Mirror symmetry and big algebras
2. Equivariant cohomology, Kirillov algebras and BNR correspondence

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Recall from lecture 1

- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \cdots \oplus K_C^{1-n}$
- $a = (a_2, \dots, a_n) \in \mathbb{A} = H^0(K_C^2) \times \cdots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & 0\dots & 0 & a_n \\ 1 & 0\dots & 0 & a_{n-1} \\ \vdots & & \vdots & \\ 0 & \dots 1 & 0 & a_2 \\ 0 & \dots & 1 & 0 \end{pmatrix} : E_0 \to E_0 K_C$ companion matrix

  $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathbb{M}^{\mathbb{C}^\times}$ *canonical uniformising Higgs bundle*
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ *Hitchin section* $\Rightarrow$ very stable
- $c \in C$, $E_k := \mathcal{O}_C \oplus K_C^{-1} \dots \oplus K_C^{-k}(c) \oplus \dots K_C^{1-n}(c)$,

  $s_c \in H^0(\mathcal{O}_C(c))$, $\Phi_k := \begin{array}{c} \begin{array}{ccc} & k & \end{array} \\ \begin{pmatrix} 0 & \dots 0 & \dots 0 \\ 1 & \dots 0 & \dots 0 \\ \vdots & & \vdots \\ 0 & \dots s_c & \dots 0 \\ \vdots & & \vdots \\ 0 & \dots & 1\ 0 \end{pmatrix} \end{array} : E_k \to E_k K_C$
- $\mathcal{H}_c^{\omega_k}(\mathcal{O}_{W_0^+}) = W_k^+ \rightsquigarrow \mathcal{E}_k = (E_k, \Phi_k)$ very stable $\rightsquigarrow$

  $h_{\mathcal{E}_k} : W_k^+ \to \mathbb{A}$ proper, finite flat
- *multiplicity algebra* of $h_{\mathcal{E}} = (h_1, .., h_N) : \mathbb{C}^N \cong W_{\mathcal{E}}^+ \to \mathbb{A} \cong \mathbb{C}^N$:

  $Q_{h_{\mathcal{E}}} := \mathbb{C}[W_{\mathcal{E}}^+ \cap h^{-1}(0)] = \mathbb{C}[W_{\mathcal{E}}^+]/(h^{-1}(\mathfrak{m}_0)) =$
  $\mathbb{C}[x_1, \dots, x_N]/(h_1, \dots, h_N)$
- $\dim Q_{h_{\mathcal{E}}} \leq \infty \Leftrightarrow \mathcal{E}$ very stable $\rightsquigarrow Q_{h_{\mathcal{E}}}$ is a graded PD ring

Hitchin map as spectrum of equivariant cohomology

- G complex reductive; $EG \to BG$ universal principal G-bundle; $H^*_{\mathrm{G}} := H^*(B\mathrm{G};\mathbb{C}) \cong \mathbb{C}[\mathfrak{g}]^{\mathrm{G}} \cong \mathbb{C}[\mathfrak{t}]^{\mathrm{W}}$
- $\mathrm{G} \circlearrowright X$ variety; $X_{\mathrm{G}} := X \times E\mathrm{G}/\mathrm{G}$ *Borel quotient*; $H^*_{\mathrm{G}}(X;\mathbb{C}) := H^*(X_{\mathrm{G}};\mathbb{C})$ *equivariant cohomology* H^*_{G}-algebra
- *equivariantly formal* $\Leftrightarrow H^*_{\mathrm{G}}(X;\mathbb{C})_0 \cong H^*(X;\mathbb{C}) \Leftrightarrow H^*_{\mathrm{G}}(X)$ is free over $H^*_{\mathrm{G}} \Rightarrow \mathrm{Spec}(H^{2*}_{\mathrm{G}}(X;\mathbb{C})) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}/\!\!/\mathrm{W}$ is proper

Theorem (Hausel, 2022)

The Hitchin map on the minuscule upward flow W_k^+ is modelled on spectrum of PGL_n-equivariant cohomology of the Grassmannian

$$\begin{array}{ccc} W_k^+ & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}),\mathbb{C})) \\ h_k \downarrow & \lrcorner & \downarrow \\ \mathbb{A} & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}) \end{array}$$

- proof by fixed point scheme in $\mathrm{Gr}(\mathrm{k},\mathrm{n}) \cong \mathrm{Gr}^{\omega_{\mathrm{k}}}$ *affine Schubert* as in (Hausel–Rychlewicz 2023)
- $\rightsquigarrow$ *multiplicity algebra* $Q_{h_k} \cong \mathbb{C}[h_k^{-1}(0)] \cong H^{2*}(\mathrm{Gr}(\mathrm{k},\mathrm{n});\mathbb{C})$ (Hausel–Hitchin 2021)

Universal bundle of Kirillov algebras

- $X_+^*(\mathrm{G}) \ni \mu$-highest weight rep. $\rho^\mu : \mathrm{G} \to \mathrm{GL}(V^\mu)$
- $C^\mu := (S(\mathfrak{g}^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \cong \mathit{Maps}(\mathfrak{g}, \mathrm{End}(V^\mu))^{\mathrm{G}}$ associative, graded $S(\mathfrak{g}^*)^{\mathrm{G}} \cong H_{\mathrm{G}}^*$-algebra: *Kirillov algebra* e.g. $M_1 := (X \mapsto \mathit{Lie}(\rho^\mu)(X))$ *small operator*
- (Kirillov 2000) C^μ commutative $\Leftrightarrow \rho^\mu$ weight multiplicity free; e.g. minuscule

Theorem (Hausel 2022)

For $\mathrm{G} \cong \mathrm{SL}_n$, ω_k *fundamental* $\exists$ *universal bundle of algebra structure on* $\rho^{\omega_k}(\mathbb{E})_c \cong \Lambda^k(\mathbb{E})_c$ *along* $W_0^+ \cong \mathbb{A}$ *modelled on* C^{ω_k}

$$\begin{array}{ccccccccc} C^{\omega_k} & \hookrightarrow & \mathrm{End}(\Lambda^k(\mathbb{E})_c) & & \mathrm{Spec}_C(C^{\omega_k}) & \twoheadleftarrow & \mathrm{Spec}_{\mathbb{A}}(\Lambda^k(\mathbb{E})_c) \\ \uparrow & \llcorner & \uparrow & \rightsquigarrow & \downarrow & \llcorner & \downarrow \\ H^{2*}_{\mathrm{SL}_n} & \hookrightarrow & \mathbb{C}[\mathbb{A}] & & \mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) & \twoheadleftarrow & \mathbb{A} \end{array}$$

- construction by applying Kirillov M-operators to Φ_a and using cyclicity of C^{ω_k} (Panyushev 2004)
- $k = 1$ familiar bundle of algebra structure from BNR corr.

Minuscule Kirillov algebra

- $\mathrm{G} = \mathrm{SL}_n$, $\omega_k := \Lambda^k \mathbb{C}^n \in X_*^+(\mathrm{SL}_n)$
- $A \in \mathfrak{sl}_n \rightsquigarrow \Lambda^k(tI - A) = t^k + t^{k-1}A_1 + \ldots A_k : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$
  define $M_i := (A \mapsto A_i) \in (Map(\mathfrak{g} \to \mathrm{End}(\Lambda^k(\mathbb{C}^n)))^{\mathrm{G}} \cong \mathcal{C}^{\omega_k}$
  e.g. $M_1 = \mathrm{Lie}(\rho^{\omega_k})$ small operator
- $(\Lambda^{n-k}(tI - A))^* = t^{n-k} + t^{n-k-1}B_1 + \ldots B_{n-k} : \Lambda^k(\mathbb{C}^n) \to \Lambda^k(\mathbb{C}^n)$
  define $N_j := (A \mapsto B_j) \in (Map(\mathfrak{g} \to \mathrm{End}(\Lambda^k(\mathbb{C}^n)))^{\mathrm{G}} \cong \mathcal{C}^{\omega_k}$
- M_i, N_j are commuting operators satisfying:
  $(t^k + t^{k-1}A_1 + \cdots + A_k)(t^{n-k} + t^{n-k-1}B_1 + \cdots + B_{n-k}) =$
  $\det(tI - A)Id_{\Lambda^k\mathbb{C}^n} = (t^n + t^{n-2}c_2 + \cdots + c_n)Id_{\Lambda^k\mathbb{C}^n}$
- $\rightsquigarrow \mathcal{C}^{\omega_k} \cong \langle M_1, \ldots, M_k, N_1, \ldots, N_{n-k}\rangle_{H^*_{\mathrm{SL}_n}} \cong H^*_{\mathrm{SL}_n}(\mathrm{Gr}(k,n), \mathbb{C})$
- recall $E_0 = \mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}$ and $\mathcal{E}_a = (E_0, \Phi_a)$ for $a \in \mathbb{A}$
- $M_i(\Phi_a) \in \mathrm{End}(\Lambda^k E_0) \otimes K^i \cong \mathrm{Hom}(K^{-i}, \mathrm{End}(\Lambda^k E_0)) \rightsquigarrow$
  $\Lambda^k(\mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}) \oplus \Lambda^k(E_0) = \Lambda^k(E_0) \oplus \Lambda^k(E_0) \to \Lambda^k(E_0)$
  defines bundle of algebra structure $\mathcal{C}^{\omega_k}(\mathcal{E}_a)$ on $\Lambda^k(E_0)$
- e.g. $k = 1 \rightsquigarrow \Phi^i_a : K^{-i} \to \mathrm{End}(E_0) \rightsquigarrow$ multiplication
  $(\mathcal{O} \oplus K^{-1} \oplus \cdots \oplus K^{1-n}) \oplus E_0 = E_0 \oplus E_0 \to E_0 \rightsquigarrow \mathcal{C}^{\omega_1}(\mathcal{E}_a)$ on E
- *spectral curve* $C_a := \mathrm{Spec}_C(\mathcal{C}^{\omega_1}(\mathcal{E}_a)) \subset \mathrm{Spec}(\mathrm{Sym}^*(K)) \cong T^*C$

Minuscule attractor as universal spectral curve

- $F \in \pi_0(\mathbb{M}^{\mathbb{C}}) \rightsquigarrow W_F^+ := \cup_{\alpha \in F} W_\alpha^+$ *attractor* of F, coisotropic
- $\mathcal{E}_k \in F_k \rightsquigarrow F_0 \cong \{\mathcal{E}_0\}$, $F_k \cong C$ for $k = 1, \ldots, n-1$
- $\rightsquigarrow W_{F_k}^+$ *minuscule attractors* $\rightsquigarrow W_{F_k}^+ \subset \mathbb{M}$ closed $\rightsquigarrow$ very stable
  conjecturally no other very stable attractor
- $tr : W_{\mathcal{E}}^+ \to \mathrm{Spec}(\langle \mathbb{C}[W_{\mathcal{E}}^+]_1 \rangle) \subset (\mathbb{C}[W_{\mathcal{E}}^+]_1)^* \cong T_{\mathcal{E}}^1 W_{\mathcal{E}}^+ \rightsquigarrow$
  $tr_F : W_F^+ \to N_F^1 \cong T^*F$ *trace map*
  alternatively (Hitchin 2021) $\theta := i_X \omega$, X gen. by $\mathbb{C}^\times$-action$\rightsquigarrow$
  $\theta|_{W_{\mathcal{E}}^+} = 0 \rightsquigarrow tr_F : W_F^+ \to T^*F$ such that $\theta = tr_F^*(\theta_{T^*F})$
- e.g. $tr_{F_k} : W_k^+ \to T^*F_k \cong T^*C$

Theorem (Hausel 2022, conjectured by Bousseau 2022)

$(h_1, tr_{F_1}) : W_{F_1}^+ \to \mathbb{A} \times T^*C \cong$ *universal spectral curve.*
In particular, $W_{F_1}^+ \cap h^{-1}(a) = C_a \subset T^*C$ *spectral curve.*

BNR correspondence from mirror symmetry

- recall BNR correspondence: $(E,\Phi) \in \mathbb{M}_{\mathrm{SL}_n}$
  $\Phi^n + a_2\Phi^{n-2} \cdots + a_n = 0$ for $h(E,\Phi) = a = (a_2,\ldots,a_n) \in \mathbb{A} \Leftrightarrow$
  $\Phi^i : K^{-i} \to \mathrm{End}(E)$ induces an action of $\mathcal{C}^{\omega_1}(\mathcal{E}_a)$ on E
  $\Leftrightarrow$ rank 1 sheaf $L_{\mathcal{E}}/C_a := \mathrm{Spec}_C(\mathcal{C}^{\omega_1}(\mathcal{E}_a)) \subset K$ spectral curve
  e.g. $L_{\mathcal{E}}$ line bundle when C_a is smooth
- $\mathcal{E} = (E,\Phi) \in h^{-1}_{\mathrm{SL}_n}(a)$ such that $C_a \subset K$ is smooth
  $S(\mathcal{O}_{\mathcal{E}})$ should be by FM transform a line bundle on $h^{-1}_{\mathrm{PGL}_n}(a)$

Theorem (Hausel 2022)

The restriction of the mirror of $\mathcal{O}_{\mathcal{E}}$ to the standard attractor $\mathcal{E} \mapsto S(\mathcal{O}_{\mathcal{E}})|_{W_{F_1^+} \cap h^{-1}(a)} = L_{\mathcal{E}}$ recovers the BNR correspondence

- generalises to $\omega_k \rightsquigarrow$ for Higgs bundle (E,Φ) we get medium Higgs fields $M_i(\Phi) : K^{-i} \to \mathrm{End}(\Lambda^k(E))$ generating an action of the bundle of algebras $\mathcal{C}^{\omega_k}(\mathcal{E}_a) \cong \Lambda^k(\mathcal{E}_a)$ on $\Lambda^k(E) \rightsquigarrow$ rank 1 module on $\mathrm{Spec}_C(\mathcal{C}^{\omega_k}(\mathcal{E}_a))$ *minuscule spectral curve* $C_a^{\omega_k} := \mathrm{Spec}_C(\mathcal{C}^{\omega_k}(\mathcal{E}_a)) \subset K \oplus K^2 \oplus \cdots \oplus K^k$

Mirror of equivariant cohomology is Kirillov algebra

- (Kapustin–Witten 2007) $\rightsquigarrow \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c) = \mathcal{H}^\mu_c(\mathcal{O}_{W_0^+})$
- when $\mathrm{G} = \mathrm{PGL}_n$ and $\mu = \omega_k \in X^*_+(\mathrm{SL}_n)$ fundamental$\rightsquigarrow$ $\mathcal{H}^{\omega_k}_c(\mathcal{O}_{W_0^+}) = \mathcal{O}_{W_k^+}$ sheaf of algebras $\rightsquigarrow$ its mirror $\Lambda^k(\mathbb{E}^\vee)_c$ should aquire a bundle of algebra structure along dual Hitchin section from fiberwise Fourier-Mukai transform

Theorem (Hausel 2022)

$$
\begin{tikzcd}
\mathrm{Spec}(C^{\omega_k}) \arrow[d] \arrow[rrrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \arrow[l, two heads] \arrow[d] \arrow[r, phantom, "\cong"] & W_k^+ \arrow[d] \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr(k,n)}, \mathbb{C})) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, two heads] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}
$$

- using (Panyushev 2004)'s $C^{\omega_k} \cong H^{2*}_{\mathrm{SL}_n}(\mathrm{Gr(k,n)}; \mathbb{C})$
- generalises - partly conjecturally - to all $\mu \in X_*^+(\mathrm{G})$
- $\rightsquigarrow$ *classical limit of geometric Satake*

Traceless Lagrangians and real forms

- recall trace map $\mathrm{tr}_F : W_F^+ \to T^*F$ for $F \in \pi_0(\mathbb{M}^{\mathbb{C}^\times})$
  preimage of the zero section: $\mathcal{L}_F := tr_F^{-1}(F)$
  $\mathcal{L} := \coprod_{F\in\pi_0(\mathbb{M}^{\mathbb{C}})} \mathcal{L}_F$ *traceless Lagrangian*
- $T_{\mathcal{E}}\mathcal{L} \cong T^0_{\mathcal{E}}\mathbb{M} \oplus T^{>1}_{\mathcal{E}}\mathbb{M}$ for $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$
- (García-Prada, Ramanan 2019) $\rightsquigarrow$ holomorphic
  anti-symplectic involution $\iota_c : \begin{array}{ccc} \mathbb{M}_{\mathrm{G}} & \to & \mathbb{M}_{\mathrm{G}} \\ (E,\Phi) & \mapsto & (E,-\Phi) \end{array}$
  such that $\mathbb{M}^{\iota_c} \cong \coprod_{\sigma'\sim\sigma_c} \mathbb{M}_{\mathrm{G}^{\sigma'}}$ holomorphic Lagrangian
- $T_{\mathcal{E}}\mathbb{M}^{\iota_c} \cong \dots \oplus T^{-2}_{\mathcal{E}}\mathbb{M} \oplus T^0_{\mathcal{E}}\mathbb{M} \oplus T^2_{\mathcal{E}}\mathbb{M} \oplus \dots$ for $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$

Theorem (Hausel 2021)

The traceless Lagrangian $\mathcal{L} \subset \mathbb{M}$ is a Lagrangian containing the Lagrangian $\mathbb{M}^{\iota_c} \subset \mathcal{L} \rightsquigarrow \mathbb{M}^{\iota_c} \cong \overline{\coprod_{F_{l,m}\in\pi_0(\mathbb{M}^{\mathbb{C}})} \mathcal{L}_{F_{l,m}}}$

- we conjecture $S(K^{\frac{1}{2}}_{\mathcal{L}_{F_k}}) \cong \mathbb{R}^1\pi_*\left(\Lambda^k(\mathbb{E}) \stackrel{\Lambda^k\Phi}{\to} \Lambda^k(\mathbb{E}) \otimes K\right)$

Motivation for the big algebras

- $\mu \in X_*^+(\mathrm{SL}_n)$ highest weight repn. $\rho^\mu : \mathrm{SL}_n \to \mathrm{GL}(V^\mu)$
- typically $\mathcal{C}^\mu = (S(\mathfrak{sl}_n) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}}$ is not commutative
- Problem: what is the mirror $\mathcal{S}(\rho^\mu(\mathbb{E}_c)) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+}) \in D^b(\mathbb{M})$?
- $\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ is pushforward of the pullback from the Hecke correspondence $\mathbb{M} \leftarrow \mathcal{H}_c^\mu \to \mathbb{M}$
  $\rightsquigarrow \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ a sheaf of algebras
  $\rightsquigarrow$ its conjectured mirror $\rho^\mu(\mathbb{E}_c)|_{{}^\vee W_0^+}$ also sheaf of algebras
  $\rightsquigarrow$ need $\mathcal{B}^\mu \subset \mathcal{C}^\mu$ *big algebra*, commutative and cyclic
  $\rightsquigarrow \mathrm{Spec}(\mathcal{B}^\mu) \to \mathrm{Spec}(H^*_{\mathrm{SL}_n})$ models $\mathrm{Spec}(\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})) \to \mathbb{A}$
- Hints for the construction:
  1. (Rozhkovskaya 2002) describes explicitly $\mathcal{C}_{\omega_1+\omega_2}(\mathfrak{sl}_3)$
     (Tai 2014) describes explicitly $\mathcal{C}_{\mu_{adj}}(\mathfrak{g})$ for $\mathfrak{g}$ simple
     they all contain a maximal cyclic commutative subalgebra
  2. the center $Z(\mathcal{C}^\mu) \subset \mathcal{C}^\mu$
     is generated by M-operators of Kirillov
  3. reminiscent of Mishchenko-Fomenko integrable systems
     maximally Poisson commutative subalgebras in $S(\mathfrak{g})$
     generated by iterated derivatives of invariant polynomials