Geometry of Eigenschemes

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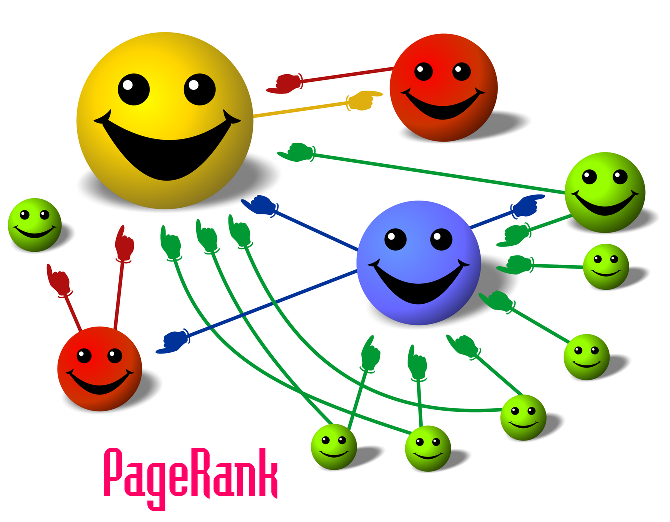
Motivation: linear algebra

Linear algebra is omnipresent both in mathematics and all of science. In its simplest form, it is about solving systems of linear equations, but surprisingly, it allows one to tackle much more complicated problems like differential equations, statistical approximations, Markov chains... Even to answer questions from pure mathematics, one rather often uses methods from linear algebra.



Solving the PageRank algorithm involves an eigenvalue problem

One of the main questions linear algebra tries to answer is the eigenvalue problem, that is finding all eigenvectors and eigenvalues of a given matrix. Because of their importance, it would be useful to assemble them into a structure which one could easily manipulate. This is what we do.

We use algebraic geometry to construct a single object (a projective scheme) carrying all the information about both eigenvalues and eigenvectors of a matrix. Moreover, even though the construction itself is very abstract, it can be intuitively visualized and some algebraic properties of the matrix can be seen geometrically, from the picture.

The abstractness of our construction is its big advantage. First of all, it enables us to use powerful tools from algebraic geometry. Secondly, it easily generalizes to a more interesting situation of a family of commuting matrices, in which we are looking for common eigenvectors. Finally, it can be done "in families", which is important for its application to vector fields.

Eigenvalues and eigenvectors

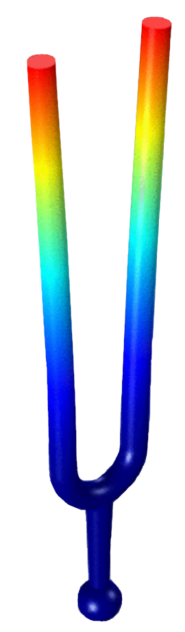
Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be a matrix. A non-zero vector $v \in \mathbb{C}^n$ is an **eigenvector** if $Av = \lambda v$ for some $\lambda \in \mathbb{C}$. Then, λ is called an **eigenvalue** of A. The **eigenspace** of λ is the linear subspace

$$V_\lambda = \{v \in \mathbb{C}^n : Av = \lambda v\}.$$

We have $V_\lambda = \ker(A - \lambda I)$ and we denote $V_\lambda^{(k)} = \ker(A - \lambda I)^k$. There is a chain of inclusions

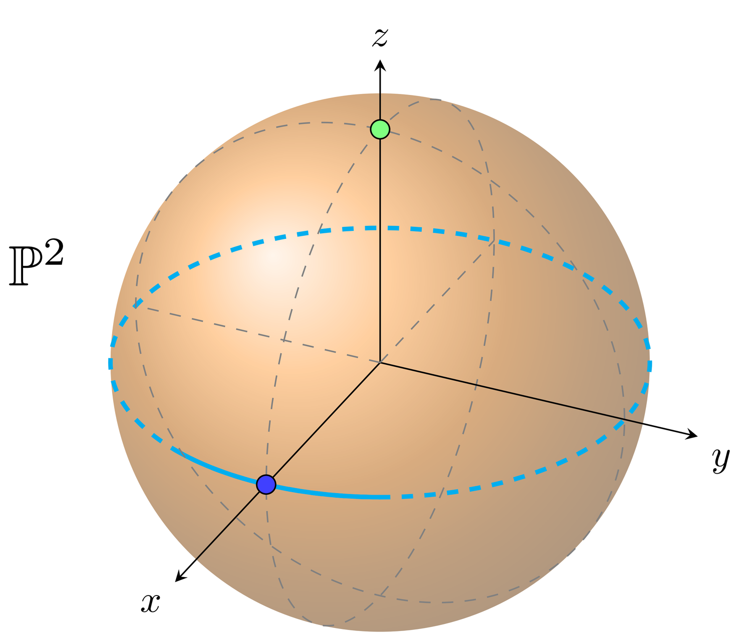
$$V_\lambda = V_\lambda^{(1)} \subseteq V_\lambda^{(2)} \subseteq \ldots \subseteq \mathbb{C}^n.$$

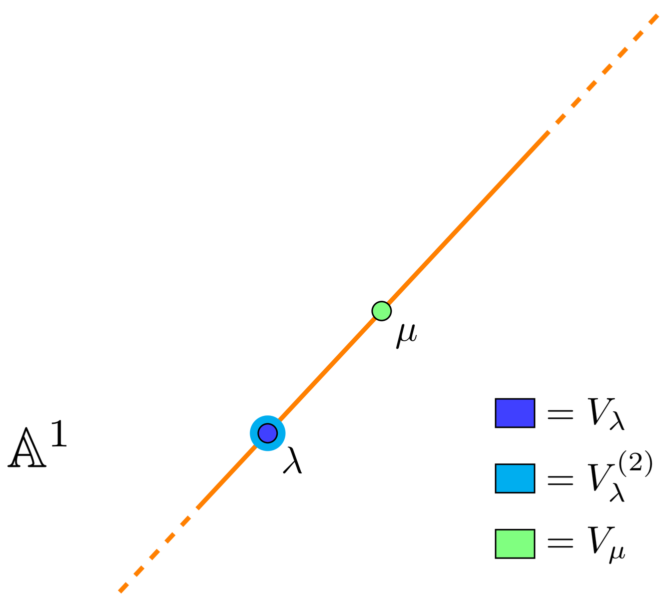
It always stabilizes ($V_\lambda^{(k)} = V_\lambda^{(k+1)}$ for k big enough) and the smallest d such that $V_\lambda^{(d)} = V_\lambda^{(d+1)}$ is called the **index** of λ.



In vibration analysis, eigenvalues are natural frequencies and eigenvectors are shapes of vibrational modes

It is more natural to think of eigenvectors in terms of lines that they span, rather than single vectors. We denote by $\mathbb{P}^n$ the n-dimensional **projective space**, it is the set of lines in $\mathbb{C}^{n+1}$ passing through the origin. We view eigenvectors as points in $\mathbb{P}^n$.





We imagine V_λ and V_μ living above the points λ and μ respectively. We make the point corresponding to λ more fuzzy to indicate a higher index

We also consider the **affine line** $\mathbb{A}^1$. This is just the one-dimensional space $\mathbb{C}$, so $\mathbb{A}^1$ is where eigenvalues live. Even though $\mathbb{C}$ is usually pictured as a plane (2-dimensional real vector space), we view $\mathbb{A}^1$ as a line, but remember that points correspond to complex numbers.

Similarly, we picture $\mathbb{C}^{n+1}$ as an $n+1$-dimensional space. Then, $\mathbb{P}^n$ can be represented as an n-dimensional sphere. A point on the sphere corresponds to the line passing through it (we implicitly identify all points with their antipodes).

As an example, consider the matrix

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

It has two distinct eigenvalues, λ and μ. Both V_λ and V_μ are one dimensional spanned by $(1,0,0)$ and $(0,0,1)$ respectively. The index of λ is 2 and of μ is 1. We have $V_\lambda^{(2)} = \mathrm{span}\langle(1,0,0),(0,1,0)\rangle$.

Visualization with algebraic geometry

First, we can generalize our setting to a tuple of commuting linear operators $f = (f_1, \ldots, f_m)$ on a finite dimensional vector space V. Then, $v \in V$ is an **eigenvector** if $f_i(v) = \lambda_i v$ for some **eigentuple** $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{C}^m$. The **eigenspace** of λ is the subspace of all eigenvectors with eigentuple λ.

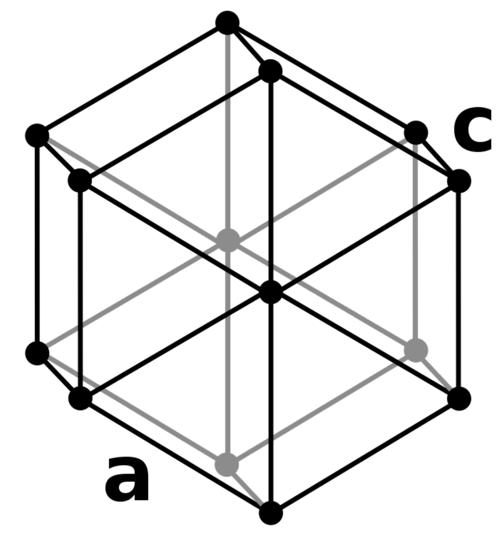
We have a map $\mathbb{C}[t_1, \ldots, t_m] \to \mathrm{End}(V^*)$ given by $t_i \mapsto f_i^*$, where $V^* = \mathrm{Hom}(V, \mathbb{C})$ is the dual space of V. We denote by R the quotient of $\mathbb{C}[t_1, \ldots, t_m]$ by its kernel. The vector space V^* is naturally a module over R. We define the **eigenscheme** of f by the formula

$$\mathbb{P}_f V = \mathrm{Proj}\,\mathrm{Sym}_R V^*.$$

We also have the **projectivisation** of V

$$\mathbb{P}V = \mathrm{Proj}\,\mathrm{Sym}\,V^*.$$

Closed points of $\mathbb{P}V$ represent lines in V.



For a hydrogen atom, $\{H, L^2, L_z\}$ form a commuting set of operators

Proposition. *The following two statements hold.*

1. *The eigenscheme $\mathbb{P}_f V$ naturally forms a closed subscheme of $\mathbb{P}V$. Its closed points represent eigenvectors of f, the nilpotent data encodes the eigenfiltration.*
2. *The canonical map $\mathbb{P}_f V \to \mathrm{Spec}\,R$ is identified with the affinization of $\mathbb{P}_f V$. Its composition with the closed embedding $\mathrm{Spec}\,R \to \mathbb{A}^m$ sends an eigenvector to its corresponding eigentuple.*

Hence, we have the following diagram, which makes precise our intuition from before.

$$\begin{tikzcd} \mathbb{P}_f V \arrow[r, hook] \arrow[d] & \mathbb{P}V \\ \mathrm{Spec}\,R \arrow[d, hook] & \\ \mathbb{A}^m & \end{tikzcd}$$

Proposition. *The closed embedding $\mathbb{P}_f V \hookrightarrow \mathbb{P}V$ induces a very ample line bundle on $\mathbb{P}_f V$. Its pushforward along $\mathbb{P}_f V \to \mathbb{A}^m$ is a $\mathbb{C}[t_1, \ldots, t_m]$-module whose underlying vector space is isomorphic to V. The action of t_i recovers the operator f_i.*

Therefore, given the geometrical picture, we can recover all f_i's. In a single operator case, this is rather obvious, but with several operators it is more surprising, because the eigenscheme of f is nothing more than the intersection of eigenschemes for each f_i. So, a priori, the more operators we take, the less information we are left with.

Applications: regularity and vector fields

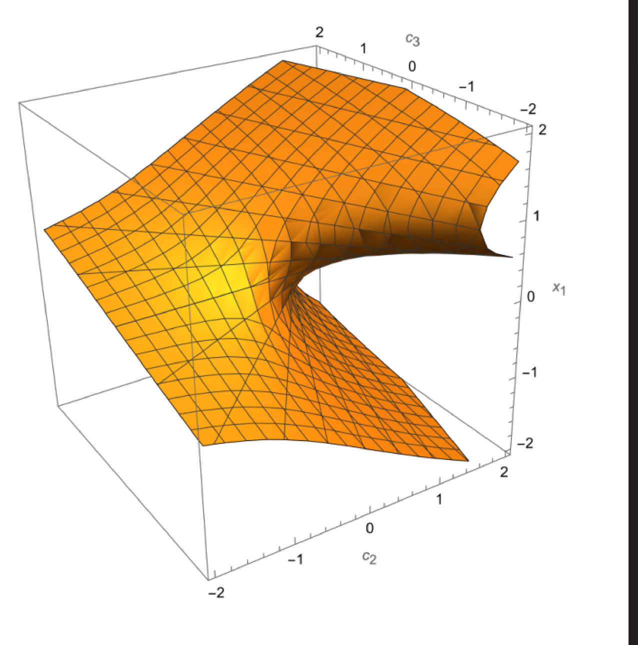
The **regularity index** $\mathrm{reg}(f)$ is the maximum dimension of eigenspaces of f. The **cyclicity index** $\mathrm{cyc}(f)$ is the minimum number of vectors that, together with their images under subsequent application of f_i's span V.

Theorem (Shekhtman). *For any tuple of commuting linear operators f, we have*

$$\mathrm{reg}(f) = \mathrm{cyc}(f^*),$$

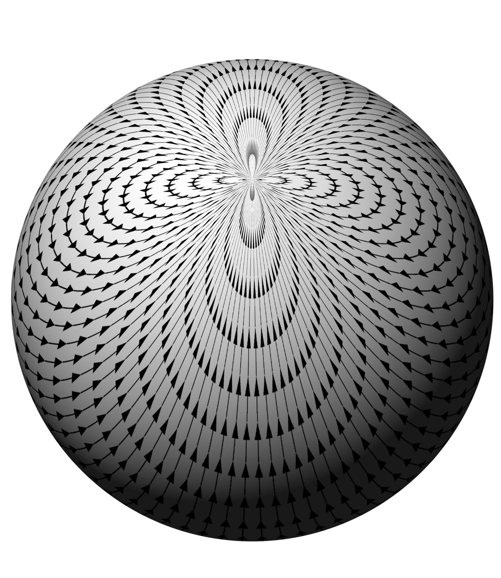
where $f^* = (f_1^*, \ldots, f_m^*)$ is the dual tuple on V^*.

We can see this fact using our geometrical picture. Indeed, $\mathrm{cyc}(f^*)$ can be interpreted as the minimum number of generators for the R-module V^*, thus $\mathrm{cyc}(f^*) = \dim \mathrm{Sym}_R V^* = \dim \mathbb{P}_f V + 1$. On the other hand, by construction, we also have $\mathrm{reg}(f) = \dim \mathbb{P}_f V + 1$. In particular, f is **regular** ($\mathrm{reg}(f) = 1$) if and only if its eigenscheme is zero-dimensional (a collection of points).



Eigenscheme in families over the Kostant section for $\mathrm{SL}_3(\mathbb{C})$

A **vector field** on a scheme X is a derivation $D : \mathcal{O}_X \to \mathcal{O}_X$ of the structure sheaf. The **zero scheme** $\mathcal{Z}_D$ of D is the closed subscheme of X cut out by the image of D.



Vector field with a single zero

Consider $X = \mathbb{P}^n$. Deriving $\mathrm{PGL}_{n+1} \cong \mathrm{Aut}(\mathbb{P}^n)$ we get $\mathfrak{sl}_{n+1} \cong \mathrm{Der}(\mathbb{P}^n)$. Hence, vector fields on $\mathbb{P}^n$ are in one-to-one correspondence with traceless matrices. For a matrix $A \in \mathfrak{sl}_{n+1}$, the associated derivation maps x_i to A_i, where x_i's are homogeneous coordinates of $\mathbb{P}^n$ and A_i is the i-th row of A treated as a linear homogeneous polynomial in x_i's. We have the following result.

Proposition. *The eigenscheme of $A \in \mathfrak{sl}_{n+1}$ agrees with the zero scheme of the associated derivation.*