

Enhanced mirror symmetry for Langlands dual Hitchin systems

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- $\mathcal{M} := \mathcal{M}_{\mathrm{PGL}_n} \ni (E, \Phi); \Phi \in \mathrm{End}_0(E) \otimes K_C$
- $$h : \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ (E, \Phi) & \mapsto & \det(x - \Phi) \end{array} \quad \text{Hitchin map}$$
- $\mathbb{C}^\times \curvearrowright \mathcal{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; *semiprojective*:
 - 1 $\mathcal{M}^{\mathbb{C}^\times}$ projective
 - 2 $\lim_{\lambda \rightarrow 0} \lambda \mathcal{E}$ exists for every $\mathcal{E} \in \mathcal{M}$
- $\mathcal{E} \in \mathcal{M}^{\mathbb{C}^\times} \rightsquigarrow W_{\mathcal{E}}^+ := \{\mathcal{F} \in \mathcal{M} \mid \lim_{\lambda \rightarrow 0} \lambda \mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W_{\mathcal{E}}^+ \subset \mathcal{M}$ locally closed $\cong T_{\mathcal{E}}^+ \mathcal{M}$
- $\lambda^*(\omega) = \lambda\omega \rightsquigarrow W_{\mathcal{E}}^+ \subset (\mathcal{M}, \omega)$ is Lagrangian
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Hitchin map as spectrum of equivariant cohomology

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- proof by fixed point scheme in $\text{Gr}(k, n) \cong \text{Gr}^{\omega_k}$

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