Multiplicity algebras, representation theory and geometry

Nhok Tkhai Shon Ngo, Hausel group, ISTernship Summer Program 2021

IST AUSTRIA
Institute of Science and Technology

Outline

My project lies in the intersection of algebra, representation theory and geometry. The main object of study is the cohomology ring of the complex Grassmannian $\mathrm{Gr}(k,n)$ – the smooth algebraic variety of k-planes in n-dimensional complex vector space. The aim was to learn more about the algebraic properties of this ring and its structures arising from the geometry of Grassmannian.

Representation theory

Imagine that we have some abstract group G. In order to understand it better we want to construct a "model" of it in some more familiar to us area. One of the simplest objects in mathematics is a linear transformation, so we can try model our group as a subset of all linear transformations of some vector space. This idea leads to the notion of a group representation.

A **representation** of a group G is a vector space V together with a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$. A **subrepresentation** of a representation V of a group G is a subspace $W \subset V$ which is invariant under all the operators $\rho(g)$, $g \in G$. A representation V of a group G is called **irreducible** if the only subrepresentations of V are 0 and V.

Why we are so interested in irreducible representations? The reason is that often the irreducible representations are those smallest "building blocks" which can be used in order to construct any representation.

Example 1. Consider $G = \mathfrak{S}_n$ – the group of permutations on of the set $\{1, 2, \ldots, n\}$. Then, there is a natural representation ρ of S_n on the $V = \mathbb{C}^n$ given by the formula

$$\rho(\sigma)\colon \mathbb{C}^n \to \mathbb{C}^n,\ \rho(\sigma)(x_1, x_2, \ldots, x_n) = \left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right).$$

It is clear that two subspaces of V defined by equations $x_1 = x_2 = \ldots = x_n$ and $x_1 + x_2 + \ldots + x_n = 0$ are subrepresentations of V. Moreover, one can show that these subrepresentations are in fact irreducible and that the initial representation is the direct sum of these two.

Example 2. In the first example the group G was finite but what happens if the group is infinite? There is a class of infinite groups, called **Lie groups**, whose representation theory is well-understood and has many applications in different branches of mathematics and physics (quantum mechanics, particle physics).

One example of a Lie group is the special linear group SL_n – the group of $n \times n$ matrices with the determinant 1. The theory of irreducible representations of SL_n has deep connections to the representations of permutation group $\mathfrak{S}_n$ and to the combinatorics of Young tableaux.

Geometry

Cohomology ring

Let M be an n-dimensional (real) smooth manifold. Then, we have a **cochain complex** of differential forms

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0,$$

where d is the exterior differentiation. A differential form $\omega \in \Omega(M)$ is called **exact** if it is equal to the exterior derivative of some differential form and is called **closed** if its exterior derivative is zero. In this case, the k-th cohomology group is defined as quotient group

$$H^k(M) = \{\text{closed forms in } \Omega^k(M)\}/\{\text{exact forms in } \Omega^k(M)\}.$$

The cohomology ring is a graded ring

$$H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$$

with a multiplication given by the wedge product.

The **Poincare duality theorem** says that for a large class of manifolds there is a nondegenerate pairing

$$H^k(M) \otimes H^{n-k}(M) \to \mathbb{R}$$

given by the integral of wedge product of two forms.

Kähler manifolds

The Kähler manifolds form a special class of manifolds possessing simultaneously three structures: Riemannian, symplectic and complex which are compatible with each other.

More precisely, a **Kähler manifold** is an n-dimensional complex manifold M with a Hermitian metric h such that the associated 2-form (also known as **Kähler form**) $\omega(u,v) = \operatorname{Im} h(u,v)$ is a real closed 2-form. In this case, (M, ω) is $2n$-dimensional symplectic manifold. The Riemannian structure on M is defined by the metric $g(u,v) = \operatorname{Re} h(u,v)$.

One particular remarkable example of a Kähler manifold is a smooth complex projective variety (e. g. the complex projective space $\mathbb{P}^n$ or the complex Grassmannian $\mathrm{Gr}(k,n)$). For such manifolds one can extend the Poincare duality theorem by the **hard Lefschetz Theorem**: the k-fold wedge product of the Kähler form ω gives an isomorphism between cohomology groups $H^{n-k}(M)$ and $H^{n+k}(M)$ i. e. the map

$$H^{n-k}(M) \to H^{n+k}(M),\ [\alpha] \mapsto [\omega^{\wedge k} \wedge \alpha]$$

is an isomorphism.

Multiplicity algebras

Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $f(0) = 0$. Then, the components $f_1, \ldots, f_n$ of the map f generate an ideal I_f in the algebra $\mathbb{C}[x_1, \ldots, x_n]$ of all polynomial maps from $\mathbb{C}^n$ to $\mathbb{C}$.

The **multiplicity algebra** of a polynomial map f is the quotient algebra $Q_f = \mathbb{C}[x_1, \ldots, x_n]/I_f$. If Q_f is finite-dimensional, then f is called a map of **finite multiplicity** and $\mu_f = \dim_{\mathbb{C}} Q_f$ is called **multiplicity** of f.

The importance of multiplicity algebra is due to the fact that it is a purely algebraic object which describes local topological properties of the map f at 0.

Theorem. *Let f be a polynomial map of finite multiplicity and suppose that f has a unique root $x = 0$ inside the closed ball $B_r(0)$. The number of preimages in $B_r(0)$ of an arbitrary sufficiently small regular value ε is equal to the multiplicity of f.*

Example. Let $m = (m_1, m_2, \ldots, m_n)$ be a n-tuple of integers. Consider the *Pham map* $\Phi^m\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by formula

$$\Phi^m(x_1, \ldots, x_n) = (x_1^{m_1}, \ldots, x_n^{m_n}).$$

It is clear that the corresponding multiplicity algebra has dimension $\mu = m_1 m_2 \ldots m_n$ and each regular value of Φ^m has exactly $m_1 m_2 \ldots m_n$ preimages.

Principal $\mathfrak{sl}_2$-triple on the cohomology of $\mathrm{Gr}(k,n)$

Cohomology of $\mathrm{Gr}(k,n)$ as multiplicity algebra

It is known that the cohomology ring of the complex Grassmannian is isomorphic to the multiplicity algebra of the polynomial map

$$r\colon (p_1, \ldots, p_k, q_1, \ldots, q_{n-k}) \mapsto (r_1, \ldots, r_n),$$

where the components of r are defined via polynomial identity

$$x^n + r_1 x^{n-1} + \ldots + r_n = (x^k + p_1 x^{k-1} + \ldots + p_k)(x^{n-k} + q_1 x^{n-k-1} + \ldots + q_{n-k}).$$

SL_n-structure

It was known that cohomology ring of the $\mathrm{Gr}(k,n)$ has a canonical structure of an irreducible SL_n-representation but the explicit action on the multiplicity algebra wasn't known.

One of the goals of my project was to better understand this structure and to find the **principal $\mathfrak{sl}_2$-triple** of this representation, i. e. three linear operators E, F and H satisfying commutation relations:

$$[H,E] = 2E,\ [H,F] = -2F,\ [E,F] = H$$

with the given E and H: we knew that in cohomology E corresponds to the operator arising in the hard Lefschetz theorem and H corresponds to the shifted cohomological degree. Thus, the main problem was to find the action of the F which we call the **Lefschetz operator**.

Results

It was found out that these operators have the following explicit differential forms:

$$\begin{aligned}
H &= 2\sum_{j=1}^{k} j p_j \frac{\partial}{\partial p_j} + 2\sum_{j=1}^{n-k} j q_j \frac{\partial}{\partial q_j} - k(n-k),\\
E &= p_1,\\
F &= \sum_{j=1}^{k} (k-j+1)(n-k+j-1) p_{j-1} \frac{\partial}{\partial p_j} -\\
&\quad - \sum_{j=1}^{n-k} (k+j-1)(n-k-j+1) q_{j-1} \frac{\partial}{\partial q_j} -\\
&\quad - \sum_{j,l=1}^{k} \left(\sum_{d=0}^{j-1} (j+l-2d-1) p_d p_{j+l-d-1} \right) \frac{\partial^2}{\partial p_j \partial p_l} +\\
&\quad + \sum_{j,l=1}^{n-k} \left(\sum_{d=0}^{j-1} (j+l-2d-1) q_d q_{j+l-d-1} \right) \frac{\partial^2}{\partial q_j \partial q_l} -\\
&\quad - 2\sum_{j=1}^{k} \sum_{l=1}^{n-k} (j+l-1) \left(\sum_{d=0}^{j-1} p_d q_{j+l-d-1} \right) \frac{\partial^2}{\partial p_j \partial q_l}.
\end{aligned}$$

It was shown that as an SL_n-module, the cohomology ring of the $\mathrm{Gr}(k,n)$ is isomorphic to $\bigwedge^k \mathbb{C}^n$ – the k-th exterior power of the standard SL_n-representation. Moreover, the corresponding isomorphism can be described through the theory of symmetric functions and Schubert calculus.