Mirror symmetry and big algebras
3. Big algebras and real forms

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Motivation for the big algebras

- $\mu \in X_*^+(\mathrm{SL}_n)$ highest weight repn. $\rho^\mu : \mathrm{SL}_n \to \mathrm{GL}(V^\mu)$
- typically $\mathcal{C}^\mu = (S(\mathfrak{sl}_n) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}}$ is not commutative
- Problem: what is the mirror $\mathcal{S}(\rho^\mu(\mathbb{E}_c)) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+}) \in D^b(\mathbb{M})$?
- $\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ is pushforward of the pullback from the Hecke correspondence $\mathbb{M} \leftarrow \mathcal{H}_c^\mu \to \mathbb{M}$
  $\rightsquigarrow$ $\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ a sheaf of algebras
  $\rightsquigarrow$ its conjectured mirror $\rho^\mu(\mathbb{E}_c)|_{{}^\vee W_0^+}$ also sheaf of algebras
  $\rightsquigarrow$ need $\mathcal{B}^\mu \subset \mathcal{C}^\mu$ *big algebra*, commutative and cyclic
  $\rightsquigarrow$ $\mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H^*_{\mathrm{SL}_n})$ models $\mathrm{Spec}(\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})) \to \mathbb{A}$
- Hints for the construction:
  1. (Rozhkovskaya 2002) describes explicitly $\mathcal{C}_{\omega_1+\omega_2}(\mathfrak{sl}_3)$
     (Tai 2014) describes explicitly $\mathcal{C}_{\mu_{adj}}(\mathfrak{g})$ for $\mathfrak{g}$ simple
     they all contain a maximal cyclic commutative subalgebra
  2. the center $Z(\mathcal{C}^\mu) \subset \mathcal{C}^\mu$
     is generated by M-operators of Kirillov
  3. reminiscent of Mishchenko-Fomenko integrable systems
     maximally Poisson commutative subalgebras in $S(\mathfrak{g})$
     generated by iterated derivatives of invariant polynomials

Construction of the big algebra

- $\{X_i\}$ basis for $\mathfrak{sl}_n$, $\{X^i\}$ dual basis wrt Killing form
- $$\begin{array}{cccl} D: \mathcal{C}^\mu & \to & \mathcal{C}^\mu = (S(\mathfrak{sl}_n^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \\ A & \mapsto & \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i} \end{array}$$ Kirillov's D-operator
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^*_{\mathrm{SL}_n}$ invariant polynomial
  $t^n + c_2(a)t^{n-2} + \cdots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M-operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \ldots, M_{n-1} \rangle_{H^*_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *medium algebra* $\cong Z(\mathcal{C}^\mu)$
- $B_{k,i-k} := D^k(c_i) \in \mathcal{C}^\mu$ *big operators* of age k and degree $i-k$
- $\mathcal{B}^\mu := \langle 1_{V^\mu}, \{B_{k,i-k}\}_{k,i} \rangle_{H^*_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *big algebra* $\supset \mathcal{M}^\mu$

Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic, finite-free $/H^*_{\mathrm{SL}_n}$ and maximal

- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathrm{G}}$ as a Gaudin algebra from (Feigin–Frenkel, 1992)
- cyclicity follows from (Feigin–Frenkel–Rybnikov 2006)
  in quantization of Mishchenko-Fomenko integrable systems
  $\rightsquigarrow \mathcal{B}^\mu$ is some sort of quantum integrable system

Geometric properties of $\mathcal{B}^\mu$

- $\mathrm{Gr} := \mathrm{PGL}_n((z))/\mathrm{PGL}_n[[z]]$ affine Grassmannian of PGL_n
- $\mathrm{Gr}^\mu := \overline{\mathrm{PGL}_n[[z]]z^\mu} \subset \mathrm{Gr}$ affine Schubert; $\mathrm{Gr}^{\omega_k} = \mathrm{Gr}(k,n)$
- (Bezrukavnikov–Finkelberg 2008) describe the graded $H^*_{\mathrm{PGL}_n}$-algebra $H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ & $IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ as module over it $\rightsquigarrow$

Corollary (Hausel 2022)

$H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu$ as $H^*_{\mathrm{PGL}_n}$-algebras
$\mathrm{End}_{H^*_{\mathrm{PGL}}(\mathrm{Gr}^\mu)}(IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \cong \mathcal{C}^\mu$
$IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{B}^\mu$ as $\mathcal{M}^\mu$-modules
$\rightsquigarrow$ (conj. unique) graded $H^*_{\mathrm{PGL}_n}$-algebra structure on $IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$

Conjecture (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{B}^\mu) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\rho^\mu(\mathbb{E}_c^\vee)) \arrow[l, twoheadrightarrow] \arrow[d] \arrow[r, phantom, "\cong"] \arrow[dl, phantom, very near start, "\llcorner"] & \mathcal{H}_c^\mu(W_0^+) \arrow[r, twoheadrightarrow] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Spec}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, twoheadrightarrow] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}$$

Finer structure of big algebras

- (Yakimova 2022) $\rightsquigarrow$ construction of $\mathcal{B}^\mu$ generalises to other simple G, but explicit generators only in classical types and G_2
- equivariant multiplicity $m^\mu(t) := m(\mathcal{B}^\mu)(t) = P_t(IH^{2*}(\mathrm{Gr}^\mu)) =$
  $D^\mu(t) := \prod_{\beta\in\Delta^+} \frac{(1-t^{\langle\mu+\rho,\beta^\vee\rangle})}{(1-t^{\langle\rho,\beta^\vee\rangle})}$ *Dynkin polynomial* unimodal
  $\Leftarrow$ principal $\mathrm{SL}_2 \subset \mathrm{G}$ or $\Leftarrow$ Hard Lefschetz for $IH^{2*}(\mathrm{Gr}^\mu)$
  $M_1 \in \mathcal{B}^\mu \cong IH^{2*}_{\mathrm{G}}(\mathrm{Gr}^\mu)$ represents the equivariant Kähler class
- $\mathrm{Spec}(\mathcal{M}^\mu) = \bigcup_{X_*^+\ni\lambda\leq\mu} \mathrm{Spec}(\mathcal{M}^\mu_\lambda)$ irreducible components.
  $\mathcal{M}^\mu_\lambda \cong H^*_{\mathrm{G}}(\mathrm{G}/P_\lambda,\mathbb{C}) \cong H^*_{\mathrm{G}}(\mathrm{Gr}^\lambda_\mu,\mathbb{C})$; $\mathrm{Gr}^\mu = \coprod_{\mu\leq\lambda}\mathrm{Gr}^\mu_\lambda$ BB-dec.
- The mutliplicity algebra of $\mathcal{M}^\mu_\lambda \subset \mathcal{B}^\mu_\lambda$ endows
  $IH^*(W^\mu_\lambda) \cong \mathcal{B}^\mu_\lambda/(\mathcal{M}^\mu_{\lambda+})$ a graded ring
  ringifying Kazhdan-Lusztig polynomials $IP_t(W^\mu_\lambda)$.
  E.g. $IH^*(W^\mu_0) \cong \mathcal{B}^\mu/(\mathcal{M}^\mu_+) \rightsquigarrow$
  geometrizes (Brylinski 1989)'s limits of weight spaces

Involutions on big algebras

- G complex semisimple; $\sigma : \mathrm{G} \to \mathrm{G}$ anti-holomorphic involution; G^σ real form; $\mathrm{K} \subset \mathrm{G}^\sigma$ maximal compact
- examples: σ_c compact; σ_s split real form
- $\theta := \theta_\sigma : \mathrm{G} \to \mathrm{G}$ holomorphic *Cartan* involution; $\mathrm{G}^\theta \cong \mathrm{K}_\mathbb{C}$
- (García-Prada, Ramanan 2019) $\rightsquigarrow$ holomorphic anti-symplectic involution $\iota : \mathbb{M}_\mathrm{G} \to \mathbb{M}_\mathrm{G}$ such that $\mathbb{M}^\iota \cong \coprod_{\sigma' \sim \sigma} \mathbb{M}_{\mathrm{G}^{\sigma'}}$ holomorphic Lagrangian
- e.g. $\sigma \sim \sigma_c$ then $\iota_c(E, \Phi) = (E, -\Phi)$
  $\sigma \sim \sigma_s$ then $\iota_s(E, \Phi) = (\theta_s(E), -\theta_s(\Phi))$, induced from $\theta_s : \mathrm{G} \to \mathrm{G}$ Chevalley involution.
- Motivating problem: for $\mathcal{E} \in \mathbb{M}^{\iota\mathbb{C}^\times}$ describe Hitchin map $h^\sigma_\mathcal{E} : W^{\iota+}_\mathcal{E} := W^+_\mathcal{E} \cap \mathbb{M}^\iota \to \mathbb{A}^\iota$
  Abstract problem: $\mu = \theta(\mu)$ describe $\iota : \mathcal{B}^\mu \to \mathcal{B}^\mu$ compute $\mathcal{B}^\mu_\iota := \mathcal{B}^\mu / (x - \iota(x))_{x \in \mathcal{B}^\mu}$ *coinvariant algebra* over $(H^*_\mathrm{G})_\theta \Leftrightarrow$ fixed point scheme $\mathrm{Spec}(\mathcal{B}^\mu)^\iota \cong \mathrm{Spec}(\mathcal{B}^\mu_\iota)$ over $\mathrm{Spec}((H^*_\mathrm{G})_\theta)$
- Observation: $\iota_c = (-1)^{deg} : \mathcal{B}^\mu \to \mathcal{B}^\mu$ and $\iota_s = (-1)^{age} : \mathcal{B}^\mu \to \mathcal{B}^\mu$ for self-dual $\mu = \theta_s(\mu)$

Algebraic conjectures on involutions

- $\mu = \theta(\mu)$ is *non-degenerate* when $\mathcal{B}^{\mu}_{\iota}$ is finite-free over $(H^*_{\mathrm{G}})_\theta$ in this case *multiplicity* $m^{\mu}_{\theta} := rank_{(H^*_{\mathrm{G}})_\theta}(\mathcal{B}^{\mu}_{\iota})$ and *equivariant multiplicity* $m^{\mu}_{\theta}(t) = P_t(\mathcal{B}^{\mu}_{\iota}/(H^*_{\mathrm{G}})_{\theta+})$
- $D^{\mu}_{\theta}(t) := \displaystyle\prod_{[\beta]\in(\Delta^+/\theta)^\circ} \frac{\prod_{\langle\mu+\rho,[\beta^\vee]\rangle\in 2\mathbb{Z}}(1-t^{\langle\mu+\rho,\beta^\vee\rangle})}{\prod_{\langle\rho,[\beta^\vee]\rangle\in 2\mathbb{Z}}(1-t^{\langle\rho,\beta^\vee\rangle})}$

Conjecture (Hausel 2023)

1. $\mu = \theta(\mu)$ *is non-degenerate* $\Leftrightarrow D^{\mu}_{\theta}(1) \neq 0$ *assumed for the rest*
2. $m^{\mu}_{\theta} = D^{\mu}_{\theta}(1) > 0$
3. $\check{\sigma} : \check{\mathrm{G}} \to \check{\mathrm{G}}$ *quasi-split* $[\check{\theta}] = [\theta] \in \mathrm{Out}(\check{\mathrm{G}}) \cong \mathrm{Out}(\mathrm{G}) \rightsquigarrow$ $\exists\, \check{\mathrm{G}}^{\sigma}$*-invariant Hermitian form* h^{μ} *on* V^{μ}*, s.t.* $|sig(h^{\mu})| = D^{\mu}_{\theta}(1)$
4. $\sigma = \sigma_c$ *then* $D^{\mu}_{\theta}(1) = D^{\mu}(-1) = sig(IH^*(\mathrm{Gr}^{\mu}))$
5. $\sigma = \sigma_s$ *then* $m^{\mu}_{\theta} \leq m^{\mu}_{0}$ *and* $m^{\mu}_{\theta} = m^{\mu}_{0} \Leftrightarrow \theta_s$ *acts trivially on* V^{μ}_{0}
6. $m^{\mu}_{\theta}(t) = D^{\mu}_{\theta}(t) \in \mathbb{N}[t]$*, palindromic* $\Leftarrow \mathcal{B}^{\mu}_{\iota}$ *Gorenstein, even CI*

- (3) $\rightsquigarrow$ (Vogan et al. 2018) and (Karpelevich 1955)
(4) $\rightsquigarrow$ (Lusztig-Yun 2013) (5) $\rightsquigarrow$ (Millson-Toledano 2004)

Geometric conjectures on involutions

- $\sigma : \mathrm{G} \to \mathrm{G}$ quasi-split $\rightsquigarrow \exists\, H \subset \mathrm{G}^\sigma$ maximal split subgroup $\rightsquigarrow \exists\, \tau : \mathrm{G} \to \mathrm{G}$ such that $\mathrm{G}^{\theta_\tau} \cong H_{\mathbb{C}} \rightsquigarrow (H^*_{\mathrm{G}})_{\theta_\tau} \cong H^*_H \cong H^*_{\mathrm{G}^\tau}$ $\rightsquigarrow \theta_\tau : \mathrm{G} \to \mathrm{G}$ distinguished ($\Leftrightarrow$ preserves a pinning) call τ *quasi-compact* e.g. when σ split then τ is compact
  when σ non-split then $(\mathrm{G}, \mathrm{G}^{\theta_\tau}) =$
  $(\mathrm{SL}_{2n+1}, \mathrm{SO}_{2n+1}), (\mathrm{SL}_{2n}, \mathrm{Sp}_n), (\mathrm{SO}_{2n}, \mathrm{SO}_{2n-1})$ or (E_6, F_4)

Conjecture (Hausel 2023)

1. $\mu = \tau(\mu) = \theta(\mu) \rightsquigarrow \tau^* = \iota \in \mathrm{Aut}_2(IH^*_{\mathrm{G}}(\mathrm{Gr}^\mu)) \cong \mathrm{Aut}_2(\mathcal{B}^\mu)$
2. *τ non-compact, μ non-degenerate $\rightsquigarrow \mathcal{B}^\mu_\iota \cong IH^*_{\mathrm{G}^\tau}((\mathrm{Gr}^\mu)^\tau)$ and*
3. *$(\mathrm{Gr}^\mu)^\tau \subset \mathrm{Gr}^\mu$ Lagrangian $\rightsquigarrow deg_t(D^\mu_{\theta_\tau}(t)) \leq deg(D^\mu(t))/2$*

- $\tau^* \in \mathrm{Aut}_2(IH^*(\mathrm{Gr}^\mu)) \cong \mathrm{Aut}_2(V^\mu)$ can be computed from the Geometric-Satake $/\mathbb{R}$ (Richarz–Zhu, 2015)$\rightsquigarrow$
  $\exists\, \delta_\tau \in {}^L\mathrm{G} \cong \check{\mathrm{G}} \rtimes \mathbb{Z}/2\mathbb{Z} \rightsquigarrow \mathrm{Ad}(\delta_\tau) \in \mathrm{Aut}_2(\check{\mathrm{G}}) \rightsquigarrow \tau^* \in \mathrm{Aut}_2(V^\mu)$
- expect $\mathrm{Ad}(\delta_\tau) = \theta_{\check{\sigma}} \in \mathrm{Aut}_2(\check{\mathrm{G}})$ quasi-split $\rightsquigarrow |sig(h^\mu)| = D^\mu_\theta(1)$
- minuscule μ by (Gonzalez–Hausel, Elkner 2023) in H^*_{G} of Grassmannians, even spheres and Cayley projective planes

Principal endoscopy and transfer for big algebras

- G complex reductive, $\kappa : \mathrm{G} \to \mathrm{G}$ distinguished corresponds to a *folding* of the Dynkin diagram
- endoscopy group $\check{\mathrm{G}}_\kappa := \widecheck{\mathrm{G}_0^\kappa}$

Conjecture (Hausel 2023)

$\mu \in X^*_+(\check{\mathrm{G}})^\kappa \cong X^*_+(\check{\mathrm{G}}_\kappa) \rightsquigarrow \kappa : \mathcal{B}^\mu(\check{\mathrm{G}}) \to \mathcal{B}^\mu(\check{\mathrm{G}})$ *s.t.* $\mathcal{B}^\mu(\check{\mathrm{G}})_\kappa \cong \mathcal{B}^\mu(\check{\mathrm{G}}_\kappa)$

- $\rightsquigarrow \mu \geq \lambda \in X^*_+(\check{\mathrm{G}})^\kappa \cong X^*_+(\check{\mathrm{G}}_\kappa)$, V^μ irrep of $\check{\mathrm{G}}$, W^μ irrep of $\check{\mathrm{G}}_\kappa \rightsquigarrow$
  $\mathrm{tr}\left(\kappa : V^\mu_\lambda \to V^\mu_\lambda\right) = \dim(\mathcal{B}^\mu_\lambda(\check{\mathrm{G}})_\kappa) = \dim(\mathcal{B}^\mu_\lambda(\check{\mathrm{G}}_\kappa)) = \dim W^\mu_\lambda \rightsquigarrow$
  $\mathrm{tr}\left(\kappa : V^\mu_\lambda \to V^\mu_\lambda\right) = \dim W^\mu_\lambda$ (Jantzen 1973)'s twining formula
- e.g. when $\sigma_c \nsim \sigma_s$ then $\kappa = \iota_c \iota_s = (-1)^{deg+age}$
  Conjecture reflects geometry (García-Prada–Ramanan 2019)
  of symplectic involution $\kappa = \iota_c \iota_s \in \mathrm{Aut}_2(\mathbb{M}_\mathrm{G})$
  $\mathbb{M}_\mathrm{G}^\kappa = \coprod_{\sigma \sim \sigma_s} \mathbb{M}_{\mathrm{G}^{\theta_\sigma}}$ hyperkähler $\rightsquigarrow$
  $(\mathcal{B}^\mu(\check{\mathrm{G}})_{\iota_s})_{\iota_c} \cong (\mathcal{B}^\mu(\check{\mathrm{G}})_{\iota_c})_{\iota_s} \cong \mathcal{B}^\mu(\check{\mathrm{G}}_\kappa)_{\iota_c} \rightsquigarrow D^\mu_{\theta_s}(-1) = D^\mu(\check{\mathrm{G}}_\kappa)(-1)$
  also $\Leftarrow$ curious $D^\mu_{\theta_c}(t) D^\mu_{\theta_s}(t) / D^\mu(t) = D^\mu_{\theta_c}(\check{\mathrm{G}}_\kappa)(t)^2 / D^\mu(\check{\mathrm{G}}_\kappa)(t)$
- for principal $\mathrm{H} \subset \mathrm{G}$ we expect a transfer $\mathcal{B}^\mu(\check{\mathrm{G}}) \twoheadrightarrow \mathcal{B}^\mu(\check{\mathrm{H}})$ e.g.
  $\mathrm{SL}_2 \subset \mathrm{G}$ principal: $\mathrm{SL}_2(V) \to \mathrm{SL}(Sym^k(V))$, $G_2 \subset \mathrm{SO}_7 (\subset \mathrm{SO}_8)$

Open directions

- $a \in \mathbb{A}$, $C_a^\mu := \mathrm{Spec}_C(\mathcal{B}^\mu(\mathcal{E}_a)) \subset K^{d_1} \oplus \cdots \oplus K^{d_N}$ *big spectral curve*
  $(E, \Phi) \in h^{-1}(a) \rightsquigarrow$ rank 1 module / $\mathcal{B}^\mu(\mathcal{E}_a)$, rank 1 sheaf / C_a^μ
  Is there a big BNR correspondence?
- If yes $\rightsquigarrow$ could transfer rank 1 $\mathcal{B}^\mu(\check{\mathrm{G}})(\mathcal{E}_a)$- module to rank 1 $\mathcal{B}^\mu(\check{\mathrm{G}}_\kappa)(\mathcal{E}_a)$-module $\rightsquigarrow$ endoscopic transfer of Higgs bundles $\mathbb{M}_{\check{\mathrm{G}}} \supset h_{\check{\mathrm{G}}}^{-1}(\mathbb{A}_{\check{\mathrm{G}}_\kappa}) \to \mathbb{M}_{\check{\mathrm{G}}_\kappa}$, expected as the mirror of $\mathbb{M}_{\mathrm{G}^\kappa} \subset \mathbb{M}_{\mathrm{G}}$
- mirror of $\mathbb{M}_{\mathrm{G}^\sigma} \subset \mathbb{M}_{\mathrm{G}}$ could be given by transfer of rank 1 $\mathcal{B}^\mu(\mathcal{E}_a)$ module to rank 1 $\mathcal{B}_\iota^\mu(\mathcal{E}_a)$-module.
  What is a rank 1 $\mathcal{B}_\iota^\mu(\mathcal{E}_a)$-module?
- $\mathrm{U}(n, n+1)^\vee \cong \mathrm{Sp}_n \subset \mathrm{GL}_{2n+1}$ is not principal $\rightsquigarrow$ there must be a non-trivial line bundle on $\mathbb{M}_{\mathrm{U}(n,n+1)} \subset \mathbb{M}_{\mathrm{GL}_{2n+1}}$ so that its mirror is supported on $\mathbb{M}_{\mathrm{Sp}_n} \subset \mathbb{M}_{\mathrm{GL}_{2n+1}}$
- $\mathrm{G} \curvearrowright M \rightsquigarrow C^\mu(M) := Maps(M, \mathrm{End}(V^\mu))^{\mathrm{G}}$ Kirillov algebra. Is there big $\mathcal{B}^\mu(M) \subset C^\mu(M)$? $\mu : M \to \mathfrak{g}^* \cong \mathfrak{g}$ Hamiltonian $\rightsquigarrow$ $\mathcal{B}^\mu(\mathfrak{g}) \to \mathcal{B}^\mu(M)$. When is it surjective?
- We can embed $\mathrm{Spec}(\mathcal{B}^\mu) \subset \mathbb{P}(V^\mu) \times \mathfrak{g}/\!/\mathrm{G}$. Does $\mathcal{B}^\mu$ map to fixed point scheme of any G-orbit closure in $\mathbb{P}(V^\mu)$? Do we have a surjective map $\mathcal{B}^\mu \twoheadrightarrow H_{\mathrm{G}}^*(\overline{\mathrm{G}/\mathrm{G}^\theta}^{wond})$?