

Mirror symmetry and big algebras

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Motivation from mirror symmetry

- \mathbb{M}_G moduli of G -Higgs bundles on curve C ; G cplx reductive
- classical limit (Donagi–Pantev 2012) of mirror symmetry:

$$\mathcal{S} : D^b(\mathbb{M}_G) \sim D^b(\mathbb{M}_{G^\vee})$$

generically Fourier-Mukai transform $\mathbb{M}_G \xrightarrow{h_G} \mathbb{A} \cong \mathbb{A}^\vee \xleftarrow{h_{G^\vee}} \mathbb{M}_{G^\vee}$

- (Kapustin–Witten 2007) dominant $\mu \in \Lambda^+(G^\vee)$, $c \in C$

Hecke operators: $\mathcal{H}_c^\mu : D^b(\mathbb{M}_G) \xrightarrow{D^b(\mathcal{H}_c^\mu)} D^b(\mathbb{M}_G)$ and

Wilson operators: $\mathcal{W}_c^\mu : D^b(\mathbb{M}_{G^\vee}) \xrightarrow{\otimes \rho^\mu(\mathbb{E}_c^\vee)} D^b(\mathbb{M}_{G^\vee})$

intertwine: $\mathcal{H}_c^\mu \circ \mathcal{S} = \mathcal{S} \circ \mathcal{W}_c^\mu$

- test for $\mathcal{O}_{\mathbb{M}_{G^\vee}} \in D_{coh}^b(\mathbb{M}_{G^\vee})$:
 $\mathcal{H}_c^\mu(\mathcal{S}(\mathcal{O}_{\mathbb{M}_{G^\vee}})) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+}) = \mathcal{S}(\mathcal{W}_c^\mu(\mathcal{O}_{\mathbb{M}_{G^\vee}})) = \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c)$
- the Hecke transform of the Hitchin section $\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ is supported at a union of Lagrangian upward flows

Lagrangian upward flows in \mathbb{M}

- $\mathbb{M} := \mathbb{M}_{\text{PGL}_n} \ni (E, \Phi); \Phi \in \text{End}_0(E) \otimes K_C$
- $h : \mathbb{M} \rightarrow \mathbb{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n)$ Hitchin map
 $(E, \Phi) \mapsto \det(x - \Phi)$
- $\mathbb{C}^\times \mathbb{C}\mathbb{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; *semiprojective*:
 - 1 $\mathbb{M}^{\mathbb{C}^\times}$ projective
 - 2 $\lim_{\lambda \rightarrow 0} \lambda \mathcal{E}$ exists for every $\mathcal{E} \in \mathbb{M}$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times} \rightsquigarrow W_{\mathcal{E}}^+ := \{\mathcal{F} \in \mathbb{M} \mid \lim_{\lambda \rightarrow 0} \lambda \mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W_{\mathcal{E}}^+ \subset \mathbb{M}$ locally closed $\cong T_{\mathcal{E}}^+ \mathbb{M}$
- $\lambda^*(\omega) = \lambda\omega \rightsquigarrow W_{\mathcal{E}}^+ \subset (\mathbb{M}, \omega)$ is Lagrangian
- $\mathbb{M} = \coprod_{\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}} W_{\mathcal{E}}^+$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow W_{\mathcal{E}}^+$ closed $\Leftrightarrow h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \rightarrow \mathbb{A}$ proper
- e.g. $\mathcal{E} = (E, 0)$ very stable $\Leftrightarrow E$ very stable (Laumon 1988)
- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \rightarrow \mathbb{A}$$

becomes explicit!

Examples of very stable Higgs bundles

- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \dots \oplus K_C^{1-n}$
- $a = (a_2, \dots, a_n) \in \mathbb{A} = H^0(K_C^2) \times \dots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & \dots & 0 & a_n \\ 1 & 0 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & a_2 \\ 0 & \dots & 0 & 1 \end{pmatrix} : E_0 \rightarrow E_0 K_C$ companion matrix
- $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathbb{M}^{\mathbb{C}^x}$ canonical uniformising Higgs bundle
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ Hitchin section \Rightarrow very stable
- $c \in C$, $E_k := \mathcal{O}_C \oplus K_C^{-1} \dots \oplus K_C^{-k}(c) \oplus \dots \oplus K_C^{1-n}(c)$,

$$s_c \in H^0(\mathcal{O}_C(c)), \Phi_k := \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & s_c & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 1 \end{pmatrix} : E_k \rightarrow E_k K_C$$

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_k := (E_k, \Phi_k)$ is very stable.

- proof by noticing $W_k^+ := W_{\mathcal{E}_k}^+ = \mathcal{H}_c^{\omega_k}(W_0^+)$
 ω_k k th fundamental character of SL_n , minuscule

Hitchin map as spectrum of equivariant cohomology

- G complex reductive; $EG \rightarrow BG$ universal principal G -bundle;
 $H_G^* := H^*(BG; \mathbb{C}) \cong \mathbb{C}[g]^G \cong \mathbb{C}[t]^W$
- $G \subset X$ variety; $X_G := X \times EG/G$ Borel quotient;
 $H_G^*(X; \mathbb{C}) := H^*(X_G; \mathbb{C})$ equivariant cohomology H_G^* -algebra
- *equivariantly formal* $\Leftrightarrow H_G^*(X; \mathbb{C})_0 \cong H^*(X; \mathbb{C}) \Leftrightarrow H_G^*(X)$ is free over $H_G^* \Rightarrow \text{Spec}(H_G^{2*}(X; \mathbb{C})) \rightarrow \text{Spec}(H_G^{2*}) \cong \mathfrak{t} // W$ is proper

Theorem (Hausel, 2022)

The Hitchin map on the minuscule upward flow W_k^+ is modelled on spectrum of PGL_n -equivariant cohomology of the Grassmannian

$$\begin{array}{ccc} W_k^+ & \twoheadrightarrow & \text{Spec}(H_{\text{PGL}_n}^{2*}(\text{Gr}(k, n), \mathbb{C})) \\ h_k \downarrow & \lrcorner & \downarrow \\ \mathbb{A} & \twoheadrightarrow & \text{Spec}(H_{\text{PGL}_n}^{2*}) \end{array}$$

- proof by fixed point scheme in $\text{Gr}(k, n) \cong \text{Gr}^{\omega_k}$ affine Schubert
- \leadsto multiplicity algebra $\mathbb{C}[h_k^{-1}(0)] \cong H^{2*}(\text{Gr}(k, n); \mathbb{C})$
(Hausel–Hitchin 2021)

Universal bundle of Kirillov algebras

- $X_+^*(G) \ni \mu$ -highest weight rep. $\rho^\mu : G \rightarrow GL(V^\mu)$
- Kirillov algebra:
 $C^\mu := (S(\mathfrak{g}^*) \otimes \text{End}(V^\mu))^G \cong \text{Maps}(\mathfrak{g}, \text{End}(V^\mu))^G$
associative $S(\mathfrak{g}^*)^G \cong H_G^*$ -algebra
- (Kirillov 2000) C^μ commutative $\Leftrightarrow \rho^\mu$ weight multiplicity free;
e.g. minuscule

Theorem (Hausel 2022)

For $G \cong SL_n$, ω_k fundamental \exists universal bundle of algebra structure on $\rho^{\omega_k}(\mathbb{E})_c \cong \Lambda^k(\mathbb{E})_c$ along $W_0^+ \cong \mathbb{A}$ modelled on C^{ω_k}

$$\begin{array}{ccccccc} C^{\omega_k} & \hookrightarrow & \text{End}(\Lambda^k(\mathbb{E})_c) & & \text{Spec}(C^{\omega_k}) & \leftarrow & \text{Spec}_{\mathbb{A}}(\Lambda^k(\mathbb{E})_c) \\ \uparrow & \lrcorner & \uparrow & \sim & \downarrow & \lrcorner & \downarrow \\ H_{SL_n}^{2*} & \hookrightarrow & \mathbb{C}[\mathbb{A}] & & \text{Spec}(H_{SL_n}^{2*}) & \leftarrow & \mathbb{A} \end{array}$$

- construction by applying Kirillov M -operators to Φ_a and using cyclicity of C^{ω_k} (Panyushev 2004)
- $k = 1$ familiar bundle of algebra structure from BNR corr.

Mirror of equivariant cohomology is Kirillov algebra

- (Kapustin–Witten 2007) $\rightsquigarrow \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$
- when $G = \mathrm{PGL}_n$ and $\mu = \omega_k \in X_+^*(\mathrm{SL}_n)$ fundamental \rightsquigarrow
 $\mathcal{H}_c^{\omega_k}(\mathcal{O}_{W_0^+}) = \mathcal{O}_{W_k^+}$ sheaf of algebras \rightsquigarrow its mirror $\Lambda^k(\mathbb{E}^\vee)_c$
 should acquire a bundle of algebra structure along dual Hitchin
 section from fiberwise Fourier-Mukai transform

Theorem (Hausel 2022)

$$\begin{array}{ccccc}
 & & \cong & & \\
 & \xrightarrow{\quad\quad\quad} & & \xrightarrow{\quad\quad\quad} & \\
 \mathrm{Spec}(C^{\omega_k}) & \leftarrow \mathrm{Spec}_{\mathbb{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \cong W_k^+ & \rightarrow & \mathrm{Spec}(H_{\mathrm{PGL}_n}^{2*}(\mathrm{Gr}(k, n), \mathbb{C})) & \\
 \downarrow \quad \lrcorner & \downarrow & \downarrow & \lrcorner & \downarrow \\
 \mathrm{Spec}(H_{\mathrm{SL}_n}^{2*}) & \leftarrow \mathbb{A}^\vee & \cong \mathbb{A} & \rightarrow & \mathrm{Spec}(H_{\mathrm{PGL}_n}^{2*}) \\
 & & \cong & & \\
 & \xrightarrow{\quad\quad\quad} & & \xrightarrow{\quad\quad\quad} &
 \end{array}$$

- using (Panyushev 2004)'s $C^{\omega_k} \cong H_{\mathrm{SL}_n}^{2*}(\mathrm{Gr}(k, n); \mathbb{C})$
- generalises - partly conjecturally - to all $\mu \in X_*^+(G)$
- \rightsquigarrow classical limit of geometric Satake

Construction of the big algebra

- $\{X_i\}$ basis for \mathfrak{sl}_n , $\{X^i\}$ dual basis wrt Killing form
- $D : C^\mu \rightarrow C^\mu = (S(\mathfrak{sl}_n^*) \otimes \text{End}(V^\mu))^G$ Kirillov's D -operator
- $A \mapsto \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i}$
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\text{SL}_n} \cong H_{\text{SL}_n}^*$ invariant polynomial
- $t^n + c_2(a)t^{n-2} + \dots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M -operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \dots, M_{n-1} \rangle_{H_{\text{SL}_n}^*} \subset C^\mu$ *medium algebra* $\cong Z(C^\mu)$
- $B_{k,i-k} := D^k(c_i) \in C^\mu$ *big operators* of age k and degree $i - k$
- $\mathcal{B}^\mu := \langle 1_{V^\mu}, \{B_{k,i-k}\}_{k,i} \rangle_{H_{\text{SL}_n}^*} \subset C^\mu$ *big algebra* $\supset \mathcal{M}^\mu$

Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset C^\mu$ is commutative, cyclic, finite-free / $H_{\text{SL}_n}^*$ and maximal

- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}^*) \otimes U(\mathfrak{g}))^G$ as a Gaudin algebra from (Feigin–Frenkel, 1992)
- cyclicity follows from (Feigin–Frenkel–Rybnikov 2006) in quantization of Mishchenko–Fomenko integrable systems $\rightsquigarrow \mathcal{B}^\mu$ is a quantum integrable system

Geometric properties of \mathcal{B}^μ

- $\text{Gr} := \overline{\text{PGL}_n((z))} / \text{PGL}_n[[z]]$ affine Grassmannian of PGL_n
- $\text{Gr}^\mu := \overline{\text{PGL}_n[[z]]z^\mu} \subset \text{Gr}$ affine Schubert; $\text{Gr}^{\omega_k} = \text{Gr}(k, n)$
- (Bezrukavnikov–Finkelberg 2008) describe the graded $H_{\text{PGL}_n}^*$ -algebra $H_{\text{PGL}_n}^*(\text{Gr}^\mu)$ & $IH_{\text{PGL}_n}^*(\text{Gr}^\mu)$ as module over it \leadsto

Corollary (Hausel 2022)

$H_{\text{PGL}_n}^*(\text{Gr}^\mu) \cong \mathcal{M}^\mu$ as $H_{\text{PGL}_n}^*$ -algebras

$\text{End}_{H_{\text{PGL}_n}^*(\text{Gr}^\mu)}(IH_{\text{PGL}_n}^*(\text{Gr}^\mu)) \cong \mathcal{C}^\mu$

$IH_{\text{PGL}_n}^*(\text{Gr}^\mu) \cong \mathcal{B}^\mu$ as \mathcal{M}^μ -modules

\leadsto (conj. unique) graded $H_{\text{PGL}_n}^*$ -algebra structure on $IH_{\text{PGL}_n}^*(\text{Gr}^\mu)$

Conjecture (Hausel 2022)

$$\begin{array}{ccccc}
 & & \cong & & \\
 & \text{Spec}(\mathcal{B}^\mu) & \leftarrow \text{Spec}_{\mathbb{A}^\vee}(\rho^\mu(\mathbb{E}_c^\vee)) \cong \mathcal{H}_c^\mu(W_0^+) & \rightarrow & \text{Spec}(IH_{\text{PGL}_n}^{2*}(\text{Gr}^\mu)) \\
 & \downarrow & \lrcorner & \downarrow & \lrcorner \\
 \text{Spec}(H_{\text{SL}_n}^{2*}) & \leftarrow & \mathbb{A}^\vee & \cong \mathbb{A} & \rightarrow & \text{Spec}(H_{\text{PGL}_n}^{2*}) \\
 & & & \cong & & \\
 & & & & &
 \end{array}$$

Finer structure of big algebras

- (Yakimova 2022) \rightsquigarrow construction of \mathcal{B}^μ generalises to other simple G , but explicit generators only in classical types and G_2
- $\text{Spec}(\mathcal{M}^\mu) = \bigcup_{X^+ \ni \lambda \leq \mu} \text{Spec}(\mathcal{M}_\lambda^\mu)$ irreducible components.
 $\mathcal{M}_\lambda^\mu \cong H_G^*(G/P_\lambda, \mathbb{C}) \cong H_G^*(\text{Gr}_\mu^\lambda, \mathbb{C})$; $\text{Gr}^\mu = \coprod_{\mu \leq \lambda} \text{Gr}_\lambda^\mu$ BB-dec.
- multiplicity algebra of $\mathcal{M}_\lambda^\mu \subset \mathcal{B}_\lambda^\mu$ endows
 $IH^*(W_\lambda^\mu) \cong \mathcal{B}_\lambda^\mu / (\mathcal{M}_{\lambda^+}^\mu)$ a graded ring
ringifying Kazhdan-Lusztig polynomials $IP_t(W_\lambda^\mu)$
 $IH^*(W_0^\mu) \cong \mathcal{B}^\mu / (\mathcal{M}_+^\mu) \rightsquigarrow$ (Brylinski 1989) limits of weight spaces
- G complex reductive, $\kappa \in \text{Aut}(G)$ distinguished
corresponds to a folding of the Dynkin diagram
- endoscopy group $\check{G}_\kappa := \widetilde{G}_0^\kappa$

Conjecture (Hausel 2023)

$$\mu \in X_+^*(\check{G})^\kappa \cong X_+^*(\check{G}_\kappa) \rightsquigarrow \kappa : \mathcal{B}^\mu(\check{G}) \rightarrow \mathcal{B}^\mu(\check{G}) \text{ s.t. } \mathcal{B}^\mu(\check{G})_\kappa \cong \mathcal{B}^\mu(\check{G}_\kappa)$$

- $(\mathcal{B}^\mu)_\kappa := \mathcal{B}^\mu / (x - \kappa(x))_{x \in \mathcal{B}^\mu}$ coinvariant algebra
- $\Rightarrow \text{tr}(\kappa : V_\lambda^\mu \rightarrow V_\lambda^\mu) = \dim \check{V}_\lambda^\mu$ (Jantzen 1973)'s twining formula
- for principal $H \subset G$ we expect a transfer $\mathcal{B}^\mu(\check{G}) \twoheadrightarrow \mathcal{B}^\mu(\check{H})$