Higgs bundle tournaments

based on joint project with Mirko Mauri

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AN AMERICAN TOURNAMENT TREATED BY THE CALCULUS OF SYMMETRIC FUNCTIONS.

By Major P. A. MacMahon.

Part I.

1. In a tournament of n players, where each player plays every other player, there are $\frac{1}{2}n(n-1)$ games. Since each game may be won or lost there are $2^{\frac{1}{2}n(n-1)}$ events and I propose to analyse them by means of the powerful calculus of symmetric functions. The final result of the play is that the players are arranged in a definite order, each with a certain number of games to his credit. These numbers constitute a partition of the number $\frac{1}{2}n(n-1)$, and we may ask how many of the $2^{\frac{1}{2}n(n-1)}$ events will yield a given partition of $\frac{1}{2}n(n-1)$ when the players are or are not in an assigned order.

2. Consider the symmetric function

$$(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3)\ldots(\alpha_{n-1}+\alpha_n)$$

of the n quantities $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

It involves $\frac{1}{2}n(n-1)$ factors, and the terms, after carrying out the multiplication, are grouped together in monomial symmetric functions.

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- $(\deg(E), n) = 1 \rightsquigarrow$ similar combinatorics by (Villegas 2011, 2023, Reineke 2012, Rayan 2018)

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  $R := \{\pm(e_i - e_j)\}_{1 \le i < j \le n} = R_+ \coprod R_- \subset \Lambda$ type A_{n-1} root system

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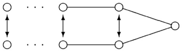
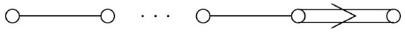
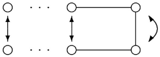
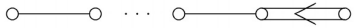
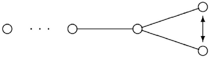
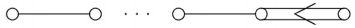
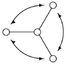
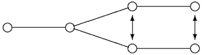
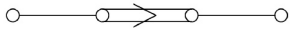
R	σ	R^σ		$\mathfrak{g}^\vee_{R^\sigma} \subset \mathfrak{g}^\vee_R$
A_{2n-1}		B_n		$\mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$
A_{2n}		C_n		$\mathfrak{so}_{2n+1} \subset \mathfrak{sl}_{2n+1}$
D_n		C_{n-1}		$\mathfrak{so}_{2n-1} \subset \mathfrak{so}_{2n}$
D_4		G_2		$\mathfrak{g}_2 \subset \mathfrak{so}_8$
E_6		F_4		$\mathfrak{f}_4 \subset \mathfrak{e}_6$

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- g.f. for score vectors of m.b.c. tournaments $\prod_{1 \le i < j \le n} (y_i y_j + x_i x_j)(x_i y_j + x_j y_i)|_{y_i = 1} \overset{D_n\text{-Weyl}}{=} \sum_{\lambda \in \Lambda} \dim V_\lambda^\rho x^\lambda$
- D_n Dynkin auto: $\tau := (n, n+1) : [2n] \to [2n]$, $[2n]/\tau = [2n-1]$

Tournament folding $D_n \rightsquigarrow C_{n-1}$

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  $\prod_{i=1}^{i-1}(y_i^2 + x_i^2) \prod_{1\leq i<j<n}(y_i y_j + x_i x_j)(x_i y_j + x_j y_i)$

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- compatible with tournament folding $A_{2n-2} \rightsquigarrow C_{n-1}$