

Mirror Symmetry with Branes by Equivariant Verlinde Formulas

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ABSTRACT. We find an agreement of equivariant indices of semi-classical homomorphisms between pairwise mirror branes in the GL_2 -Higgs moduli space on a Riemann surface. On one side we have the components of the Lagrangian brane of $U(1,1)$ -Higgs bundles, whose mirror was proposed by Nigel Hitchin to be certain even exterior powers of the hyperholomorphic Dirac bundle on the SL_2 -Higgs moduli space. The agreement arises from a mysterious functional equation. This gives strong computational evidence for Hitchin's proposal.

1. Introduction

In the conference ‘Hitchin 70: Differential Geometry and Quantization’ in Aarhus in September 2016, the first and third authors gave talks on recent results on the understanding of equivariant Verlinde formulas on Higgs moduli spaces. Here, we explain how such techniques can be used to compute equivariant indices of semi-classical homomorphisms between branes on Higgs moduli spaces and how, in turn, this could be used to give non-trivial computational evidence for Hitchin's mirror symmetry proposals in [Hi2, §7].

Hitchin introduced the moduli space of Higgs bundles on a Riemann surface [Hi1] in 1987. In 2006, the work of Kapustin and Witten [KW] proposed an understanding of the geometric Langlands programme using the S -duality of four-dimensional supersymmetric Yang–Mills theory. One of the main statements is that S -duality induces mirror symmetry between the moduli space $\mathcal{M}_{\text{DR}}(G)$ of flat G -connections on C and $\mathcal{M}_{\text{DR}}(G^L)$ for the Langlands dual group G^L . The precise statement is still missing, but we have more understanding [DP1] in the semi-classical limit where the mirror symmetry should reduce to a relative Fourier–Mukai type equivalence between $D(\mathcal{M}(G))$ and $D(\mathcal{M}(G^L))$, where $\mathcal{M}(G)$ denotes certain moduli space of G -Higgs bundles on C . Even this duality is not completely understood, as we lack the description of the Fourier–Mukai transform on the most singular fibres of the Hitchin map. Consequently, we have few global results proving aspects of this duality.

One global computational result was achieved in [HT] in 2002. There, as a consequence of mirror symmetry, a conjecture was proposed for the agreement of certain Hodge numbers of $\mathcal{M}(\text{SL}_n)$ and $\mathcal{M}(\text{PGL}_n)$, which were proved there for $n = 2, 3$. This conjecture for every n has just recently been settled in [GWZ], using p -adic integration.

Here, we are studying another global approach where deeper properties of the conjectured mirror symmetry could be computationally verified. We set $G = \text{GL}_2$, and C a smooth complex projective curve of genus $g > 1$. The components of the $\mathcal{M}(\text{U}(1, 1))$ Higgs moduli space inside the $\mathcal{M}(\text{GL}_2)$ moduli space can be labeled by certain characteristic classes, as explained in [Hi2]. There are g such components and we will denote these Lagrangians by

$$L_0, \dots, L_{g-1} \subset \mathcal{M}(\text{GL}_2).$$

To be more precise, L_i is the locus of Higgs bundles of the form

$$M_1 \oplus M_2 \xrightarrow{\Phi} M_1 K_C \oplus M_2 K_C,$$

where M_j are line bundles with $\text{deg}(M_1) = g - 1 - i$, $\text{deg}(M_2) = i - g + 1$ and Φ is off-diagonal. For $i = 0, \dots, g - 2$, the Lagrangians are located in the stable locus $\mathcal{M}(\text{GL}_2)^s$ and so are smooth, and L_i is isomorphic to an affine bundle over the $2i$ th symmetric product of C times the Jacobian of degree $g - 1 - i$ line bundles on C . The last component, L_{g-1} , is special as it contains strictly semi-stable points as well.

As explained in [Hi2, §7], L_i is a BAA brane whose mirror should be a BBB brane, i.e. a hyperholomorphic sheaf on the mirror $\mathcal{M}(\text{GL}_2)$ (cf. also [BS] and [GW, §6]). Its support will be $\mathcal{M}(\text{SL}_2) \subset \mathcal{M}(\text{GL}_2)$ and, roughly speaking, it is constructed from a universal bundle \mathbf{E} on $\mathcal{M}(\text{SL}_2) \times C$ by taking $\mathbf{V} := R^1\pi_*(\mathbf{E})$ by the projection π to the first factor. Then, Hitchin proposes that the brane L_i should be mirror to $\Lambda^{2i}\mathbf{V}$. This was checked to be correct on a generic fibre $\chi_a := \chi^{-1}(a)$ of the Hitchin map in the sense that $\mathcal{O}_{L_i}|_{\chi_a}$ is Fourier–Mukai dual to $\Lambda^{2i}\mathbf{V}|_{\chi_a}$. Additionally, the fact that $\Lambda^{2i}\mathbf{V}$ carries a natural hyperholomorphic structure gave more credence to the proposal that $\Lambda^{2i}\mathbf{V}$ should be the mirror of L_i . However, it is unclear how to extend the relative Fourier–Mukai transform to the most singular fibres of the Hitchin map, in particular to the nilpotent cone χ_0 , and so we did not have direct checks of the correctness to the proposal that L_i is indeed relative Fourier–Mukai dual to $\Lambda^{2i}\mathbf{V}$.

There have recently been several works computing equivariant Verlinde formulas, on the moduli space of Higgs bundles. Let \mathcal{L} be the determinant line bundle on $\mathcal{M}(G)$, where G , is a simply connected simple group, such as $G = \mathrm{SL}_n$. The natural

$$\mathbb{T} := \mathbb{C}^\times$$

action on $\mathcal{M}(G)$ by scaling the Higgs field lifts to a \mathbb{T} -equivariant structure on \mathcal{L} . Although $H^*(\mathcal{M}(G); \mathcal{L}^k)$ is infinite-dimensional, the induced \mathbb{T} -action will have finite-dimensional weight spaces and non-trivial only for non-positive weights:

$$H^*(\mathcal{M}(G); \mathcal{L}^k) = \bigoplus_{j \in \mathbb{Z}_{\geq 0}} H^*(\mathcal{M}(G); \mathcal{L}^k)^{-j}.$$

We define for the \mathbb{T} -equivariant line bundle \mathcal{L}^k (and, similarly, later for a \mathbb{T} -equivariant sheaf) the equivariant index as

$$\chi_{\mathbb{T}}(\mathcal{M}(G); \mathcal{L}^k) = \sum_{i,j} (-1)^i \dim(H^i(\mathcal{M}(G), \mathcal{L}^k)^j) t^{-j} \in \mathbb{Z}[[t]]. \tag{1.1}$$

In a series of papers [GP, GPYY, AGP] following insights from studying path integrals in certain supersymmetric quantum field theories and TQFT techniques, precise formulas have been achieved for (1.1). Equivalent formulas have been also found in [HL]. There is also a more combinatorial understanding of the results for $G = \mathrm{SL}_n$ as residue formulas in a TQFT framework in [HSz].

In this paper, we compute similar characters but only on $\mathcal{M}(\mathrm{GL}_2)$ and, eventually, $\mathcal{M}(\mathrm{SL}_2)$. Namely, we will consider the (generically) vector bundles $\Lambda^{2i}\mathbf{V}$ mentioned above, with support in $\mathcal{M}(\mathrm{SL}_2)$ together with a \mathbb{T} -equivariant structure. We will also consider the analogues of the determinant bundle \mathcal{L} on $\mathcal{M}(\mathrm{GL}_2)$ with a \mathbb{T} -equivariant structure and, in particular, the \mathbb{T} -equivariant coherent sheaves $\mathcal{L}^2 \mathcal{O}_{L_i}$ on $\mathcal{M}(\mathrm{GL}_2)$ (for the detailed constructions, see Section 2). With the notation $\mathcal{L}_i := \mathcal{L}^2 \mathcal{O}_{L_i}$ for $0 \leq i < g - 1$ and $\Lambda_j := \Lambda^{2j}\mathbf{V}$, our main result is the following:

Theorem 1.2 *For $0 \leq i, j < g - 1$ we have*

$$\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_j \otimes \Lambda_i) = \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_j) \in \mathbb{Z}[[t]]. \tag{1.2}$$

In Corollary 6.3, we will explain how to extend this symmetry for the case of $i = g - 1$.

We can interpret the left-hand side of (1.2) as the equivariant index of derived homomorphisms from $(\Lambda^{2i}\mathbf{V})^\vee \cong \Lambda^{2i}\mathbf{V}$ to $\mathcal{L}^2 \mathcal{O}_{L_j}$, and the right-hand side as the equivariant index of derived homomorphisms from $\mathcal{L}^2 \mathcal{O}_{L_i}^\vee \cong \mathcal{L}^2 \mathcal{O}_{L_i}$ to $\Lambda^{2j}\mathbf{V}$. For more details on this see Section 7. This way, our Theorem 1.2 provides computational evidence for Hitchin’s proposal in [Hi2, §7].

Interestingly, most of the ingredients used here have already appeared in the first author’s PhD dissertation [Ha1], which was written under the supervision of Nigel Hitchin. In a sense, our computation could have been done almost 20 years ago. The present paper thus outlines the importance of ideas from physics, in particular the work of Kapustin and Witten [KW] in 2006 and further mathematical insights by Hitchin [Hi2, §7] in 2013.

The plan of the paper is as follows. In Section 2 we introduce the \mathbb{C}^\times -equivariant coherent sheaves \mathcal{L}_i and Λ_i on $\mathcal{M}(\mathrm{GL}_2)$. In Section 3, we compute the equivariant index $\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_j)$ for $i < g - 1$, by reducing the computation via Hirzebruch–Riemann–Roch to an integral on the $2i$ th symmetric product of C , which we evaluate using a residue formula due to Zagier. In Section 4, we find a change of variables which will imply our main symmetry observation for $i, j < g - 1$. In Section 5, following the technique of Teleman and Woodward [TW], we compute the equivariant index on the whole moduli stack and then, in Section 6, we prove Corollary 6.3. Finally, in Section 7, we sketch how this should be a consequence of mirror symmetry.

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2. Background

Let C be a smooth complex projective curve of genus $g > 1$. Let $\mathcal{M}(\mathrm{GL}_2)$ denote the moduli space of semi-stable degree zero rank 2 Higgs bundles on C . We recall [BGG, Hi2, Sch] that the $U(1, 1)$ -Higgs bundles in $\mathcal{M}(\mathrm{GL}_2)$ are degree zero rank 2 Higgs bundles

$$E \xrightarrow{\Phi} E \otimes K_C \tag{2.1}$$

such that $\Phi \neq 0$ and

$$E \xrightarrow{-\Phi} E \otimes K_C$$

is equivalent with (2.1). In other words, there exists an $a : E \rightarrow E$ automorphism of E such that

$$\Phi \circ a = -\Phi.$$

When $\Phi \neq 0$, this implies that such a Higgs bundle has the form

$$M_1 \oplus M_2 \xrightarrow{\begin{pmatrix} 0 & \phi_2 \\ \phi_1 & 0 \end{pmatrix}} (M_1 \oplus M_2) \otimes K_C, \tag{2.2}$$

which consists of a direct sum of two line bundles and an off-diagonal Higgs field. We can assume

$$\deg M_2 \leq \deg M_1 = g - 1 - i$$

for some integer $i \leq g - 1$. When $i < g - 1$, then

$$\phi_1 \in H^0(C; M_1^{-1} M_2 K_C)$$

cannot be 0 because of semi-stability. As

$$\deg(M_1^{-1} M_2 K_C) = 2i,$$

we have that $i \geq 0$.

The locus of Higgs bundles of the form (2.2) with $\deg(M_1) = g - 1 - i$ for $0 \leq i \leq g - 1$ is denoted by

$$L_i \subset \mathcal{M}(\mathrm{GL}_2).$$

With an argument similar to the one below for SL_2 , one can show that for $0 \leq i < g - 1$ the locus $L_i \subset \mathcal{M}(\mathrm{GL}_2)$ is isomorphic with a total space of a vector bundle over $\mathcal{J}_i \times C_{2i}$. We denote by \mathcal{J}_i the Jacobian of degree i line bundles on C , by C_j the j th symmetric product of the curve C and $\tilde{i} := g - 1 - i$.

We will also need to define the sheaf \mathcal{L} . Recall the tensor product map

$$\tau : \mathcal{M}(\mathrm{SL}_2) \times T^* \mathcal{J}_0 \rightarrow \mathcal{M}(\mathrm{GL}_2),$$

which is a Galois cover, with Galois group $\mathcal{J}[2]$ comprising the 2-torsion points on the Jacobian. We choose $\mathcal{L} \in \mathrm{Pic}(\mathcal{M}(\mathrm{GL}_2))$ such that

$$\tau^*(\mathcal{L}) = \mathcal{L}_{\mathrm{SL}_2} \boxtimes \mathcal{O}_{T^* \mathcal{J}_0},$$

where $\mathcal{L}_{\mathrm{SL}_2}$ is the determinant line bundle on $\mathcal{M}(\mathrm{SL}_2)$ constructed below (2.8). \mathcal{L} is not unique, because

$$\tau^* : \mathrm{Pic}(\mathcal{M}(\mathrm{GL}_2)) \rightarrow \mathrm{Pic}(\mathcal{M}(\mathrm{SL}_2) \times \mathcal{J}_0)$$

has a kernel isomorphic to

$$\mathrm{Pic}(\mathcal{M}(\mathrm{GL}_2))[2] \cong \mathcal{J}_0[2].$$

However, when restricted to $\mathcal{M}(\mathrm{U}(1, 1))$, the ambiguity disappears. For $0 \leq i < g - 1$, we will denote

$$\mathcal{L}_i := \mathcal{L}^2 \otimes \mathcal{O}_{L_i}$$

a coherent sheaf on $\mathcal{M}_{\mathrm{GL}_2}$, while, for $i = g - 1$, we define

$$\mathcal{L}_{g-1} := (\mathcal{L}^2 \oplus \mathcal{L}^2) \otimes \mathcal{O}_{L_{g-1}}. \tag{2.3}$$

In our computation below, we will restrict to the part

$$L'_i := L_i \cap \mathcal{M}(\mathrm{SL}_2) \subset \mathcal{M}(\mathrm{SL}_2)$$

inside the moduli space $\mathcal{M}(\mathrm{SL}_2)$ of semi-stable rank 2 Higgs bundles with trivial determinant and trace-free Higgs field. In other words, L'_i is a component of the moduli space of $\mathrm{SU}(1, 1)$ -Higgs bundles in $\mathcal{M}(\mathrm{SL}_2)$. This means to set $M := M_1$ and choose $M_2 = M^{-1}$ in (2.2).

Now we give a more detailed description of $L'_i \subset \mathcal{M}(\mathrm{SL}_2)$ for $0 \leq i \leq g - 2$. For similar discussions, see [Ha1, Lemma 6.1.2] and [HT2, (6.1)].

Define maps $\mathcal{J}'_i \rightarrow \mathcal{J}_{2i}$ by sending $M \mapsto M^{-2}K$ and the Abel–Jacobi map $C_{2i} \rightarrow \mathcal{J}_{2i}$ by sending $D \mapsto L(D)$. Then, we construct the fibred product of these maps:

$$F'_i := C_{2i} \times_{\mathcal{J}_{2i}} \mathcal{J}'_i. \tag{2.4}$$

We have the two projections $\mathrm{pr}_{\mathcal{J}'_i} : F'_i \rightarrow \mathcal{J}'_i$ and $\mathrm{pr}_{C_{2i}} : F'_i \rightarrow C_{2i}$. By construction, F'_i is isomorphic to the moduli space of complexes $M \xrightarrow{\phi} M^{-1}K_C$, with $\mathrm{deg}(M) = \bar{i}$ and $\phi \in H^0(\Sigma; M^{-2}K_C)$, in other words, of nilpotent SL_2 -Higgs bundles

$$M \oplus M^{-1} \xrightarrow{\phi} MK_C \oplus M^{-1}K_C$$

given by

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix}.$$

Let

$$\Delta_{2i} \subset C \times C_{2i}$$

denote the universal divisor, and \mathcal{P}'_i a normalized universal bundle on $C \times \mathcal{J}'_i$. By abuse of notation, we will denote their pullbacks by $\mathrm{pr}_{C_{2i}}$ and, respectively, $\mathrm{pr}_{\mathcal{J}'_i}$ to $C \times F'_i$, with the same Δ_{2i} and \mathcal{P}'_i . As $i < g - 1$, we have

$$\mathrm{deg}(K_C^2(-D)) > 2g - 2$$

for $D \in C_{2i}$ and so

$$H^1(C; K_C^2(-D)) = 0.$$

Consequently,

$$E_i := \text{pr}_{C^*} (K_C^2(-\Delta_{2i}))$$

is a vector bundle on F'_i .

E_i parametrizes $SU(1, 1) \subset SL_2$ -Higgs bundles as follows. First, note that $\mathcal{P}_i^2 K_C^{-1}(\Delta_{2i})$ restricted to $x \times C$ is a trivial line bundle for every $x \in F'_i$ and thus, by the push-pull formula, there is a line bundle $M_{F'_i}$ on F'_i such that

$$\mathcal{O}(\Delta_{2i}) \cong M_{F'_i} K_C \mathcal{P}_i^{-2}. \quad (2.5)$$

Consider the rank 2 vector bundle

$$\mathbf{E}_i := \mathcal{P}_i \oplus \mathcal{P}_i^{-1} M_{F'_i} \quad (2.6)$$

on $C \times F'_i$ and the Higgs field

$$\Phi_i = \begin{pmatrix} 0 & \phi'_i \\ \phi_i & 0 \end{pmatrix}$$

with

$$\phi_i \in H^0(C \times E_i; \mathcal{P}_i^{-2} M_{F'_i} K_C) = H^0(C \times E_i; \mathcal{O}(\Delta_{2i}))$$

given by the universal divisor Δ_{2i} and

$$\phi'_i \in H^0(C \times E_i; \mathcal{P}_i^2 L_{E_i}^{-1} K_C) = H^0(C \times E_i; K_C^2(-\Delta_{2i}))$$

the tautological section.

Thus, (\mathbf{E}_i, Φ_i) is a family of stable SL_2 -Higgs bundles parametrized by (the total space of) E_i . Thus, we get the embedding

$$\iota_i : E_i \rightarrow \mathcal{M}(SL_2)^s \subset \mathcal{M}(SL_2).$$

We denote its image

$$L'_i := \iota_i(E_i) \subset \mathcal{M}(SL_2)$$

and note that it parametrizes all Higgs bundles of the form $M \oplus M^{-1} \xrightarrow{\Phi} MK_C \oplus M^{-1}K_C$ with Higgs field

$$\Phi = \begin{pmatrix} 0 & \phi' \\ \phi & 0 \end{pmatrix}$$

and such that $\deg M = \bar{i}$. This precisely agrees with a component of the $SU(1,1)$ -Higgs moduli space as described in [Sch] (see also [Hi2, §7] and [Hi1, §10]). Thus, L'_i is a connected smooth Lagrangian subvariety in the stable locus of $\mathcal{M}(SL_2)$. Note that there is one more component of $\mathcal{M}(SU(1,1))$ inside $\mathcal{M}(SL_2)$, namely L'_{g-1} . However, it will no longer be contained in the stable locus and will not be smooth.

We are now going to construct the vector bundles Λ_j on L'_i for $j = 0, \dots, g - 1$. Hitchin [Hi2, §7] defines

$$\Lambda_j := \Lambda^{2j}\mathbf{V}$$

locally, where \mathbf{V} is the Dirac bundle on $U \rightarrow \mathcal{M}(SL_2)^s$ for an étale open U . It is constructed as

$$\mathbf{V} := R^1 \text{pr}_{C*}(\mathbf{E}_U \xrightarrow{\Phi_U} \mathbf{E}_U K_C)$$

where

$$\mathbf{E}_U \xrightarrow{\Phi_U} \mathbf{E}_U K_C$$

is a universal SL_2 -Higgs bundle on $C \times U$. The universal Higgs bundle does not exist on the whole $C \times \mathcal{M}(SL_2)^s$ but on an étale open covering, and the obstruction to glue vanishes for the even exterior powers $\Lambda_j = \Lambda^{2j}\mathbf{V}$. We note that \mathbf{V} is a vector bundle on $\mathcal{M}(SL_2)^s$ as, for stable degree zero Higgs bundles (E, Φ) , the zeroth and second hypercohomology of $E \xrightarrow{\Phi} EK_C$ vanishes by [Ha2, Corollary 3.5].

Finally, we will extend the construction of \mathbf{V} to the whole of $\mathcal{M}(SL_2)$ as a complex of coherent sheaves by defining it for an étale open $U \rightarrow \mathcal{M}(SL_2)$ by the formula

$$\mathbf{V} := R\text{pr}_{C*}(\mathbf{E}_U \xrightarrow{\Phi_U} \mathbf{E}_U K_C)[1].$$

Then we define

$$\Lambda_j := \Lambda^{2j}\mathbf{V}$$

as a complex of coherent sheaves on the whole $\mathcal{M}(SL_2)$.

When we restrict this construction of Λ_j to L'_i for $i < g - 1$, then we have almost a universal bundle on $C \times L'_i$. Namely, (\mathbf{E}_i, Φ_i) is a family of stable SL_2 -Higgs bundles on $C \times L'_i$. However, \mathbf{E}_i is not itself an SL_2 bundle, as

$$\det(\mathbf{E}_i) = M_{F'_i}.$$

$M_{F'_i}$ is pulled back from L'_i and ultimately from F'_i , which we will see has no square root. However, on the even exterior powers, we can just tensor with integer powers of $\det(\mathbf{E}_i)$ to get the restriction of Λ_j to L'_i and so we get

$$\Lambda_j|_{L'_i} \cong M_{F'_i}^{-j} \Lambda^{2j}(R^1 \text{pr}_{L'_i*}(\mathbf{E}_i \xrightarrow{\Phi_i} \mathbf{E}_i K_C)), \tag{2.7}$$

which gives the desired vector bundle on L'_i .

We will also need \mathbb{T} -equivariant structure on Λ_j . To construct this, we note that Hitchin's construction can be done on \mathbb{T} -equivariant étale open $U \rightarrow \mathcal{M}(\mathrm{SL}_2)^s$, where, from the universal property, one gets a \mathbb{T} -equivariant structure on $\mathbf{E}_U \xrightarrow{\Phi_U} \mathbf{E}_U K_C$ (cf. [HT1, §4]). They will then glue to yield a \mathbb{T} -equivariant structure on Λ_j . The \mathbb{T} -action we endow Λ_j is the one twisted by the trivial line bundle with a weight $-j$ \mathbb{T} -action on it.

To see the \mathbb{T} -equivariant structure on $\Lambda_j|_{L'_i}$ in (2.7), we endow \mathcal{P}_i with the trivial \mathbb{T} -action and $M_{F'_i}$ with a weight minus one \mathbb{T} -action, which will give the \mathbb{T} -equivariant structure on \mathbf{E}_i in (2.6). Then (\mathbf{E}_i, Φ_i) becomes a \mathbb{T} -equivariant Higgs bundle, provided we twist the second term $\mathbf{E}_i \otimes K_C$ with a weight one \mathbb{T} -equivariant trivial line bundle. The $\Lambda_j|_{L'_i}$ will then inherit a \mathbb{T} -equivariant structure, which we will further twist with a weight $-j$ trivial line bundle.

Finally, we will need to construct the \mathbb{T} -equivariant determinant line bundle $\mathcal{L}_{\mathrm{SL}_2}$ on $\mathcal{M}(\mathrm{SL}_2)$ (which, by abuse of notation, we will abbreviate to \mathcal{L}). It is again constructed [Qi] by gluing together

$$\Lambda^{2g-2}(\mathrm{Rpr}_{C*}(\mathbf{E}_U)[1]) \tag{2.8}$$

from open sets $U \rightarrow \mathcal{M}(\mathrm{SL}_2)$ in an étale open covering.

Again on L'_i , we can compute it from our family (\mathbf{E}_i, Φ_i) and get

$$\mathcal{L}|_{L'_i} = M_{F'_i}^{-g+1} \Lambda^{2g-2}(\mathrm{Rpr}_{C*}(\mathbf{E}_i)[1]). \tag{2.9}$$

We can now define our other family of \mathbb{T} -equivariant sheaves,

$$\mathcal{L}'_i := \iota_{i*} \mathcal{L}^2|_{L'_i}, \tag{2.10}$$

on $\mathcal{M}(\mathrm{SL}_2)$, supported on L'_i . For a variable s , we also denote

$$\Lambda_{s^2} := \bigoplus_{j=0}^{2g-2} s^{2j} \Lambda_j. \tag{2.11}$$

3. Computation for $i < g - 1$

First, we start with a definition:

Definition 3.1 Let X be a semi-projective \mathbb{T} -variety (see [HV]) and \mathcal{F} an \mathbb{T} -equivariant sheaf on it. Then define

$$\chi_{\mathbb{T}}(X; \mathcal{F}) := \sum_{i,j} (-1)^i \dim(H^i(X; \mathcal{F})^{-j}) t^j,$$

where $H^i(X; \mathcal{F})^{-j}$ is the $-j$ -th weight space of the induced \mathbb{T} action on $H^i(X; \mathcal{F})$.

We expect that, for a semi-projective X , the equivariant index satisfies $\chi_{\mathbb{T}}(X; \mathcal{F}) \in \mathbb{Z}((t))$. The following then is one of our main computational tools.

Proposition 3.2 For $0 \leq i < g - 1$ and denoting $v = t^{\frac{1}{2}}$, we have

$$\begin{aligned} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_{s^2}) &= \chi_{\mathbb{T}}(L'_i; \mathcal{L}'_i \otimes \Lambda_{s^2}) \\ &= \frac{1}{2\pi i} \oint_{|z^2 - v^2| = \epsilon} z^{4i} \frac{\left((1 + \frac{s}{zv})(1 + svz) \right)^{g-1+i} \left((1 + \frac{sv}{z})(1 + \frac{sz}{v}) \right)^i}{(1 - v^2 z^2)^{2i+g-1} (1 - v^2/z^2)^{2i-g+1}} \\ &\quad \left(4 + \frac{\frac{sv}{z}}{1 + \frac{sv}{z}} + \frac{svz}{1 + svz} + \frac{4v^2 z^2}{1 - v^2 z^2} + \frac{\frac{4v^2}{z^2}}{1 - \frac{v^2}{z^2}} - \frac{\frac{s}{zv}}{1 + \frac{s}{zv}} - \frac{\frac{sz}{v}}{1 + \frac{sz}{v}} \right)^g \frac{dz}{z} \in \mathbb{Z}[[t, s]]. \end{aligned} \tag{3.3}$$

Remark 3.4 In the right-hand side of (3.3), v is treated as a fixed complex number close to 1, and ϵ is a sufficiently small positive real number. In particular, the contour $|z^2 - v^2| = \epsilon$ will have two components around the two square roots $\pm v$ of t . Thus, the result could also be computed algebraically as the sum of the residues of the differential form when z equals the two square roots of t .

Proof As L'_i is the total space of the \mathbb{T} -equivariant vector bundle E_i , where the \mathbb{T} -action has weight two, we have that

$$\begin{aligned} \chi_{\mathbb{T}}(L'_i; \mathcal{L}'_i \otimes \Lambda_{s^2}) &= \chi_{\mathbb{T}}(F'_i; \mathcal{L}^2 \otimes \Lambda_{s^2} \otimes \mathrm{Sym}_{t^2}(E_i^*)) \\ &= \int_{F'_i} \mathrm{ch}(\mathcal{L}^2) \mathrm{ch}(\Lambda_{s^2}) \mathrm{ch}(\mathrm{Sym}_{t^2}(E_i^*)) \mathrm{td}(T_{F'_i}). \end{aligned} \tag{3.5}$$

Recall [HT1, §5] that we have

$$c_1(\Delta_{2i}) = 2i \otimes \sigma + \sum_{l=1}^{2g} \xi_l \otimes e_l + \eta \otimes 1 \in H^2(C_{2i} \times C),$$

where σ is generator of $H^2(C)$, and e_1, \dots, e_{2g} are canonical symplectic generators of $H^1(C)$. Thus, $\eta \in H^2(C_{2i})$, and $\xi_1, \dots, \xi_{2g} \in H^1(C_{2i})$. A theorem of Macdonald [Mac, (6.3)] asserts that the cohomology ring $H^*(C_m)$ is generated by η and the ξ_j . It is convenient to introduce

$$\theta_j = \xi_j \xi_{j+g}$$

and

$$\theta = \sum_{j=1}^g \theta_j \in H^2(C_{2i}).$$

We note the identity

$$\left(\sum_{l=1}^{2g} \xi_l \otimes e_l \right)^2 = -2\theta\sigma \quad (3.6)$$

used below.

By Grothendieck–Riemann–Roch we find that

$$\begin{aligned} \text{ch}(E_i) &= \text{pr}_{F_i'} \left(\text{ch}(K_C^2(-\Delta_{2i})) \text{td}(C) \right) \\ &= \text{pr}_{F_i'} \left(\exp \left((4g - 4 - 2i) \otimes \sigma - \sum_{l=1}^{2g} \xi_l \otimes e_l - \eta \otimes 1 \right) (1 + (1-g)\sigma) \right) \\ &= \text{pr}_{F_i'} \left(\left((4g - 4 - 2i - \theta) \exp(-\eta) \otimes \sigma - \sum_{l=1}^{2g} \exp(-\eta) \xi_l \otimes e_l + \exp(-\eta) \otimes 1 \right) (1 + (1-g)\sigma) \right) \\ &= \text{pr}_{F_i'} \left(\left((4g - 4 - 2i - \theta + 1 - g) \exp(-\eta) \right) \otimes \sigma + \sum_{l=1}^{2g} \exp(-\eta) \xi_l \otimes e_l + \exp(-\eta) \otimes 1 \right) \\ &= (3g - 3 - 2i - \theta) \exp(-\eta) \\ &= (2g - 3 - 2i) \exp(-\eta) + \sum_{i=1}^g \exp(-\eta - \theta_i). \end{aligned} \quad (3.7)$$

Thus, introducing $\zeta = \exp(\eta)$, we have

$$\begin{aligned} \text{ch}(\text{Sym}_{t^2} E_i^*) &= \frac{1}{(1 - t^2 \zeta)^{2i-1} \prod_{i=1}^g (1 - t^2 \zeta \exp(\theta_i))} \\ &= \frac{1}{(1 - t^2 \zeta)^{2i+g-1} \exp\left(-\frac{t^2 \zeta \theta}{1 - t^2 \zeta}\right)} \end{aligned} \quad (3.8)$$

as

$$\begin{aligned} \prod_{i=1}^g (1 - t^2 \zeta \exp(\theta_i)) &= \prod_{i=1}^g (1 - t^2 \zeta (1 + \theta_i)) = \prod_{i=1}^g (1 - t^2 \zeta - t^2 \zeta \theta_i) \\ &= (1 - t^2 \zeta)^g \prod_{i=1}^g \left(1 - \frac{t^2 \zeta \theta_i}{1 - t^2 \zeta} \right) = (1 - t^2 \zeta)^g \prod_{i=1}^g \exp\left(-\frac{t^2 \zeta \theta_i}{1 - t^2 \zeta}\right) \\ &= (1 - t^2 \zeta)^g \exp\left(-\frac{t^2 \zeta \theta}{1 - t^2 \zeta}\right). \end{aligned}$$

Similar computation, using the formula for total Chern class [Mac, (14.5)], yields

$$\text{td}(T_{C_{2i}}) = \left(\frac{\eta}{1 - 1/\zeta} \right)^{2i-g+1} \exp\left(\frac{\theta}{\zeta - 1} - \frac{\theta}{\eta} \right). \quad (3.9)$$

To compute the remaining Chern characters in (3.5), we recall that

$$c_1(\mathcal{P}_{\bar{i}}) = \bar{i} \otimes \sigma + \sum_{l=1}^{2g} \tau_l \otimes e_l \in H^2(\mathcal{J}_{2\bar{i}} \times C).$$

Comparing with (2.5), we see that

$$\text{ch}(\mathcal{P}_{2\bar{i}}^2) = \text{ch}(\Delta_i) \text{ch}(K_C^{-1}) \text{ch}(M_{F'_i}^{-1}),$$

and the fact that $M_{F'_i}$ is pulled back from F'_i implies that on F'_i we have

$$\tau_l = \xi_l/2$$

and

$$\text{ch}(M_{F'_i}) = \zeta. \tag{3.10}$$

We need now to compute the \mathbb{T} -equivariant Chern character of \mathbf{E}_i . From (2.6) and the fact that \mathbb{T} acts on the second summand with weight minus one, we get

$$\begin{aligned} \text{ch}_{\mathbb{T}}(\mathbf{E}_i) &= \text{ch}(\mathcal{P}_{\bar{i}}) + t \text{ch}(\mathcal{P}_{\bar{i}}^{-1}) \text{ch}(M_{F'_i}) = \exp\left(\bar{i} \otimes \sigma + \sum_{l=1}^{2g} \xi_l/2 \otimes e_l\right) \\ &\quad + t\zeta \exp\left(-\bar{i} \otimes \sigma - \sum_{l=1}^{2g} \xi_l/2 \otimes e_l\right). \end{aligned}$$

It follows from Grothendieck–Riemann–Roch that

$$\begin{aligned} \text{ch}_{\mathbb{T}}(\mathbf{V}_i) &= -\text{ch}_{\mathbb{T}}(\text{Rpr}_{C*}(\mathbf{E}_i \xrightarrow{\Phi_i} \mathbf{E}_i K_C)) = \text{pr}_{C*}(\text{ch}_{\mathbb{T}}(\mathbf{E}_i)(-1 + t^{-1} \text{ch}(K_C)) \text{td}(C)) \\ &= \text{pr}_{C*} \left(\left(\exp\left(\bar{i} \otimes \sigma + \sum_{l=1}^{2g} \xi_l/2 \otimes e_l\right) + t\zeta \exp\left(-\bar{i} \otimes \sigma - \sum_{l=1}^{2g} \xi_l/2 \otimes e_l\right) \right) \right. \\ &\quad \left. (-1 + t^{-1} + (g-1)(1+t^{-1})\sigma) \right) \\ &= \text{pr}_{C*} \left(\left((1 + (\bar{i} - \theta/4) \otimes \sigma) + t\zeta (1 + (-\bar{i} - \theta/4) \otimes \sigma) \right) \right. \\ &\quad \left. (-1 + t^{-1} + (g-1)(1+t^{-1})\sigma) \right) \\ &= ((\bar{i} - \theta/4) + t\zeta(-\bar{i} - \theta/4))(t^{-1} - 1) + (1 + t\zeta)(g-1)(t^{-1} + 1) \\ &= \bar{i}(t^{-1} - 1)(1 - t\zeta) - \theta/4(t^{-1} - 1)(1 + t\zeta) + (g-1)(t^{-1} + 1)(1 + t\zeta) \\ &= \bar{i}(t^{-1} - 1)(1 - t\zeta) - e^{\theta/4}(t^{-1} - 1)(1 + t\zeta) + g(t^{-1} - 1)(1 + t\zeta) \\ &\quad + (g-1)(t^{-1} + 1)(1 + t\zeta). \end{aligned}$$

Thus, we have

$$\begin{aligned} \text{ch}_{\mathbb{T}}(\Lambda_s(\mathbf{V}_i)) &= \left(\frac{(1 + \frac{s}{i})(1 + st\zeta)}{(1 + s)(1 + s\zeta)} \right)^{\bar{i}} \prod_{i=1}^g \frac{(1 + se^{\theta_i/4})(1 + st\zeta e^{\theta_i/4})}{(1 + \frac{s}{i}e^{\theta_i/4})(1 + s\zeta e^{\theta_i/4})} \left(\frac{(1 + \frac{s}{i})(1 + s\zeta)}{(1 + s)(1 + st\zeta)} \right)^g \\ &\quad ((1 + s/t)(1 + s\zeta)(1 + s)(1 + st\zeta))^{g-1} \\ &= \frac{\exp\left(\frac{s\theta/4}{1+s} + \frac{st\zeta\theta/4}{1+st\zeta}\right)}{\exp\left(\frac{\frac{s}{i}\theta/4}{1+\frac{s}{i}} + \frac{s\zeta\theta/4}{1+s\zeta}\right)} ((1 + s/t)(1 + st\zeta))^{g-1+\bar{i}} ((1 + s)(1 + s\zeta))^{\bar{i}}. \end{aligned} \tag{3.11}$$

Let

$$s' := \frac{s}{\zeta^{1/2}}.$$

Recalling the extra equivariant twist we endowed Λ_i in the paragraphs after (2.7), we have

$$\text{ch}_{\mathbb{T}}(\Lambda_{s^2}) = (\text{ch}_{\mathbb{T}}(\Lambda_{s'}(\mathbf{V}_i)) + \text{ch}_{\mathbb{T}}(\Lambda_{-s'}(\mathbf{V}_i)))/2. \tag{3.12}$$

Finally, we compute

$$\begin{aligned} &\text{ch}_{\mathbb{T}}(\text{Rpr}_{C_*}(\mathbf{E}_i)[1]) \\ &= \text{pr}_{C_*} \left(- \left(\exp \left(\bar{i} \otimes \sigma + \sum_{l=1}^{2g} \xi_l/2 \otimes e_l \right) + t\zeta \exp \left(-\bar{i} \otimes \sigma - \sum_{l=1}^{2g} \xi_l/2 \otimes e_l \right) \right) \right. \\ &\quad \left. (1 + (1 - g)\sigma) \right) \\ &= \text{pr}_{C_*} \left(- (1 + (\bar{i} - \theta/4) \otimes \sigma) - t\zeta (1 + (-\bar{i} - \theta/4) \otimes \sigma) \right) (1 + (1 - g)\sigma) \\ &= \bar{i}(t\zeta - 1) + \theta/4(1 + t\zeta) + (g - 1)(t\zeta + 1) \\ &= \bar{i}(t\zeta - 1) + \sum_{i=1}^g e^{\theta_i/4}(1 + t\zeta) - (t\zeta + 1); \end{aligned}$$

thus,

$$\text{ch}_{\mathbb{T}}(\det(\text{Rpr}_{C_*}(\mathbf{E}_i)[1])) = (t\zeta)^{\bar{i}+g-1} e^{\theta/2}$$

and so, by (2.9) and (3.10),

$$\text{ch}_{\mathbb{T}}(\mathcal{L}|_{L'_i}) = t^{\bar{i}} \zeta^{\bar{i}} e^{\theta/2}. \tag{3.13}$$

Recall the following integral formula of Zagier’s from [Th, (7.2)]. For any power series $A(x) \in \mathbb{C}[[x]]$ and $B(x) \in \mathbb{C}[[x]]$,

$$\int_{C_n} A(\eta) \exp(B(\eta)\sigma) = \text{Res}_{x=0} \left(\frac{A(x)(1 + xB(x))^g}{x^{n+1}} dx \right).$$

We also let

$$z = e^{\frac{x-u}{2}} \in \mathbb{C}[[x, u]],$$

in particular $dx = \frac{2dz}{z}$. Here, u is such that

$$e^{-u} = t.$$

We also introduce

$$v := e^{\frac{-u}{2}}.$$

By recalling that $\text{pr}_{C_{2i}} : F'_i \rightarrow C_{2i}$ is a 2^{2g} cover, we will now compute

$$\begin{aligned} & \int_{F'_i} \text{ch}_{\mathbb{T}}(\mathcal{L}'_i) \text{ch}_{\mathbb{T}}(\Lambda_{s'}(\mathbf{V}_i)) \text{ch}_{\mathbb{T}}(\text{Sym}_{t^2} E_i^*) \text{td}(T_{F'_i}) \\ &= \int_{F'_i} (t\zeta)^{2i} \frac{\exp\left(\theta + \frac{s'\theta/4}{1+s'} + \frac{s't\zeta\theta/4}{1+s't\zeta} + \frac{t^2\zeta\theta}{1-t^2\zeta} + \frac{\theta}{\zeta-1}\right) \left((1+s')(1+s't\zeta)\right)^{g-1+i} \left((1+s')(1+s'\zeta)\right)^i \eta^{2i-g+1}}{\exp\left(\frac{s'\theta/4}{1+s'} + \frac{s'\zeta\theta/4}{1+s'\zeta} + \frac{\theta}{\eta}\right) (1-t^2\zeta)^{2i+g-1} (1-1/\zeta)^{2i-g+1}} \\ &= 2^{2g} \text{Res}_{x=0} z^{4i} \frac{\left((1 + \frac{s}{zv})(1 + svz)\right)^{g-1+i} \left((1 + \frac{sv}{z})(1 + \frac{sz}{v})\right)^i x^{-g}}{(1 - v^2 z^2)^{2i+g-1} (1 - v^2/z^2)^{2i-g+1}} \\ & \quad \left(1 + x \left(1 + \frac{sv}{4z} + \frac{svz/4}{1 + svz} + \frac{v^2 z^2}{1 - v^2 z^2} + \frac{v^2}{z^2} - \frac{s}{4zv} - \frac{sz}{4v} - \frac{1}{x}\right)\right)^g dx \\ &= \text{Res}_{x=0} z^{4i} \frac{\left((1 + \frac{s}{zv})(1 + svz)\right)^{g-1+i} \left((1 + \frac{sv}{z})(1 + \frac{sz}{v})\right)^i}{(1 - v^2 z^2)^{2i+g-1} (1 - v^2/z^2)^{2i-g+1}} \\ & \quad \left(4 + \frac{sv}{z} + \frac{svz}{1 + svz} + \frac{4v^2 z^2}{1 - v^2 z^2} + \frac{4v^2}{z^2} - \frac{s}{zv} - \frac{sz}{v}\right)^g dx \\ &= \frac{1}{2\pi i} \oint_{|z-v|=\epsilon} z^{4i} \frac{\left((1 + \frac{s}{zv})(1 + svz)\right)^{g-1+i} \left((1 + \frac{sv}{z})(1 + \frac{sz}{v})\right)^i}{(1 - v^2 z^2)^{2i+g-1} (1 - v^2/z^2)^{2i-g+1}} \\ & \quad \left(4 + \frac{sv}{z} + \frac{svz}{1 + svz} + \frac{4v^2 z^2}{1 - v^2 z^2} + \frac{4v^2}{z^2} - \frac{s}{zv} - \frac{sz}{v}\right)^g \frac{2dz}{z}. \end{aligned}$$

We notice that

$$\int_{F'_i} \text{ch}_{\mathbb{T}}(\mathcal{L}_i) \text{ch}_{\mathbb{T}}(\Lambda_{-s'}(V_i)) \text{ch}_{\mathbb{T}}(\text{Sym}_{t^2} E_i^*) \text{td}(T_{F'_i})$$

will yield the same result with z replaced by $-z$.

Thus, (3.12) implies the proposition. \square

4. The Symmetry for $i, j < g - 1$

First, we rewrite the right-hand side of (3.3) in a more manageable form.

Introduce the notations

$$f(z, s, v) = \frac{z^4(1 - v^2/z^2)^2(1 + \frac{s}{zv})(1 + svz)}{(1 - v^2z^2)^2(1 + \frac{sv}{z})(1 + \frac{sz}{v})} \tag{4.1}$$

and

$$h(z, s, v) = z \frac{\partial f}{\partial z} \frac{(1 + \frac{s}{zv})(1 + svz)(1 + \frac{sv}{z})(1 + \frac{sz}{v})}{(1 - v^2/z^2)(-v^2z^2 + 1)}.$$

We can compute

$$z \frac{\partial}{\partial z} \frac{f(z, s, v)}{f(z, s, v)} = \left(4 + \frac{\frac{sv}{z}}{1 + \frac{sv}{z}} + \frac{svz}{1 + svz} + \frac{4v^2z^2}{1 - v^2z^2} + \frac{\frac{4v^2}{z^2}}{1 - \frac{v^2}{z^2}} - \frac{\frac{s}{zv}}{1 + \frac{s}{zv}} - \frac{\frac{sz}{v}}{1 + \frac{sz}{v}} \right). \tag{4.2}$$

With this notation, for $i < g - 1$, we can rewrite the right-hand side of (1.1) as

$$\frac{1}{2\pi i} \oint_{|(z^2 - v^2)(1 + \frac{s}{zv})(1 + svz)| = \epsilon} \frac{h^{g-1} df}{f^i f},$$

because the integrand has no pole at $z = -s/v$ or $z = -1/vs$. Thus, for $0 \leq i < g - 1$ and $0 \leq j \leq g - 1$, (3.3) gives

$$\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_j) = \frac{1}{(2\pi i)^2} \oint_{|s| = \epsilon} \oint_{|f| = \epsilon} \frac{h^{g-1} df ds}{f^i s^{2j} f s}. \tag{4.3}$$

Proposition 4.4 *For any integers i and j we have*

$$\frac{1}{(2\pi i)^2} \oint_{|s| = \epsilon} \oint_{|f| = \epsilon} \frac{h^{g-1} df ds}{f^i s^{2j} f s} = \frac{1}{(2\pi i)^2} \oint_{|s| = \epsilon} \oint_{|f| = \epsilon} \frac{h^{g-1} df ds}{f^j s^{2i} f s}.$$

Proof To show that h is ‘symmetric’ in f and s^2 , we consider

$$w = \left(\frac{(sz + v)(vz + s)}{(szv + 1)(sv + z)} \right)^{\frac{1}{2}}.$$

Then, using (4.5), we find

$$f(w, f(z, s, v)^{\frac{1}{2}}, v) = s^2$$

and that

$$h(w, f(z, s, v)^{\frac{1}{2}}, v) = h(z, s, v).$$

One can check these functional equations by noting that h and f depend only on z^2, s^2 and zs ; thus, the substitutions $z = w$ and $s = f^{\frac{1}{2}}$ can be done polynomially.

Thus, consider $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ by

$$\varphi(z, s) = (w, f(z, s, v)^{\frac{1}{2}})$$

defined on a dense open subset of the source, by avoiding the poles of f and w , and also choosing compatible branches of the two square roots, so that

$$wf^{\frac{1}{2}} = \frac{z^2(1 - v^2/z^2)(vz + s)}{(1 - v^2z^2)(sv + z)}. \tag{4.5}$$

Then we see that

$$\begin{aligned} \varphi^* \left(\frac{df}{f} \right) &= \frac{ds^2}{s^2} = \frac{2ds}{s}, \\ \varphi^* \left(\frac{ds}{s} \right) &= \frac{df^{\frac{1}{2}}}{f^{\frac{1}{2}}} = \frac{1}{2} \frac{df}{f}, \end{aligned}$$

and so

$$\varphi^* \left(\frac{df \wedge ds}{fs} \right) = \frac{ds \wedge df}{sf}.$$

Thus, we can compute

$$\begin{aligned} \oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} &= \oint_{|f|=|s^2|=\epsilon} \varphi^* \left(\frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} \right) = \oint_{|f|=|s^2|=\epsilon} \frac{h^{g-1}}{s^i f^{2j}} \frac{ds \wedge df}{sf} \\ &= \oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{s^i f^{2j}} \frac{df \wedge ds}{fs}, \end{aligned}$$

proving the proposition. □

Remark 4.6 We can observe that

$$f(1/z, s, v) = f(z, s, v)^{-1}$$

and that

$$\frac{h(1/z, s, v)}{f(1/z, s, v)} = \frac{h(z, s, v)}{f(z, s, v)}.$$

Consequently, after the change of variables $z \mapsto 1/z$, we have

$$\begin{aligned} \oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} &= \oint_{|s^2|=|1/f|=\epsilon} \frac{h^{g-1}}{f^{2g-2-i} s^{2j}} \frac{-df \wedge ds}{fs} \\ &= \oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^{2g-2-i} s^{2j}} \frac{df \wedge ds}{fs}, \end{aligned} \tag{4.7}$$

where we used the residue theorem and the fact that the poles of the integrand are located at the zeros of f and $1/f$. Using Proposition 4.4, this yields

$$\oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} = \oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2(2g-2-j)}} \frac{df \wedge ds}{fs},$$

which is expected in light of the role of s in labelling the various exterior powers of \mathbf{V} as in (2.11).

Remark 4.8 We can also notice that, for $i < 0$,

$$\oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} = 0, \tag{4.9}$$

because the integrand has no pole at zeroes of f . From Proposition 4.4 and (4.7), we get that, for $j < 0$ and $j > 2g - 2$,

$$\oint_{|s^2|=|f|=\epsilon} \frac{h^{g-1}}{f^i s^{2j}} \frac{df \wedge ds}{fs} = 0. \tag{4.10}$$

For $0 \leq i < g - 1$, this follows from the fact that \mathbf{V} is a vector bundle of rank $4g - 4$ on $\mathcal{M}(\mathrm{GL}_2)^s$. We will also give an interpretation of this for $i = g - 1$ below in Remark 6.2.

Finally, by combining the previous Proposition 4.4 with (4.3), we have now the following result, which is identical to our Theorem 1.2 from the introduction.

Corollary 4.11 For $0 \leq i, j < g - 1$, we have

$$\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_j) = \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_j \otimes \Lambda_i)$$

5. Computation over the Moduli Stack

In this section, we will continue the study of the equivariant indices of Λ_j , but now over the *moduli stack* of Higgs bundles. This means that we will also incorporate the contribution of unstable Hitchin pairs. The benefit is that we can now apply the very powerful techniques developed by Teleman and Woodward in [TW].

As this approach has the advantage of being easily generalizable to other Lie groups, we will first take G to be a general connected finite-dimensional reductive group over \mathbb{C} , and will only specify it to be $SL(2, \mathbb{C})$ or $GL(2, \mathbb{C})$ at a later stage. We assume $\pi_1(G)$ has no torsion. Let G_0 be a real form of G , whose fundamental group again is assumed to be free. We consider the moduli space of G_0 -Higgs bundles, following [GGM]. The Lie algebra of G_0 is denoted as \mathfrak{g}_0 and satisfies $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{g}$. It admits the Cartan decomposition

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{m}.$$

Here, \mathfrak{h} is the Lie algebra of the maximal compact subgroup $H \subset G_0$ and, as the adjoint action of H preserves the Cartan decomposition, \mathfrak{m} will carry an H -representation. We denote the set of weights as $\mathfrak{X}(\mathfrak{m})$. Over the curve C , a G_0 -Higgs bundle is a pair (E, φ) , where E is a holomorphic principal $H^{\mathbb{C}}$ -bundle over C and

$$\varphi \in H^0(C, E(\mathfrak{m}^{\mathbb{C}}) \otimes K_C),$$

where $E(\mathfrak{m}^{\mathbb{C}}) := E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$ is the $\mathfrak{m}^{\mathbb{C}}$ -bundle associated to E .

One can define the moduli space $\mathcal{M}(G_0)$ of semi-stable G_0 -Higgs bundles, over which there is again a \mathbb{T} -action acting by scaling the Higgs field, and one can consider the equivariant index of various vector bundles over it. To apply the result of [TW], we will first work with the moduli stack of G_0 -Higgs bundles, which is defined as follows. Denote $E^*(\mathfrak{h}^{\mathbb{C}})$ the vector bundle associated to the universal $H^{\mathbb{C}}$ -bundle over $C \times \text{Bun}_{H^{\mathbb{C}}}$, and π the projection $C \times \text{Bun}_{H^{\mathbb{C}}} \rightarrow \text{Bun}_{H^{\mathbb{C}}}$. Then the moduli stack of G_0 -Higgs bundles is

$$\mathfrak{M}(G_0) = \underline{\text{Spec}}_{\text{Bun}_{H^{\mathbb{C}}}} \text{Sym} \left(R\pi_*(E^*(\mathfrak{h}^{\mathbb{C}}))[1] \right).$$

Over $\mathfrak{M}(G_0)$, we say a line bundle L has level $k \in H^4(BH^{\mathbb{C}}, \mathbb{Z})$ if $c_1(L)$ is the image of k under the injective transgression homomorphism $H^4(BH^{\mathbb{C}}, \mathbb{Z}) \rightarrow H^2(\text{Bun}_{H^{\mathbb{C}}}, \mathbb{Z})$; k defines a quadratic form on \mathfrak{h} and, following [TW], we call a line bundle *admissible* if $k > -h$ has a quadratic form with $h := -\frac{1}{2} \text{Tr}_{\mathfrak{h}}$. For a line bundle L of level k , we will usually refer to it as \mathcal{L}^k , with slightly abusive use of notation, to make the level explicit. When $H^4(BH^{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}$, k is an integer, and \mathcal{L}^k will be the k th tensor power of the level-one line bundle \mathcal{L} ; \mathbb{T} also acts on $\mathfrak{M}(G_0)$ by scaling the Higgs field, and we will consider admissible line bundles that are equivariant, whose equivariant index is what we are after. But, before presenting the formula for this index, we first need to introduce some notations.

Consider the function on the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}^{\mathbb{C}}$,

$$D : \quad \xi \mapsto \frac{k+h}{2} (\xi, \xi) - \text{Tr}_{\mathfrak{h}} \left(\text{Li}_2(te^{\xi}) \right),$$

and the map from T , the Cartan of $H^{\mathbb{C}}$, to T^{\vee} :

$$\chi_t = e^{(k+h)(\cdot, \cdot)} \prod_{\mu \in \mathfrak{X}(\mathfrak{m})} (1 - te^{\mu})^{\mu}.$$

We let F_t be the preimage of $e^{2\pi i\rho}$ in the regular part of T . It is obvious that F_t is invariant under the action of the Weyl group W_H of H . Combined with the Killing form, D defines an endomorphism on \mathfrak{t} which we denote as H_t^\dagger ; finally, we define

$$\theta_t := \frac{\prod_\alpha (1 - e^\alpha) \prod_{\mu \in \mathfrak{R}(\mathfrak{m})} (1 - te^\mu)}{\det H_t^\dagger},$$

where α in the first product in the numerator runs over all roots of \mathfrak{h} . Then we have the following theorem.

Theorem 5.1 (Equivariant Verlinde formula for real groups) *The equivariant index of \mathfrak{L}^k over $\mathfrak{M}(G_0)$ is given by*

$$\chi_{\mathbb{T}}(\mathfrak{M}(G_0), \mathfrak{L}^k) = \sum_{f \in F_t/W_H} \theta_t(f)^{1-g}. \tag{5.2}$$

Proof From the map $\mathfrak{M}(G_0) \rightarrow \text{Bun}_{\mathbb{H}^{\mathbb{C}}}$, we have

$$\chi_{\mathbb{T}}(\mathfrak{M}(G_0), \mathfrak{L}^k) = \chi \left(\text{Bun}_{\mathbb{H}^{\mathbb{C}}}, \mathfrak{L}^k \otimes \Lambda_{-t} R\pi_* (E^*(\mathfrak{h}^{\mathbb{C}})) \right).$$

Here, χ on the right-hand side denotes the usual (non-equivariant) index of a K-theory class over $\text{Bun}_{\mathbb{H}^{\mathbb{C}}}$. Also, we have abusively used \mathfrak{L}^k to denote its restriction to $\text{Bun}_{\mathbb{H}^{\mathbb{C}}}$, and used the identity $\text{Sym}_t(V[1]) = \Lambda_{-t}V$ for arbitrary class K-theory V . Then, the right-hand side of the equation can be handled by Theorem 2.15 of [TW]. The computation is a straightforward modification of that in the proof of Theorem 7 in [AGP], with the adjoint representation now replaced with \mathfrak{m} . \square

Remark 5.3 In fact, the theorem holds for more general non-compact subgroups $G_0 \subset G$ that are not real forms of G . For example, it holds when $G_0 = G$ is the entire group, for which the index formula just becomes the equivariant Verlinde formula for G studied in [GP, GPYY, AGP, HL].

Example 5.4 ($G_0 = \text{SL}(2, \mathbb{R})$) In this case, $H = \text{U}(1)$, and $\mathfrak{m}^{\mathbb{C}}$ is the two-dimensional representation that decomposes as $\mathbb{C}_{+2} \oplus \mathbb{C}_{-2}$, where the subscripts denote the weights of the $\text{U}(1)$ -action. We parametrize H by an angle ϑ in $[0, 2\pi)$. Then F_t is the set of solutions to the equation

$$e^{2ki\vartheta} \left(\frac{1 - te^{-2i\vartheta}}{1 - te^{2i\vartheta}} \right)^2 = 1,$$

and

$$\theta_t = \frac{(1 - te^{2i\vartheta})(1 - te^{-2i\vartheta})}{\det H_t^\dagger},$$

where H_t^\dagger is the Hessian of

$$D: \vartheta \mapsto -k\vartheta^2 - (\text{Li}_2(te^{2i\vartheta}) + \text{Li}_2(te^{-2i\vartheta})).$$

Example 5.5 ($G_0 = U(1, 1)$) This is very similar to the previous case. We now have $H = U(1) \times U(1)$ and $\mathfrak{m}^\mathbb{C} = \mathbb{C}_{1,-1} \oplus \mathbb{C}_{-1,1}$, where subscripts again denote weights under $U(1) \times U(1)$. We can identify H with diagonal 2×2 matrices $\text{diag}\{e^{i\vartheta_1}, e^{i\vartheta_2}\}$, and it is convenient to consider the combination $\vartheta = (\vartheta_1 - \vartheta_2)/2$, $\vartheta' = (\vartheta_1 + \vartheta_2)/2$, both in $[0, 2\pi)$. The level is given by two integers k and k' , corresponding to, respectively, the non-abelian and abelian parts of $U(1, 1)$. Now, F_t is given by the solutions to the following set of equations:

$$\begin{aligned} e^{2ki\vartheta} \left(\frac{1 - te^{-2i\vartheta}}{1 - te^{2i\vartheta}} \right)^2 &= 1, \\ e^{2k'i\vartheta'} &= 1 \end{aligned}$$

and

$$\theta_t = \frac{(1 - te^{2i\vartheta})(1 - te^{-2i\vartheta})}{\det H_t^\dagger},$$

where H_t^\dagger is the Hessian of

$$D: \vartheta \mapsto -k\vartheta^2 - k'\vartheta'^2 - (\text{Li}_2(te^{2i\vartheta}) + \text{Li}_2(te^{-2i\vartheta})).$$

It is easy to relate the $U(1, 1)$ case to the previous case of $G_0 = \text{SL}(2, \mathbb{R})$, as the equations for ϑ are the same, while the new variable ϑ' takes value in $\{0, \frac{\pi}{k}, \frac{2\pi}{k}, \dots, \frac{(2k-1)\pi}{k}\}$. Using

$$\det H_t^{\dagger, U(1,1)} = \det H_t^{\dagger, \text{SL}(2, \mathbb{R})} \cdot k',$$

we have

$$\chi_{\mathbb{T}}(\mathfrak{M}_{U(1,1)}, \Omega^k) = k'^g \cdot \chi_{\mathbb{T}}(\mathfrak{M}_{\text{SL}(2, \mathbb{R})}, \Omega^k).$$

In our study of mirror symmetry between moduli spaces of Higgs bundles, it is also useful to consider indices of exterior powers of \mathbf{V} . On the moduli space, \mathbf{V} can only be defined locally, but it exists globally on the stack $\mathfrak{M}(\text{SL}(2, \mathbb{C}))$ as a hypercohomology complex $\text{Rpr}_{\mathbb{C}^*}(\mathbf{E} \xrightarrow{\Phi} \mathbf{E} \otimes K_{\mathbb{C}})[1]$, which we will continue to denote as \mathbf{V} .

There are several moduli stacks we would like to consider, with natural morphisms between them:

$$\begin{array}{ccc} \text{Bun}_{\mathbb{C}^\times} & \rightarrow & \text{Bun}_{\text{SL}(2, \mathbb{C})} \\ \downarrow & & \downarrow \\ \mathfrak{M}(\text{SU}(1, 1)) & \rightarrow & \mathfrak{M}(\text{SL}(2, \mathbb{C})). \end{array}$$

The vertical morphisms are closed immersions, while the horizontal ones are not. The morphism at the bottom will also become a closed immersion if $\mathfrak{M}(\mathrm{SU}(1, 1))$ is replaced with $\mathfrak{M}'(\mathrm{SU}(1, 1)) = \mathfrak{M}(\mathrm{SU}(1, 1))/W$, where $W = \mathbb{Z}/2\mathbb{Z}$ is the Weyl group of $\mathrm{SL}(2, \mathbb{C})$. The complex \mathbf{V} can be restricted to $\mathfrak{M}'(\mathrm{SU}(1, 1))$ as well as be pulled back to every corner of the above diagram. To avoid clutter, we will continue denoting the resulting complexes on $\mathfrak{M}'(\mathrm{SU}(1, 1))$ and $\mathfrak{M}(\mathrm{SU}(1, 1))$ as \mathbf{V} , while we will denote the pullback to $\mathrm{Bun}_{\mathbb{C}^\times}$ and $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})}$ in a different font as \mathfrak{V} . On the other hand, we will use \mathfrak{L}^k for the k th tensor power of the determinant line bundle that also exists on all the five stacks mentioned above.

To study the index of \mathfrak{V} and its exterior powers, it is useful to have the following proposition.

Proposition 5.6 *In the rational equivariant K-theory of $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})}$, we have*

$$\mathfrak{V} = (v^{-1} + v)(g - 1)\mathbf{E}_x + (v^{-1} - v)\mathbf{E}_C,$$

where \mathbf{E}_x and \mathbf{E}_C are obtained from the rank 2 universal bundle \mathbf{E} over $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})} \times \mathbb{C}$ by, respectively, restricting to a point $x \in \mathbb{C}$ and slanting with the fundamental class of \mathbb{C} .

Proof As $\mathfrak{V} = v^{-1}R\pi_*(\mathbf{E} \otimes K_C) - vR\pi_*(\mathbf{E})$, the proposition directly follows from two identities in the K-theory of $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})}$,

$$R\pi_*(\mathbf{E}) = (1 - g)\mathbf{E}_x + \mathbf{E}_C,$$

and

$$R\pi_*(\mathbf{E} \otimes K_C) = (g - 1)\mathbf{E}_x + \mathbf{E}_C. \quad \square$$

Now we consider the index of $\Lambda_s \mathbf{V}$ on the moduli stack of $\mathrm{SU}(1, 1)$ -Higgs bundles $\mathfrak{M}(\mathrm{SU}(1, 1))$. As the maximal compact subgroup of $\mathrm{SU}(1, 1)$ is $\mathrm{U}(1)$, we use the morphism

$$p : \mathfrak{M}(\mathrm{SU}(1, 1)) \rightarrow \mathrm{Bun}_{\mathbb{C}^\times} = \mathrm{Pic}(\mathbb{C}) \times B\mathbb{C}^\times$$

to reduce the index computation over $\mathfrak{M}(\mathrm{SU}(1, 1))$ to an index computation over $\mathrm{Bun}_{\mathbb{C}^\times}$. To give the index formula explicitly, we parametrize the complexified maximal compact subgroup $\mathrm{H}^{\mathbb{C}} = \mathbb{C}^\times$ by z , and denote the set of solutions of

$$z^{2k} \left(\frac{1 - v^2 z^{-2}}{1 - v^2 z^2} \right)^2 \left(\frac{1 + sv^{-1}z^{-1}}{1 + sv^{-1}z} \right) \left(\frac{1 + svz}{1 + svz^{-1}} \right) = 1 \tag{5.7}$$

as $F'_{t,s}$. Define

$$\theta'_{t,s} := \frac{1}{\det H'_{t,s}} \cdot \frac{(1 - v^2 z^2)(1 - v^2 z^{-2})}{(1 + sv^{-1}z)(1 + sv^{-1}z^{-1})(1 + svz)(1 + svz^{-1})}, \tag{5.8}$$

where $\det H_{t,s}^{\dagger}$ is given by

$$\left[\left(2k + \frac{4v^2 z^2}{1 - v^2 z^2} + \frac{4v^2 z^{-2}}{1 - v^2 z^{-2}} - \frac{sv^{-1}z}{1 + sv^{-1}z} - \frac{sv^{-1}z^{-1}}{1 + sv^{-1}z^{-1}} + \frac{svz}{1 + svz} + \frac{svz^{-1}}{1 + svz^{-1}} \right) \right]. \quad (5.9)$$

Then we have

Theorem 5.10 *For \mathcal{L}^k a line bundle on $\mathfrak{M}(\mathrm{SU}(1, 1))$ with level $k \geq 0$, the \mathbb{T} -equivariant index of $\mathcal{L}^k \otimes \Lambda_s \mathbf{V}$ is given by*

$$\chi_{\mathbb{T}} \left(\mathfrak{M}(\mathrm{SU}(1, 1)), \mathcal{L}^k \otimes \Lambda_s \mathbf{V} \right) = \sum_{f \in F'_{t,s}} \theta'_{t,s}(f)^{1-g}. \quad (5.11)$$

Proof When $s = 0$, the theorem becomes the Verlinde formula for $\mathrm{SU}(1, 1)$ -Higgs bundles discussed earlier. We now highlight the effect of turning on s .

The first step is to rewrite $\Lambda_s \mathfrak{Y}$ on $\mathrm{Bun}_{\mathbb{C}^\times}$ into a form that Theorem 2.15 of [TW] can handle. To achieve this, we will start with rewriting $\Lambda_s \mathfrak{Y}$ over $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})}$. Recall first that, from the universal $\mathrm{SL}(2, \mathbb{C})$ -bundle over $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})} \times \mathbb{C}$ and a $\mathrm{SL}(2, \mathbb{C})$ representation R , one can obtain a vector bundle, denoted as $E(R)$. We obviously have $\mathbf{E} = E(R_F)$, where R_F is the two-dimensional fundamental representation of $\mathrm{SL}(2, \mathbb{C})$. One can obtain classes $E_x(R)$ and $E_C(R)$ in the K-theory of $\mathrm{Bun}_{\mathrm{SL}(2, \mathbb{C})}$ similar to the construction of \mathbf{E}_x and \mathbf{E}_C . Using the identity

$$\Lambda_s = \exp \left[- \sum_{p>0} \frac{(-s)^p \psi^p}{p} \right]$$

with ψ^p denoting the p th Adams operation and the relation $\psi^p E_C(R) = E_C(\psi^p R)/p$, one can rewrite

$$\Lambda_s \mathfrak{Y} = E_x(\Lambda_s((v + v^{-1})R_F))^{\otimes(g-1)} \otimes \exp \left[- \sum_{p=1}^{\infty} \frac{(-s)^p E_C(\psi^p((v^{-1} - v)R_F))}{p^2} \right]. \quad (5.12)$$

Similarly, on $\mathrm{Bun}_{\mathbb{C}^\times}$, we denote $E(R_n)$ the line bundle associated to the universal line bundle via the weight- n representation over $\mathrm{Bun}_{\mathbb{C}^\times} \times \mathbb{C}$. As before, we can construct K-theory classes $E_x(R_n)$ and $E_C(R_n)$ over $\mathrm{Bun}_{\mathbb{C}^\times}$. From (5.12), one can deduce the following identity in the K-theory of $\mathrm{Bun}_{\mathbb{C}^\times}$:

$$\begin{aligned} \Lambda_s \mathfrak{Y} &= E_x(\Lambda_s((v + v^{-1})(R_1 + R_{-1})))^{\otimes(g-1)} \\ &\otimes \exp \left[- \sum_{p=1}^{\infty} \frac{((-s/v)^p - (sv)^p) E_C(R_p + R_{-p})}{p^2} \right]. \end{aligned} \quad (5.13)$$

Since

$$\mathfrak{M}(\mathrm{SU}(1, 1)) = \underline{\mathrm{Spec}}_{\mathrm{Bun}_{\mathbb{C}^\times}} \mathrm{Sym}(R\pi_*E(R_2 + R_{-2}))[1],$$

we have the equality

$$\chi_{\mathbb{T}}\left(\mathfrak{M}(\mathrm{SU}(1, 1)), \mathcal{L}^k \otimes \Lambda_s \mathbf{V}\right) = \chi\left(\mathrm{Bun}_{\mathbb{C}^\times}, \mathcal{L}^k \otimes \Lambda_s \mathfrak{Y} \otimes \Lambda_{-t} R\pi_*E(R_2 + R_{-2})\right),$$

and the theorem follows from a direct computation using Theorem 2.15 of [TW] applied to $\mathrm{Bun}_{\mathbb{C}^\times}$. With the parametrization of the complexified maximal compact subgroup of $\mathrm{SU}(1, 1)$ by z , the character of R_n is given by z^n . Then Theorem 2.15 of [TW] tells us that the first factor $E_x(\Lambda_s((v + v^{-1})(R_1 + R_{-1}))^{\otimes(g-1)})$ in (5.13) will result in an additional s -dependent factor,

$$\mathrm{Tr}_{R_s}(z),$$

in $\theta_{s,t}'^{1-g}$, with R_s being the virtual representation $(\Lambda_s((v + v^{-1})(R_1 + R_{-1}))^{\otimes(g-1)})$. It is easy to see that the character of R_s is given by

$$\left[(1 + svz)(1 + svz^{-1})(1 + sv^{-1}z^{-1})(1 + sv^{-1}z)\right]^{g-1}.$$

On the other hand, the second factor in (5.13) will modify both $\theta'_{s=0,t}$ and $F'_{s=0,t}$. The modification to the former is only through $\det H_{t,s}^{\dagger}$, which is now given by the Hessian of the function

$$D'_s = D'_{s=0} + \sum_{p=1}^{\infty} \frac{((-sv)^p - (-s/v)^p) \mathrm{Tr}_{\psi^p(R_1+R_{-1})}(z)}{p^2}.$$

Since $\mathrm{Tr}_{\psi^p(R_1+R_{-1})}(z) = z^p + z^{-p}$, the summation over p in the above can be written as the sum of four terms,

$$\begin{aligned} \sum_{p=1}^{\infty} \frac{((-sv)^p - (-s/v)^p) (z^p + z^{-p})}{p^2} &= \mathrm{Li}_2(-svz^{-1}) + \mathrm{Li}_2(-svz) - \mathrm{Li}_2(-sv^{-1}z) \\ &\quad - \mathrm{Li}_2(-sv^{-1}z^{-1}). \end{aligned}$$

Computing the Hessian of D'_s gives the desired $\det H_{t,s}^{\dagger}$.

Now we determine the set $F'_{s,t}$. When $s = 0$, it is given by the set of solutions to the ‘Bethe ansatz equations’,

$$z^{2k} \left(\frac{1 - v^2 z^{-2}}{1 - v^2 z^2} \right)^2 = 1.$$

When s is turned on, from Theorem 2.15 of [TW], a new factor will be included in the left-hand side of the above equation, given by

$$\exp \left[- \sum_{p=1}^{\infty} \frac{((-sv)^p - (-s/v)^p)(z^p - z^{-p})}{p} \right] = \left(\frac{1 + sv^{-1}z^{-1}}{1 + sv^{-1}z} \right) \left(\frac{1 + svz}{1 + svz^{-1}} \right).$$

Taking into account all three modifications proves the theorem. □

In previous sections, we studied the index over the smooth part of the ‘real slice’ $\mathfrak{M}'(\mathrm{SU}(1, 1)) \subset \mathfrak{M}(\mathrm{SL}(2, \mathbb{C}))$, for which we have the following.

Theorem 5.14 *The equivariant index of $\mathfrak{L}^k \otimes \Lambda_s \mathbf{V}$ on $\mathfrak{M}'(\mathrm{SU}(1, 1))$,*

$$\chi_{\mathbb{T}} \left(\mathfrak{M}'(\mathrm{SU}(1, 1)), \mathfrak{L}^k \otimes \Lambda_s \mathfrak{V} \right) = \frac{1}{2} \sum_{f \in F'_{t,s}} \theta'_{t,s}(f)^{1-g}, \tag{5.14}$$

is equal to one-half of the equivariant index $\chi_{\mathbb{T}}(\mathfrak{M}(\mathrm{SU}(1, 1)), \mathfrak{L}^k \otimes \Lambda_s \mathbf{V})$ on $\mathfrak{M}(\mathrm{SU}(1, 1))$.

Proof This theorem follows from the facts that $\mathfrak{M}'(\mathrm{SU}(1, 1)) = \mathfrak{M}(\mathrm{SU}(1, 1))/W$, and $\mathfrak{L}^k \otimes \Lambda_s \mathbf{V}$ is invariant under the action of the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$. □

6. Symmetry for $i = g - 1$

To compare the formula (5.15) for $k = 2$ with the formula (4.3), we rewrite it in the same residue form. First, we note that the left-hand side of (5.7) for $k = 2$ equals $f(z, s, v)$ from (4.1). We also see that (5.9) agrees with

$$z \frac{\partial}{\partial z} f(z, s, v),$$

$$f(z, s, v),$$

which is computed in (4.2). Finally, we see from (5.8) that

$$\theta'_{t,s} = \frac{f(z, s, v)}{h(z, s, v)}.$$

By recalling that $F'_{t,s}$ denotes the set of solutions of $f = 1$, which are all simple zeroes, we can compute

$$\begin{aligned} \sum_{\varphi \in F'_{t,s}} \theta'_{t,s}(\varphi)^{1-g} &= \sum_{a \in F'_{t,s}} \left(\frac{h(a, s, v)}{f(a, s, v)} \right)^{g-1} = \frac{1}{2\pi i} \oint_{|1-f|=\epsilon} \left(\frac{h}{f} \right)^{g-1} \frac{1}{1-f} \frac{-df}{f} \\ &= \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f} \right)^{g-1} \frac{1}{1-f} \frac{df}{f} + \frac{1}{2\pi i} \oint_{|z^2 v^2 - 1|=\epsilon} \left(\frac{h}{f} \right)^{g-1} \frac{1}{1-f} \frac{df}{f} \\ &= \frac{1}{2\pi i} \oint_{|f|=\epsilon} \sum_{j=0}^{\infty} \left(\frac{h}{f} \right)^{g-1} f^j \frac{df}{f} - \frac{1}{2\pi i} \oint_{|z^2 v^2 - 1|=\epsilon} \sum_{j=1}^{\infty} \left(\frac{h}{f} \right)^{g-1} f^{-j} \frac{df}{f} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi i} \oint_{|f|=\epsilon} \sum_{j=0}^{\infty} \left(\frac{h}{f}\right)^{g-1} f^j \frac{df}{f} + \frac{1}{2\pi i} \oint_{|f|=\epsilon} \sum_{j=1}^{\infty} \left(\frac{h}{f}\right)^{g-1} f^{-j} \frac{df}{f} \\
 &= \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f}\right)^{g-1} \frac{df}{f} + \sum_{i=0}^{g-2} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_{s^2}) + \frac{1}{2\pi i} \oint_{|f|=\epsilon} \sum_{j=1}^{\infty} \left(\frac{h}{f}\right)^{g-1} f^j \frac{df}{f} \\
 &= \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f}\right)^{g-1} \frac{df}{f} + 2 \sum_{i=0}^{g-2} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_{s^2}).
 \end{aligned}$$

In the third equation, we used the residue theorem and the fact that the poles of the integrand are the zeroes of $1 - f$, f and $z^2v^2 - 1$. In the fourth equation we see that the power series expansions converge on the respective contours. In the fifth equation, we used the residue theorem again, noting that the only poles of the integrand are the zeroes of f and $z^2v^2 - 1$. In the sixth, we used (4.3), (4.9) and (4.7). Finally, in the seventh equation, we used (4.3) and (4.9) again.

Thus, from (5.14),

$$\begin{aligned}
 \chi_{\mathbb{T}}(\mathfrak{M}'(\mathrm{SU}(1, 1)), \mathcal{L}^2 \otimes \Lambda_s \mathfrak{Y}) &= \frac{1}{2} \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f}\right)^{g-1} \frac{df}{f} \\
 &+ \sum_{j=1}^{g-1} \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f}\right)^{g-1} f^j \frac{df}{f}. \tag{6.1}
 \end{aligned}$$

Remark 6.2 From the observation in Remark 4.8, we see that, even though \mathbf{V} and, consequently, Λ_s are no longer vector bundles on $\mathfrak{M}'(\mathrm{SU}(1, 1))$, cohomologically, \mathbf{V} behaves like a rank $4g - 4$ vector bundle on $\mathfrak{M}'(\mathrm{SU}(1, 1))$. Interestingly, the same observation about the virtual Dirac bundle \mathbf{V} was the crucial point in the proof of the vanishing of the intersection form on the rank 2 odd degree Higgs moduli space in [Ha1, Ha2].

From (6.1), we also get

$$2\chi_{\mathbb{T}}(\mathfrak{M}'(\mathrm{SU}(1, 1)), \mathcal{L}^2 \otimes \Lambda_s \mathfrak{Y}) - 2 \sum_{i=0}^{g-2} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_{s^2}) = \frac{1}{2\pi i} \oint_{|f|=\epsilon} \left(\frac{h}{f}\right)^{g-1} \frac{df}{f}.$$

We can now combine Theorem 5.14 and Proposition 4.4 with the computation above and deduce the following:

Corollary 6.3 For $0 \leq j < g - 1$, we have

$$\begin{aligned}
 &2\chi_{\mathbb{T}}(\mathfrak{M}'(\mathrm{SU}(1, 1)), \mathcal{L}^2 \otimes \Lambda^{2j} \mathfrak{Y}) - 2 \sum_{i=0}^{g-2} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i \otimes \Lambda_j) \\
 &= \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_j \otimes \Lambda_{g-1}).
 \end{aligned}$$

Remark 6.4 For a completely symmetrical picture, we would like to see the left-hand side agree with $\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_{g-1} \otimes \Lambda_j)$ with the special convention for \mathcal{L}_{g-1} in (2.3). It seems, however, that the non-semi-stable locus in $\mathfrak{M}(\mathrm{SU}(1, 1))$ will contribute non-trivially to the equivariant index. We expect that a modification of the extension of the Dirac complex $\Lambda_s \mathbf{V}$ to $\mathfrak{M}(\mathrm{SL}_2)$ will solve this problem.

7. Reflection of Mirror Symmetry

If \mathcal{F}_1 and \mathcal{F}_2 are \mathbb{T} -equivariant coherent sheaves on a semi-projective (cf. [HV]) \mathbb{T} -variety X , then we can define (cf. [Gu, (2.23),(3.4)]) the *equivariant Euler form* as

$$\begin{aligned} \chi_{\mathbb{T}}(X; \mathcal{F}_1, \mathcal{F}_2) &= \sum_{k,l} \dim(H^k(R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2))^l) (-1)^k t^{-l} \\ &= \sum_{k,l} \dim(\mathrm{Hom}_{\mathbf{D}_{coh}(X)}(\mathcal{F}_1, \mathcal{F}_2[k])^l) (-1)^k t^{-l} \\ &= \sum_{k,l} \dim(\mathrm{Ext}^k(X; \mathcal{F}_1, \mathcal{F}_2)^l) (-1)^k t^{-l} \in \mathbb{C}((t)), \end{aligned}$$

which we can compute by Hirzebruch–Riemann–Roch and localization:

$$\begin{aligned} \chi_{\mathbb{T}}(X; \mathcal{F}_1, \mathcal{F}_2) &= \int_X \mathrm{ch}_{\mathbb{T}}(\mathcal{F}_1)^* \mathrm{ch}_{\mathbb{T}}(\mathcal{F}_2) \mathrm{td}_{\mathbb{T}}(X) \\ &= \int_{X^{\mathbb{T}}} \mathrm{ch}_{\mathbb{T}}(\mathcal{F}_1)^* |_{X^{\mathbb{T}}} \mathrm{ch}_{\mathbb{T}}(\mathcal{F}_2|_{X^{\mathbb{T}}}) \mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}(N_{X^{\mathbb{T}}}^*)) \mathrm{td}(X^{\mathbb{T}}), \end{aligned}$$

where $N_{X^{\mathbb{T}}}$ is the normal bundle of $X^{\mathbb{T}}$ in X and for

$$a = a_0 + a_2 + \dots + a_{2\dim(X)} \in H^{2*}(X).$$

We define the class

$$a^* = \sum (-1)^i a_{2i} \in H^{2*}(X).$$

Corollary 7.1 *When $0 \leq i, j < g - 1$,*

$$\chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \mathcal{L}_i, \Lambda_j) = -t^{g-1} \chi_{\mathbb{T}}(\mathcal{M}(\mathrm{GL}_2); \Lambda_i, \mathcal{L}_j).$$

Proof We expect that

$$\mathcal{L}_i^{\vee} \cong \mathcal{L}_i[-4g + 3] \tag{7.2}$$

and

$$\Lambda_i^{\vee} \cong \Lambda_i[-2g], \tag{7.3}$$

as self-dualities in the derived category $D_{coh}(\mathcal{M}(\mathrm{GL}_2))$. Indeed, (7.3) holds on $\mathcal{M}(\mathrm{GL}_2)^s$ because there it is a push-forward of a vector bundle from $\mathcal{M}(\mathrm{SL}_2)^s$ of codimension $2g$ with a trivial normal bundle and, as [Hi2, §7, Remarks 1, 2] argues, the vector bundle $\Lambda_i|_{\mathcal{M}(\mathrm{SL}_2)^s}$ is self-dual, due to the quaternionic structure on it. On the other hand, (7.2) holds for $i < g - 1$, because of [Hu, Corollary 3.40], using that $\mathcal{M}(\mathrm{GL}_2)^s$ is symplectic and by the following

Lemma 7.4 *For $0 \leq i < g - 1$, we have*

$$K_{L_i} = \mathcal{L}^4|_{L_i} = \mathcal{L}_i^2|_{L_i}. \quad \square$$

Proof This follows from $c_1(K_{L'_i}|_{F'_i}) = c_1(\mathcal{L}^4|_{F'_i})$. Starting with [Mac, (14.10)] and (3.7), we compute

$$\begin{aligned} c_1(K_{L'_i}|_{F'_i}) &= c_1(K_{F'_i})c_1(E_i^*) = (g - 2i - 1)\eta + \theta + (3g - 3 - 2i)\eta + \theta \\ &= (4g - 4 - 4i)\eta + 2\theta = 4c_1(\mathcal{L}|_{F'_i}). \end{aligned}$$

from (3.13). □

To also trace how the \mathbb{T} -action intertwines these isomorphisms, we compute using [CG, Lemma 5.4.9]:

$$\begin{aligned} \mathrm{ch}_{\mathbb{T}}(\mathcal{L}_i)^*|_{L_i} &= \mathrm{ch}_{\mathbb{T}}\left(\sum_j (-1)^j \Lambda_j N_{L_i}^*\right)^* \mathrm{ch}_{\mathbb{T}}(\mathcal{L}^{-2}|_{L_i}) \\ &= \mathrm{ch}_{\mathbb{T}}\left(\sum_j (-1)^j \Lambda_j N_{L_i}\right) \mathrm{ch}_{\mathbb{T}}(\mathcal{L}^{-2}|_{L_i}) \\ &= \frac{(-1)^{4g-3} e^{c_1^{\mathbb{T}}(N_{L_i})}}{\mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}N_{L_i}^*)} \mathrm{ch}_{\mathbb{T}}(\mathcal{L}^{-2}|_{L_i}). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathrm{ch}_{\mathbb{T}}(\mathcal{L}_i)^*|_{F'_i} &= \frac{(-1)^{4g-3} e^{c_1^{\mathbb{T}}(N_{L_i}|_{F'_i})}}{\mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}N_{L_i}^*)} \mathrm{ch}_{\mathbb{T}}(\mathcal{L}^{-2}|_{F'_i}) = \frac{(-1)^{4g-3} e^{c_1^{\mathbb{T}}(E_i^*) + c_1^{\mathbb{T}}(T_{F'_i}^*) + c_1^{\mathbb{T}}(\mathcal{L}^{-2}|_{F'_i})}}{\mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}N_{L_i}^*)} \\ &= \frac{(-1)^{4g-3} e^{(g-1+2\bar{i})(\eta-u) + \theta + (g-1-2i)\eta + \theta + (g+2i)u - 2\bar{i}(\eta-u) - \theta}}{\mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}N_{L_i}^*)} \\ &= \frac{-\mathrm{ch}_{\mathbb{T}}(\mathcal{L}^2|_{F'_i})t^{-1}}{\mathrm{ch}_{\mathbb{T}}(\mathrm{Sym}N_{L_i}^*)} = -\mathrm{ch}_{\mathbb{T}}(\mathcal{L}_i|_{F'_i})t^{-1}. \end{aligned} \tag{7.5}$$

Here, we used

$$c_1^{\mathbb{T}}(E_i^*) = (g - 1 + 2\bar{i})(\eta - u) + \theta$$

from (3.7) and the facts that $\text{rank}(E_i^*) = g - 1 + 2\bar{i}$ and the \mathbb{T} -action has weight minus one on E_i^* . We also used

$$c_1^{\mathbb{T}}(T_{F_i}^*) = (g - 1 - 2i)\eta + \theta + (g + 2i)u$$

from [Mac, (14.10)] and $\text{rank}(T_{F_i}^*) = g + 2i$ and the \mathbb{T} -action has weight one on $T_{F_i}^*$. Finally,

$$c_1^{\mathbb{T}}(\mathcal{L}^{-2}|_{F_i'}) = -2\bar{i}(\eta - u) - \theta,$$

from (3.13). Similarly,

$$\text{ch}(\Lambda_j)^*|_{F_i'} = \frac{(-1)^{2g} t^{-g} \text{ch}_{\mathbb{T}}(\Lambda_j^*|_{F_i'})}{\text{ch}_{\mathbb{T}}(\text{Sym}N_{F_i'}^{F_i})} = \frac{t^{-g} \text{ch}_{\mathbb{T}}(\Lambda_j|_{F_i'})}{\text{ch}_{\mathbb{T}}(\text{Sym}N_{F_i'}^{F_i})},$$

because, from (3.11) and (3.12), we can deduce

$$\text{ch}_{\mathbb{T}}(\Lambda_j^*|_{F_i'}) = \text{ch}_{\mathbb{T}}(\Lambda_j|_{F_i'}).$$

Thus, we can compute from (7.5)

$$\begin{aligned} \chi_{\mathbb{T}}(\mathcal{M}(\text{GL}_2); \mathcal{L}_i, \Lambda_j) &= \int_{F_i'} \text{ch}_{\mathbb{T}}(\mathcal{L}_i)^*|_{F_i'} \text{ch}_{\mathbb{T}}(\Lambda_j|_{F_i'}) \text{ch}_{\mathbb{T}}(\text{Sym}N_{F_i'}^*) \text{td}(F_i') \\ &= \int_{F_i'} -t^{-1} \text{ch}(\mathcal{L}^2|_{F_i'}) \text{ch}_{\mathbb{T}}(\Lambda_j|_{F_i'}) \text{ch}_{\mathbb{T}}(\text{Sym}E_i^*) \text{td}(F_i') = -t^{-1} \chi_{\mathbb{T}}(\mathcal{M}(\text{GL}_2); \mathcal{L}_i \otimes \Lambda_j). \end{aligned}$$

On the other side, we have

$$\begin{aligned} \chi_{\mathbb{T}}(\mathcal{M}(\text{GL}_2); \Lambda_i, \mathcal{L}_j) &= \int_{F_j'} \text{ch}_{\mathbb{T}}(\Lambda_i)^*|_{F_j'} \text{ch}_{\mathbb{T}}(\mathcal{L}_j|_{F_j'}) \text{ch}_{\mathbb{T}}(\text{Sym}N_{F_j'}^*) \text{td}(F_j') \\ &= \int_{F_j'} t^{-g} \text{ch}_{\mathbb{T}}(\Lambda_i|_{F_j'}) \text{ch}_{\mathbb{T}}(\text{Sym}N_{F_j'}^*) \text{td}(F_j') = t^{-g} \chi_{\mathbb{T}}(\mathcal{M}(\text{GL}_2); \Lambda_i \otimes \mathcal{L}_j), \end{aligned}$$

and the result follows. □

Remark 7.6 Based on the Fourier–Mukai transform on the generic fibres of the Hitchin map, we expect that (the semi-classical limit of the) mirror of \mathcal{L}_i will be the shifted coherent sheaf $\Lambda_i[-g]$ and that of Λ_j the shifted coherent sheaf $\mathcal{L}_j[-3g + 3]$. This explains the sign in Corollary 7.1.

Remark 7.7 The extra t^{g-1} factor in Corollary 7.1 seems to be necessary, and the reason for its appearance could be the fact that the canonical bundle $K_{\mathcal{M}(\text{SL}_2)}$ is trivial with a weight $3g - 3$ equivariant structure and therefore has no square root for g even.

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