Enhanced mirror symmetry for Langlands dual Hitchin systems

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ETH zürich | Institute for Theoretical Studies

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dual special Lagrangian fibrations $\Leftrightarrow$ SYZ mirror symmetry

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Lagrangian upward flows in $\mathcal{M}$

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- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \to \mathcal{A}$$

becomes explicit!

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Hitchin map as spectrum of equivariant cohomology

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- *equivariantly formal* $\Leftrightarrow H^*_{\mathrm{G}}(X;\mathbb{C})_0 \cong H^*(X;\mathbb{C})$

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- when $\mathrm{G} = \mathrm{PGL}_n$ and $\mu = \omega_k \in X_+^*(\mathrm{SL}_n)$ fundamental$\rightsquigarrow$ $\mathcal{H}_c^{\omega_k}(\mathcal{O}_{W_0^+}) = \mathcal{O}_{W_k^+}$ sheaf of algebras $\rightsquigarrow$ its mirror $\Lambda^k(\mathbb{E}^\vee)_c$ should aquire a bundle of algebra structure along dual Hitchin section from fiberwise Fourier-Mukai transform

Theorem (Hausel 2022)

$$\begin{tikzcd}
\mathrm{Spec}(C_{\omega_k}) \arrow[d] & \mathrm{Spec}_{\mathcal{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \arrow[l, two heads] \arrow[r, phantom, "\cong"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & W_k^+ \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}),\mathbb{C})) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"] & \mathcal{A}^\vee \arrow[l, two heads] \arrow[r, phantom, "\cong"] & \mathcal{A} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\arrow[from=1-1, to=1-4, bend left, "\cong"]
\end{tikzcd}$$

- using (Panyushev 2004)'s $C_{\omega_k} \cong H^{2*}_{\mathrm{SL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n});\mathbb{C})$
- generalises - partly conjecturally - to all $\mu \in X_*^+(\mathrm{G})$

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- $\rightsquigarrow$ "classical limit of Geometric Satake equivalence"