Enhanced mirror symmetry for Langlands dual Hitchin systems

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SYZ mirror symmetry for Hitchin systems

- C smooth, projective, complex curve
- G complex reductive group; $\mathrm{G}^\vee$ its Langlands dual
  e.g. $\mathrm{PGL}_n^\vee \cong \mathrm{SL}_n$
- (Hitchin 1987) (Simpson 1992):
  $\mathcal{M}_\mathrm{G}$ moduli space of semi-stable G-Higgs bundles (E, Φ)
  - E principal G-bundle
  - $\Phi \in H^0(C; ad(E) \otimes K_C)$ *Higgs field*
- $h_\mathrm{G} : \mathcal{M}_\mathrm{G} \to \mathcal{A}_\mathrm{G} := \mathrm{Spec}(\mathbb{C}[\mathcal{M}_\mathrm{G}])$ *Hitchin map*
  proper, completely integrable Hamiltonian system
  i.e. fibers Lagrangians with respect to symplectic $\omega \in \Omega^2(\mathcal{M}_\mathrm{G})$
- $\exists$ hyperkähler metric $(\mathcal{M}_\mathrm{G}, J) \cong \mathcal{M}_\mathrm{DR}(\mathrm{G})$ moduli flat G-bundles
- (Hausel-Thaddeus 2003) $\mathrm{G} = \mathrm{PGL}_n$ (Donagi-Pantev 2012) $\forall$ G

$$\begin{tikzcd}
\mathcal{M}_\mathrm{DR}(\mathrm{G}) \arrow[dr, "\mathrm{h}_\mathrm{G}"'] & & \mathcal{M}_\mathrm{DR}(\mathrm{G}^\vee) \arrow[dl, "\mathrm{h}_{\mathrm{G}^\vee}"] \\
& \mathcal{A}_G \cong \mathcal{A}_{G^\vee} &
\end{tikzcd}$$

dual special Lagrangian fibrations $\Leftrightarrow$ SYZ mirror symmetry

Topological mirror symmetry

- (Hausel, Thaddeus 2003) topological mirror symmetry: $E_{st}(\mathcal{M}_{\mathrm{PGL}_n}) = E_{st}(\mathcal{M}_{\mathrm{SL}_n})$ stringy Hodge numbers agree proved for $n = 2, 3$
  ”topological shadow of equivalence of derived categories of coherent sheaves”
- (Hausel 2013) $\rightsquigarrow$ proposes attack on topological mirror symmetry using (Ngô 2010)’s techniques on the cohomology of the Hitchin fibration in his proof of Langlands-Shelstead fundamental lemma
- (Gröchenig, Wyss, Ziegler 2020) prove topological mirror symmetry for all n using p-adic integration
- (Gröchenig, Wyss, Ziegler 2020) reprove (Ngô 2010)’s geometric stabilisation with p-adic integration
- (Maulik, Shen 2021) prove topological mirror symmetry using (Ngô 2010)’s techniques

(Classical limit of) Homological mirror symmetry

- (Kapustin–Witten 2007) derive (Kontsevich 1994) homological mirror symmetry from S-duality in 4D N=4 SUSY Yang-Mills

$$\begin{array}{ccccc} \mathcal{S} : D^b_{coh}(\mathcal{M}_{\mathrm{DR}}(\mathrm{G})) & \overset{HMS}{\cong} & Fuk(\mathcal{M}_{\mathrm{DR}}(\mathrm{G}^\vee)) & \overset{KW}{\cong} & D^b_{D-mod}(Bun_{\mathrm{G}^\vee}) \\ \downarrow{\scriptstyle \lambda\to 0} & & & & \swarrow{\scriptstyle \hbar\to 0} \\ \mathcal{S} : D^b_{coh}(\mathcal{M}_{\mathrm{G}}) & \overset{CLHMS}{\cong} & D^b_{coh}(\mathcal{M}_{\mathrm{G}^\vee}) & & \end{array}$$

- $\rightsquigarrow D^b_{coh}(\mathcal{M}_{\mathrm{DR}}(\mathrm{G})) \overset{GLC}{\cong} D^b_{D-mod}(Bun_{\mathrm{G}^\vee})$

  Geometric Langlands Corr. of (Beilinson-Drinfeld, 1995)

- (Donagi–Pantev 2012) explain classical limit of HMS:

  $\mathcal{S} : D^b_{coh}(\mathcal{M}_{\mathrm{G}}) \overset{CLHMS}{\cong} D^b_{coh}(\mathcal{M}_{\mathrm{G}^\vee})$

  generically as Fourier-Mukai transform

  $D^b_{coh}(h^{-1}_{\mathrm{G}}(a)) \overset{FM}{\cong} D^b_{coh}(h^{-1}_{\mathrm{G}}(a)^\vee) \cong D^b_{coh}(h^{-1}_{\mathrm{G}^\vee}(a))$

Enhanced mirror symmetry in the classical limit

- (Kapustin-Witten 2007) introduce enhancements
- propose branes of type (B,A,A) to be mirror to (B,B,B)
- (B,A,A) branes: complex Lagrangians on $(\mathcal{M}_{\mathrm{G}}, \omega)$
  (B,B,B) branes: hyperholomorphic vector bundles in $\mathcal{M}_{\mathrm{G}^\vee}$
- propose t'Hooft and Wilson operators; in the classical limit:
  $\mathcal{H}_c^\mu : D^b_{coh}(\mathcal{M}_{\mathrm{G}}) \to D^b_{coh}(\mathcal{M}_{\mathrm{G}})$ *Hecke operator* (B,A,A)
  $$\begin{array}{rcl} \mathcal{W}_c^\mu : D^b_{coh}(\mathcal{M}_{\mathrm{G}^\vee}) & \to & D^b_{coh}(\mathcal{M}_{\mathrm{G}^\vee}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes \rho_\mu(\mathbb{E}^\vee)_c \end{array}$$ *Wilson operator* (B,B,B)
  $c \in C$; $\mu \in X_*^+(\mathrm{G}) = X^*_+(\mathrm{G}^\vee)$ dominant cocharacter;
  $\rho_\mu : \mathrm{G}^\vee \to \mathrm{GL}(V^\mu)$ μ-highest weight representation;
  $\mathbb{E}^\vee$ universal $\mathrm{G}^\vee$-bundle on $\mathcal{M}_{\mathrm{G}^\vee} \times C$
- intertwine $\mathcal{S}$: $\mathcal{H}_c^\mu \circ \mathcal{S} = \mathcal{S} \circ \mathcal{W}_c^\mu$
- test for $\mathcal{O}_{\mathcal{M}_{\mathrm{G}^\vee}} \in D^b_{coh}(\mathcal{M}_{\mathrm{G}^\vee})$:
  $\mathcal{H}_c^\mu(\mathcal{S}(\mathcal{O}_{\mathcal{M}_{\mathrm{G}^\vee}})) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+}) = \mathcal{S}(\mathcal{W}_c^\mu(\mathcal{O}_{\mathcal{M}_{\mathrm{G}^\vee}})) = \mathcal{S}(\rho_\mu(\mathbb{E}^\vee)_c)$
- ”mirror of universal bundle in irreducible representation $=$
  Hecke transformed Hitchin section ($= W_0^+$)”
- $\mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$ union of Lagrangian upward flows

Lagrangian upward flows in $\mathcal{M}$

- $\mathcal{M} := \mathcal{M}_{\mathrm{PGL}_n} \ni (E, \Phi);\ \Phi \in \mathrm{End}_0(E) \otimes K_C$
- $\begin{array}{cccc} h: & \mathcal{M} & \to & \mathcal{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E, \Phi) & \mapsto & \det(x - \Phi) \end{array}$ Hitchin map
- $\mathbb{C}^\times \circlearrowright \mathcal{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; *semiprojective*:
  1. $\mathcal{M}^{\mathbb{C}^\times}$ projective
  2. $\lim_{\lambda \to 0} \lambda\mathcal{E}$ exists for every $\mathcal{E} \in \mathcal{M}$
- $\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times} \rightsquigarrow W^+_{\mathcal{E}} := \{\mathcal{F} \in \mathcal{M} | \lim_{\lambda \to 0} \lambda\mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W^+_{\mathcal{E}} \subset \mathcal{M}$ locally closed $\cong T^+_{\mathcal{E}}\mathcal{M}$
- $\lambda^*(\omega) = \lambda\omega \rightsquigarrow W^+_{\mathcal{E}} \subset (\mathcal{M}, \omega)$ is Lagrangian
- $\mathcal{M} = \coprod_{\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times}} W^+_{\mathcal{E}}$
- $\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow W^+_{\mathcal{E}}$ closed $\Leftrightarrow h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathcal{A}$ proper
- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathcal{A}$$

becomes explicit!

Examples of very stable Higgs bundles

- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \cdots \oplus K_C^{1-n}$
- $a = (a_2, \dots, a_n) \in \mathcal{A} = H^0(K_C^2) \times \cdots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & 0\dots & 0 & a_n \\ 1 & 0\dots & 0 & a_{n-1} \\ \vdots & & \vdots & \\ 0 & \dots 1 & 0 & a_2 \\ 0 & \dots & 1 & 0 \end{pmatrix} : E_0 \to E_0 K_C$ companion matrix
- $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathcal{M}^{\mathbb{C}^\times}$ *canonical uniformising Higgs bundle*
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ *Hitchin section* $\Rightarrow$ very stable
- $c \in C$, $E_k := \mathcal{O}_C \oplus K_C^{-1} \dots \oplus K_C^{-k}(c) \oplus \dots K_c^{1-n}(c)$,

  $s_c \in H^0(\mathcal{O}_C(c))$, $\Phi_k := \begin{array}{c} \begin{array}{ccc} & k & \end{array} \\ \begin{pmatrix} 0 & \dots 0 & \dots 0 \\ 1 & \dots 0 & \dots 0 \\ \vdots & & \vdots \\ 0 & \dots s_c & \dots 0 \\ \vdots & & \vdots \\ 0 & \dots & 1\ 0 \end{pmatrix} \end{array} : E_k \to E_k K_C$

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_k := (E_k, \Phi_k)$ *is very stable.*

- proof by noticing $W_k^+ := W_{\mathcal{E}_k}^+ = \mathcal{H}_c^{\omega_k}(W_0^+)$
  ω_k kth fundamental character of SL_n, minuscule

Hitchin map as spectrum of equivariant cohomology

- G complex reductive; $E\mathrm{G} \to B\mathrm{G}$ universal principal G-bundle; $H^*_{\mathrm{G}} := H^*(B\mathrm{G};\mathbb{C}) \cong \mathbb{C}[\mathfrak{t}]^{\mathrm{W}}$
- $\mathrm{G} \circlearrowright X$ variety; $X_{\mathrm{G}} := X \times E\mathrm{G}/\mathrm{G}$ *Borel quotient*; $H^*_{\mathrm{G}}(X;\mathbb{C}) := H^*(X_{\mathrm{G}};\mathbb{C})$ *equivariant cohomology* H^*_{G}-algebra
- *equivariantly formal* $\Leftrightarrow H^*_{\mathrm{G}}(X;\mathbb{C})_0 \cong H^*(X;\mathbb{C}) \Leftrightarrow H^*_{\mathrm{G}}(X)$ is free over $H^*_{\mathrm{G}} \Rightarrow \mathrm{Spec}(H^{2*}_{\mathrm{G}}(X;\mathbb{C})) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}/\!\!/\mathrm{W}$ is proper

Theorem (Hausel, 2022)

The Hitchin map on the minuscule upward flow W^+_k is modelled on spectrum of PGL_n-equivariant cohomology of the Grassmannian

$$\begin{array}{ccc} W_k^+ & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(k,n),\mathbb{C})) \\ h_k \downarrow & \lrcorner & \downarrow \\ \mathcal{A} & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}) \end{array}$$

- proof by fixed point scheme in $\mathrm{Gr}(k,n) \cong \mathrm{Gr}^{\omega_k}$ *affine Schubert*
- $\rightsquigarrow$ *multiplicity algebra* $\mathbb{C}[h_k^{-1}(0)] \cong H^{2*}(\mathrm{Gr}(k,n);\mathbb{C})$ (Hausel–Hitchin 2021)

Universal bundle of Kirillov algebras

- $X^*_+(\mathrm{G}) \ni \mu$-highest weight rep. $\rho_\mu : \mathrm{G} \to \mathrm{GL}(V^\mu)$
- *Kirillov algebra*:
  $C_\mu := (S(\mathfrak{g}^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \cong Maps(\mathfrak{g}, \mathrm{End}(V^\mu))^{\mathrm{G}}$
  associative $S(\mathfrak{g}^*)^{\mathrm{G}} \cong H^*_{\mathrm{G}}$-algebra
- (Kirillov 2000) C_μ commutative $\Leftrightarrow \rho_\mu$ weight multiplicity free; e.g. minuscule

Theorem (Hausel 2022)

For $\mathrm{G} \cong \mathrm{SL}_n$, ω_k fundamental $\exists$ universal bundle of algebra structure on $\rho_{\omega_k}(\mathbb{E})_c \cong \Lambda^k(\mathbb{E})_c$ along $W_0^+ \cong \mathcal{A}$ modelled on C_{ω_k}

$$\begin{array}{ccccccccc} C_{\omega_k} & \hookrightarrow & \mathrm{End}(\Lambda^k(\mathbb{E})_c) & & \mathrm{Spec}(C_{\omega_k}) & \twoheadleftarrow & \mathrm{Spec}_{\mathcal{A}}(\Lambda^k(\mathbb{E})_c) \\ \uparrow & \llcorner & \uparrow & \rightsquigarrow & \downarrow & \llcorner & \downarrow \\ H^{2*}_{\mathrm{SL}_n} & \hookrightarrow & \mathbb{C}[\mathcal{A}] & & \mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) & \twoheadleftarrow & \mathcal{A} \end{array}$$

- construction by applying Kirillov operators to Φ_a and using cyclicity of C_{ω_k} (Panyushev 2004)
- $k = 1$ familiar bundle of algebra structure from BNR corr.

Mirror of equivariant cohomology is Kirillov algebra

- (Kapustin–Witten 2007) $\rightsquigarrow \mathcal{S}(\rho_\mu(\mathbb{E}^\vee)_c) = \mathcal{H}_c^\mu(W_0^+)$
- when $\mathrm{G} = \mathrm{PGL}_n$ and $\mu = \omega_k \in X^*_+(\mathrm{SL}_n)$ fundamental$\rightsquigarrow$ $\mathcal{H}_c^{\omega_k}(\mathcal{O}_{W_0^+}) = \mathcal{O}_{W_k^+}$ sheaf of algebras $\rightsquigarrow$ its mirror $\Lambda^k(\mathbb{E}^\vee)_c$ should aquire a bundle of algebra structure along dual Hitchin section from fiberwise Fourier-Mukai transform

Theorem (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(C_{\omega_k}) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathcal{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \arrow[l, two heads] \arrow[d] \arrow[r, phantom, "\cong"] \arrow[dl, phantom, very near start, "\llcorner"] & W_k^+ \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}),\mathbb{C})) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathcal{A}^\vee \arrow[l, two heads] \arrow[r, phantom, "\cong"] & \mathcal{A} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}$$

- using (Panyushev 2004)'s $C_{\omega_k} \cong H^{2*}_{\mathrm{SL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n});\mathbb{C})$
- generalises - partly conjecturally - to all $\mu \in X_*^+(\mathrm{G})$
- $\rightsquigarrow$ ”classical limit of Geometric Satake equivalence”