimensional Lie algebras

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$. The **Killing form** on $\mathfrak{g}$ is a bilinear map $\langle\cdot,\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$\langle X, Y\rangle := \operatorname{trace}([X, [Y, -]] : \mathfrak{g} \to \mathfrak{g}) \quad \text{for } X, Y \in \mathfrak{g}.$$

It appears that the Killing form is symmetric and possesses a $\mathfrak{g}$-invariance property, namely

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle \quad \text{for } X, Y, Z \in \mathfrak{g}.$$

We call $\mathfrak{g}$ **semisimple** if the Killing form on $\mathfrak{g}$ is non-degenerate. Then it gives an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, and the $\mathfrak{g}$-invariance property of $\langle\cdot,\cdot\rangle$ even assures that it is an isomorphism of $\mathfrak{g}$-modules, where for $X \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$ we define actions of $Y \in \mathfrak{g}$ as $Y * X = [Y, X]$ and $Y * f = f \circ [-, Y]$.
Semisimple Lie algebras possess a wide variety of remarkable properties. We will be interested in the structure of their representations: if $\mathfrak{g}$ is semisimple, then each finite dimensional representation $V$ of $\mathfrak{g}$ is a direct sum of **irreducible representations**, defined as not containing any nontrivial subrepresentations. Moreover, there is a certain class of subalgebras of $\mathfrak{g}_0$ of $\mathfrak{g}$, called **Cartan subalgebras**, playing an important role in classifying $\mathfrak{g}$-representations. One particular property of them is that any representation $V$ of $\mathfrak{g}$ is a direct sum of subspaces $V_\lambda = \{v \in V : \forall_{X\in\mathfrak{g}_0} X \cdot v = \alpha(X)v\}$, which we call **weight subspaces**.

## Classical family algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. In the papers [Kir1] and [Kir2], A. A. Kirillov introduced an algebra defined for any irreducible representation of $\mathfrak{g}$ as follows. Let $V$ be an irreducible representation of $\mathfrak{g}$ given by a Lie algebra map $\pi : \mathfrak{g} \to \operatorname{End} V$. Then $\pi$ defines an action on $\operatorname{End} V$ via $X \cdot H = [\pi(X), H]$ for $X \in \mathfrak{g}$, $H \in \operatorname{End} V$. We also endow the symmetric algebra $S(\mathfrak{g})$ over $\mathfrak{g}$ with the standard $\mathfrak{g}$-action satisfying Leibniz rule given by

$$\{X, Y_1 \dots Y_k\} = \sum_{i=1}^{k} [X, Y_i] Y_1 \dots Y_{i-1} Y_{i+1} \dots Y_k \quad \text{for } X \in \mathfrak{g}, Y_i \in \mathfrak{g} = S_1(\mathfrak{g}).$$

We combine these two actions to define an action on the Lie algebra $\operatorname{End} V \otimes S(\mathfrak{g})$ as

$$X \cdot (a \otimes P) = [\pi(X), a] \otimes P + a \otimes \{X, P\} \quad \text{for } X \in \mathfrak{g}, a \in \operatorname{End} V, P \in S(\mathfrak{g}).$$

**Definition.** Define $C_\pi(\mathfrak{g})$ to be the space $(\operatorname{End} V \otimes S(\mathfrak{g}))^{\mathfrak{g}}$. We call it the **classical family algebra of** $\mathfrak{g}$ **on** $V$.

It is known that $C_\pi(\mathfrak{g})$ is closed under multiplication inherited from $\operatorname{End} V \otimes S(\mathfrak{g})$, hence is an associative algebra over $\mathbb{C}$ itself. Then a natural question that we may ask is if this algebra is commutative, and if not, then what can we say about its center? It was shown in [Kir1] that $C_\pi(\mathfrak{g})$ is commutative if and only if all weight subspaces of $V$ have dimension 1. But in general, even the generators of the center of $C_\pi(\mathfrak{g})$ are not known.

## A more general setting

Our approach to studying classical family algebras is to consider the following generalized construction. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Let also $\mathfrak{h}$ be a Lie algebra with a Lie algebra map $\pi : \mathfrak{g} \to \mathfrak{h}$.

**Definition.** We define a $\mathfrak{g}$-action on the Lie algebra $\mathfrak{h} \otimes S(\mathfrak{g})$ similarly to the case $\mathfrak{h} = \operatorname{End} V$ considered before and $C_{\mathfrak{h}}(\mathfrak{g})$ to be the space $(\mathfrak{h} \otimes S(\mathfrak{g}))^{\mathfrak{g}}$. We call it the **classical family algebra of** $\mathfrak{g}$ **on** $\mathfrak{h}$.

Similarly to the case of $S(\mathfrak{g})$, the $\mathfrak{g}$-action on $\mathfrak{g}^*$ extends to a $\mathfrak{g}$-action on $S(\mathfrak{g}^*)$, which allows us to define $C^*_{\mathfrak{h}}(\mathfrak{g}) := (\mathfrak{h} \otimes S(\mathfrak{g}^*))^{\mathfrak{g}}$. It can be shown that both $C_{\mathfrak{h}}(\mathfrak{g})$ and $C^*_{\mathfrak{h}}(\mathfrak{g})$ have a natural Lie algebra structure, and even an algebra structure if $\mathfrak{h}$ is an algebra itself.

## The $D$-operator

We can define an action of $\mathfrak{g}^*$ on $S(\mathfrak{g})$ by differentiation satisfying Leibniz rule:

$$\frac{\partial X_1 \dots X_k}{\partial f} = \sum_{i=1}^{k} f(X_i) X_1 X_2 \dots X_{i-1} X_{i+1} \dots X_k,$$

which can be extended to the action of $S(\mathfrak{g}^*)$ on $S(\mathfrak{g})$. Then, assuming that $\mathfrak{h}$ is an algebra, we can define an action of $\mathfrak{h} \otimes S(\mathfrak{g}^*)$ on $\mathfrak{h} \otimes S(\mathfrak{g})$ via

$$(a \otimes P) * (b \otimes Q) = ab \otimes (\partial Q/\partial P).$$

It can be shown that it restricts to an action of $C^*_{\mathfrak{h}}(\mathfrak{g})$ on $C_{\mathfrak{h}}(\mathfrak{g})$. In particular, it sends $1 \otimes S(\mathfrak{g})^{\mathfrak{g}}$ to $C_{\mathfrak{h}}(\mathfrak{g})$.
Assume for the rest of this section that $\mathfrak{g}$ is semisimple. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form, we get that $C_{\mathfrak{h}}(\mathfrak{g})$ acts on itself by differentiation. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ and $X^1, \dots, X^n$ be the basis of $\mathfrak{g}$ dual to it. We define an element $C := \frac{1}{2}\sum_i X_i X^i \in S(\mathfrak{g})$, called the **Casimir element**. It can be shown that it lies in $S(\mathfrak{g})^{\mathfrak{g}}$, hence gives the associated differential action $D$ on $C_{\mathfrak{h}}(\mathfrak{g})$, which we call the $D$**-operator**.
Surprisingly, $D(1 \otimes S(\mathfrak{g})^{\mathfrak{g}})$ appears to lie in the center $Z$ of $C_\pi(\mathfrak{g})$. It is not true in general, however, that $\sum_{k\geq 1} D^k(1 \otimes S(\mathfrak{g})^{\mathfrak{g}}) \subset Z$, but the main hypothesis that we have is that this family commutes. Our idea of proving this uses the theory of universal enveloping algebras.

## Universal enveloping algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. It appears that there exists an associative algebra $U(\mathfrak{g})$ and a Lie algebra homomorphism $\rho : \mathfrak{g} \to U(\mathfrak{g})$ satisfying the following universal property: for any algebra $\mathfrak{h}$ and a Lie algebra map $\pi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique algebra homomorphism $\widetilde{\pi} : U(\mathfrak{g}) \to A$ such that $\widetilde{\pi} \circ \rho = \pi$. In other words, there is a natural bijection

$$\operatorname{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A) \simeq \operatorname{Hom}_{\mathrm{Alg}}(U(\mathfrak{g}), A).$$

Now, let $\mathfrak{g}$ be semisimple. Define $C(\mathfrak{g})$ to be the algebra $(U(\mathfrak{g}) \otimes S(\mathfrak{g}))^{\mathfrak{g}}$. We call it the **universal classical family algebra of** $\mathfrak{g}$. We have the following

**Proposition.** *Let $\mathfrak{h}$ be an algebra and $\pi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra map. Then the algebra map $\widetilde{\pi} \otimes \mathrm{id} : U(\mathfrak{g}) \otimes S(\mathfrak{g}) \to \operatorname{End} V \otimes S(\mathfrak{g})$ restricts to an algebra map $C(\mathfrak{g}) \to C_{\mathfrak{h}}(\mathfrak{g})$. Moreover, this map commutes with the $D$-operator.*

Therefore, if we understand the structure of $C(\mathfrak{g})$, then we will be able to say much about $C_{\mathfrak{h}}(\mathfrak{g})$ for any algebra $\mathfrak{h}$. In particular, if we prove the commutativity of $\sum_{k\geq 1} D^k(1 \otimes S(\mathfrak{g})^{\mathfrak{g}})$ for $C(\mathfrak{g})$, then we will automatically prove it for all classical family algebras.

## References

[Kir1] A. Kirillov (2000). *Family algebras.* Pages. 620. 1079-676200075. 10.1090/S1079-6762-00-00075-5.

[Kir2] A. Kirillov (2001). *Introduction to family algebras.* Moscow Mathematical Journal, 1, 49-63.