

Enhanced mirror symmetry for Langlands dual Hitchin systems

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Abstract

The first part of this paper is a survey of mathematical results on mirror symmetry phenomena between Hitchin systems for Langlands dual groups. The second part introduces and discusses multiplicity algebras of the Hitchin system on Lagrangians, and considers corresponding conjectural structures on their mirror.



1. Introduction

Considering the 2-dimensional reduction of the Yang-Mills equations in 4 dimensions, Hitchin [Hi1] in 1987 introduced and studied the moduli space of solutions to certain self-duality equations on a Riemann surface. The moduli space turns out to have an "extremely rich geometric structure".

The moduli space \mathcal{M} of solutions for a complex reductive structure group (Hitchin first considered $= \mathrm{SL}_2$), carries a canonical hyperkähler metric g with complex structures I, J and K , and corresponding Kähler forms ω_I, ω_J and ω_K . In complex structure I it agrees with the moduli space $\mathcal{M}_{\mathrm{Dol}}$ of Higgs bundles - or Hitchin pairs - (E, Φ) , where E is a - bundle and the Higgs field $\Phi \in H^0(C; \mathrm{ad}(E) \otimes K)$ is a section of the adjoint bundle twisted by the canonical bundle on a complex curve C :

$$(\mathcal{M}, I) \cong \mathcal{M}_{\mathrm{Dol}}.$$

Under an isomorphism induced by multiplying the Higgs field with i the Kähler manifolds $(\mathcal{M}, I, \omega_I)$ and $(\mathcal{M}, J, \omega_J)$ are isomorphic. In turn, they are both isomorphic with the moduli space $\mathcal{M}_{\mathrm{DR}}$ of flat -connections on the curve C :

$$(1.1) \quad (\mathcal{M}, J) \cong (\mathcal{M}, K) \cong \mathcal{M}_{\mathrm{DR}}.$$

The notation $\mathcal{M}_{\mathrm{Dol}}$ for Dolbeault and $\mathcal{M}_{\mathrm{DR}}$ for De Rham non-abelian cohomologies follows [Si] who introduced the viewpoint of non-abelian Hodge theory in the study of \mathcal{M} .

In turn, Hitchin [Hi2] introduced the Hitchin map

$$h : \mathcal{M} \rightarrow \mathcal{A}.$$

In the $= \mathrm{GL}_n$ case this is just the characteristic polynomial of the Higgs field

$$h(E, \Phi) = \det(x - \Phi) \in \mathcal{A} := \bigoplus_{i=1}^n H^0(C; K^i).$$

For general one needs to consider invariant polynomials on $\mathfrak{g} := \mathrm{Lie}()$ and compute them on the Higgs field. He proved in [Hi2] that h is an algebraically completely integrable Hamiltonian system with respect to the I -holomorphic symplectic form $\omega_C := \omega_I + i\omega_K$. Thus h is sometimes referred to as the *Hitchin system*. This means that $\dim(\mathcal{A}) = \dim(\mathcal{M})/2$ and that the component functions of h are independent and Poisson commute. Additionally, the Hitchin map is proper, which was proved by Hitchin for SL_2 in [Hi2], for GL_n by Nitsure [Ni] and Simpson [Si] and by Faltings [Fa] for general . The complete integrability and properness of the Hitchin map together imply that its generic fiber is a torsor over an Abelian variety. In particular topologically they are isomorphic to compact tori.

Due to the flexibility of their constructions (choice of curves and structure groups - but also various types of ramification data) Hitchin systems have been related to most of the known integrable systems [DM, Do]. They thus play a central role in the field of integrable systems.

Our main interest in this survey will be how mirror symmetry and Langlands duality relate to the Hitchin system. In 2003 the paper [HT] mathematically related the Hitchin system for Langlands dual groups to mirror symmetry. In particular it formulated a topological mirror symmetry conjecture for certain SL_n and PGL_n Hitchin systems.

In 2007 Kapustin-Witten [KW] placed the Hitchin system in the framework of a certain supersymmetric 4-dimensional Yang-Mills theory reduced to 2 dimensions. It also offered a detailed understanding of mirror symmetry and the geometrical Langlands program as a reduction of S -duality in 4 dimensions. This led to many papers, such as [BS, BGH, Hi3, HMP, HH] discussing pairs of mirror branes in Langlands dual Hitchin systems. The last two papers emphasised a further structure on \mathcal{M} , namely a canonical \mathbb{T} -action given by $(E, \Phi) \mapsto (E, \lambda\Phi)$.

In 2010 Ngô [Ng] proved the Fundamental Lemma in the Langlands program, via a detailed understanding of the cohomology of certain singular fibers of the Hitchin map. In [Ha] similarities between the topological mirror symmetry conjecture of [HT] and such cohomological results of Ngô were discussed and were conjecturally related. In [GWZ1] Gröchenig–Wyss–Ziegler proved the topological mirror symmetry conjecture of [HT] using an arithmetic p -adic integration technique. In turn in [GWZ2] the same authors managed to reprove Ngô’s cohomological results with these new p -adic techniques. More recently Maulik–Shen [MS] managed to complete some of the suggestions of [Ha] and derived a proof of the topological mirror symmetry conjecture from Ngô’s results.

First we will discuss some of the background to these developments, and then in §3 we will explain some unpublished results about enhanced mirror symmetry for Langlands dual groups at the tip of the nilpotent cone.

2. Background

2.1. Mirror symmetry

Three aspects of mirror symmetry will be relevant for us: topological and homological mirror symmetry and Strominger–Yau–Zaslow mirror symmetry.

Mirror symmetry in a nutshell relates the complex geometry of a Calabi-Yau X with complex structure I_X and Kähler, in particular symplectic, 2-form ω_X to the symplectic geometry of a same dimensional mirror Calabi-Yau (Y, I_Y, ω_Y) . Originally [GP] 3-dimensional examples of such a correspondence appeared in string theory and mirror symmetry as the statement that the physics of a certain 2-dimensional type A non-linear sigma model with target (X, ω_X) matches that of a 2-dimensional type B non-linear sigma model with target (Y, I_Y) .

The study of mathematical aspects of this mirror symmetry has been one of the central subjects in modern symplectic/ complex algebraic geometry. The first mathematical aspect of the mirror relationship is the agreement of Hodge numbers $h^{n-p,q}(X) = h^{p,q}(Y)$, which we call *topological mirror symmetry*.

The formulation of Kontsevich [Ko] in 1994 of *homological mirror symmetry*

$$Fuk(X, \omega_X) \cong D^b(Y, I_Y)$$

- the agreement of the Fukaya category of X and the derived category of coherent sheaves on Y - gave a profound mathematical conjecture for what mirror symmetry should mean.

The early 90's saw several constructions of conjectured mirror pairs in [CLS, Ba1, BB]. In 1996 [SYZ] Strominger–Yau–Zaslow suggested a way to construct the mirror of a Calabi-Yau 3-fold X out of the geometry of X . They argued that there should be fibrations of X and Y over the same base $B \cong S^3$:

$$(2.1) \quad \begin{array}{ccc} X & & Y \\ & \searrow & \swarrow \\ & B & \end{array}$$

so that the generic fibers are dual special Lagrangian 3-tori. Here $L \subset X$ special Lagrangian means that L is Lagrangian $\omega_X|_L = 0$ and additionally $Im(\Omega_X)|_L = 0$ the imaginary part of the Calabi-Yau volume form vanishes on L . In turn then Y should be thought of as the moduli space of certain objects in $Fuk(X, \omega_X)$, generically defined by special Lagrangian 3-tori equipped with a $U(1)$ -local system. A mathematical formulation of Strominger–Yau–Zaslow was pursued by Gross–Siebert [GS1], with many accomplishments and recent breakthroughs [GS2, GHKK].

Between mirror Calabi–Yau 3-folds a complete construction of dual special Lagrangian fibrations (2.1) is still missing. The Higgs bundle moduli spaces for Langlands dual groups, where the Hitchin systems will automatically give us such dual special Lagrangian fibrations, is a natural example, albeit in a geometrically different scenario from the original [SYZ].

2.2. Geometric Langlands correspondence

In the works of [Dr1],[La1],[BD] a geometric version of the Langlands correspondence has been proposed. Recall that the Langlands program in number theory (see [Ge, Fr] for some introductory ideas) for a reductive group over a number field relates automorphic data for n -like modular forms for SL_2 - with spectral data n -like a Galois representation on the cohomology of an elliptic curve for SL_2 - for a Langlands dual L group. The conjectures can be formulated over the other kind of global field as well: the function field of a curve over a finite field. The conjectures become more tractable in this case as the algebraic geometry of curves can be efficiently used. Here we will consider the even more geometric version of this program for function fields of a curve over the complex numbers.

Over \mathbb{C} the Langlands dual ${}^L := {}^L$ of a complex reductive group is simple to construct. The classification of complex reductive groups is via their root datum $(X, \Phi, X^\vee, \Phi^\vee)$ - consisting of a rank n lattice X , a root system and coroot system $\Phi \subset X$ and $\Phi^\vee \subset X^\vee$ in the dual lattice X^\vee , satisfying certain properties- attached to all rank n complex reductive groups. The Langlands dual of then is the reductive group ${}^\vee$ whose root datum is the dual root datum $(X^\vee, \Phi^\vee, X, \Phi)$. For example $GL_n^\vee \cong GL_n$ and $SL_n^\vee \cong PGL_n$.

The geometric Langlands correspondence of Beilinson-Drinfeld [BD] for a smooth projective curve C proposes to construct from a n -local system on C - a geometric analogue of a Galois representation- a holonomic D -module on the moduli stack of bundles $Bun_{n, \vee}$ - a geometric analogue of an automorphic form. The main property of this construction is that the holonomic D -module must be an eigensheaf of certain Hecke operators. [BD] succeed in this construction for a certain set of n -local systems on C , the so-calledopers.

We will see below how Beilinson-Drinfeld's picture can be understood also as enhanced mirror symmetry between the hyperkähler, and thus in particular Calabi-Yau, moduli space of flat connections on \mathcal{M}_{DR} and the moduli space of flat \vee connections $\mathcal{M}_{\text{DR}}^\vee$.

2.3. SYZ mirror symmetry for Langlands dual Hitchin systems

The starting point of [HT] was the observation that for $= \text{SL}_n$ and $\vee = \text{PGL}_n$ the two Hitchin systems $h : \mathcal{M}_{\text{Dol}} \rightarrow \mathcal{A}$ for -Higgs bundles and $\mathcal{M}_{\text{Dol}}^\vee \rightarrow \mathcal{A}^\vee$ for \vee Higgs bundles have the same base $\mathcal{A} \cong \mathcal{A}^\vee$ and in the diagram

$$(2.2) \quad \begin{array}{ccc} \mathcal{M}_{\text{Dol}} & & \mathcal{M}_{\text{Dol}}^\vee \\ & \searrow h & \swarrow h^\vee \\ & \mathcal{A} & \end{array}$$

the generic fibers are dual Abelian varieties. The fibers are holomorphic Lagrangian with respect to the holomorphic symplectic forms $\omega_{\mathbb{C}} := \omega_J + i\omega_K$, i.e. $\omega_{\mathbb{C}} = 0$, thus both $\omega_J = \omega_K = 0$, along the fibers because of the complete integrability of the Hitchin systems.

The same maps in complex structure J then yield

$$(2.3) \quad \begin{array}{ccc} \mathcal{M}_{\text{DR}} & & \mathcal{M}_{\text{DR}}^\vee \\ & \searrow h & \swarrow h^\vee \\ & \mathcal{A} & \end{array}$$

where the generic fibers are special Lagrangian fibrations. This means that ω_J vanishes on the fibers, thus they are Lagrangian. Additionally ω_K as well as the imaginary part of the J -holomorphic Calabi-Yau form $(\omega_K + i\omega_I)^{2d}$ also vanish along the fibers.

To find a mathematically testable form of mirror symmetry the paper [HT] formulated *topological mirror symmetry* between the mirror Calabi-Yau's: \mathcal{M}_{DR} and $\mathcal{M}_{\text{DR}}^\vee$.

2.4. Topological mirror symmetry for Langlands dual Hitchin systems

To formulate this version of mirror symmetry from [HT] we will be a bit more precise about our moduli spaces. For SL_n we consider $\mathcal{M} := \mathcal{M}_{\text{Dol}}$ the moduli space of stable Hitchin pairs of rank n fixed bundles E on C of fixed determinant line bundle of degree 1 with trace-free Higgs fields $\Phi \in H^0(C; \text{End}_0(E) \otimes K)$. For PGL_n we consider the action of $\Gamma := \text{Jac}_C[n]$, the group of order n line bundles on C , on the SL_n moduli space \mathcal{M} and define $\mathcal{M}^\vee := \mathcal{M}/\Gamma$. Then \mathcal{M} will be a smooth quasi-projective variety, while \mathcal{M}^\vee is a quasi-projective orbifold. For our considerations we will need extra twisting structure on the moduli spaces in the form of a gerbe

$$\alpha \in H^2(\mathcal{M}^\vee, \text{U}(1)) \cong H^2(\mathcal{M}, \text{U}(1))^\Gamma$$

which can be constructed using the universal bundle on \mathcal{M} .

One can then define certain mixed Hodge numbers

$$h^{p,q}(\mathcal{M}) = h^{p,q}(H_c^{p+q}(\mathcal{M}))$$

for the smooth \mathcal{M} and α -twisted stringy Hodge numbers for the orbifold $\mathcal{M}^\vee = \mathcal{M}/\Gamma$:

$$h_{st,\alpha}^{p,q}(\mathcal{M}^\vee) := \sum_{\gamma \in \Gamma} h^{p-F(\gamma); q-F(\gamma)}(H^{p+q-2F(\gamma)}(\mathcal{M}^\vee; L_\alpha)^\Gamma),$$

where $F(\gamma)$ is the fermionic shift, defined from the action of γ on the tangent space of \mathcal{M} at a γ -fixed point.

With these we can formulate our topological mirror symmetry conjecture:

Conjecture 2.1 ([HT]). *We have an agreement of Hodge numbers $h^{p,q}(\mathcal{M}) = h_{st,\alpha}^{p,q}(\mathcal{M}^\vee)$.*

Note that as it stands the conjecture is about the Hodge numbers of the Higgs moduli spaces $\mathcal{M} = \mathcal{M}_{\text{Dol}}$. However it was proved in [HT] that the Hodge numbers of \mathcal{M}_{Dol} and \mathcal{M}_{DR} agree, and so (2.1) is also about the agreement of (mixed) Hodge numbers of the proposed mirrors \mathcal{M}_{DR} and $\mathcal{M}_{\text{DR}}^\vee$. This way we can interpret Conjecture 2.1 as topological mirror symmetry for our SYZ mirror pair \mathcal{M}_{DR} and $\mathcal{M}_{\text{DR}}^\vee$.

Conjecture 2.1 was proved for SL_2 and SL_3 in [HT]. More recently in 2020 the general case of SL_n was settled by Gröchenig–Wyss–Ziegler [GWZ1]. They used an arithmetic p -adic integration technique, pioneered by Batyrev [Ba2] to check some topological mirror symmetry conjectures in the usual mirror symmetry.

In [Ha] we observed a curious similarity. Namely Ngô in his proof [Ng] of the Fundamental Lemma in the Langlands program, reduced the Fundamental Lemma to the agreement of the number of points of certain singular Hitchin fibers over finite fields and in turn to the agreement of certain Hodge numbers of singular fibers of the Hitchin fibration. In [Ha][§5.4] we argued that Ngô’s cohomological result is a relative version of the topological mirror symmetry conjecture along the Hitchin fibration. A strategy was also proposed to deduce topological mirror symmetry using Ngô’s techniques. This proposal has been recently completed by Maulik–Shen in [MS] in 2020 giving a new proof of the topological mirror symmetry Conjecture 2.1 using Ngô’s techniques.

Finally, in 2020, Gröchenig–Wyss–Ziegler in [GWZ2] managed to extend their p -adic integration techniques from [GWZ1] for Higgs moduli spaces of general reductive groups and in turn they found a new proof of Ngô’s cohomological result.

2.5. Geometric Langlands as enhanced Homological Mirror Symmetry

In 2007 Kapustin–Witten [KW] put forward a detailed circle of ideas amounting to a physics derivation of the geometric Langlands Correspondence as an enhanced mirror symmetry. They argued that a well-studied S -duality (or electro-magnetic or Montonen–Olive duality [MO]) in a certain four-dimensional $N = 4$ supersymmetric Yang–Mills theory, when reduced to two dimensions, yields an enhanced mirror symmetry, which in turn recovers the geometric Langlands correspondence as formulated by [BD].

In this two-dimensional reduction, Montonen–Olive duality becomes an equivalence of a type B sigma model with target the moduli space \mathcal{M}_{DR} of flat -connections on a complex curve C and a type A sigma model with target $\mathcal{M}_{\text{DR}}^\vee$, the moduli space of flat $^\vee$ -connections on C . As a consequence the category of boundary conditions in the two theories should be equivalent

$$(2.4) \quad \mathcal{S} : D^b(\mathcal{M}_{\text{DR}}) \simeq \text{Fuk}(\mathcal{M}_{\text{DR}}^\vee),$$

which can be interpreted as Kontsevich’s homological mirror symmetry conjecture applied to the mirror pair \mathcal{M}_{DR} and $\mathcal{M}_{\text{DR}}^{\vee}$.

Kapustin–Witten [KW] explained that this equivalence of categories has more structure due to the hyperkähler targets, and more symmetries due to their origin in 4-dimensions than the usual homological mirror symmetry, which arises from an equivalence of two 2-dimensional sigma models. They use these additional ideas to construct from a flat connection α in \mathcal{M}_{DR} , considered by its skyscraper sheaf $\mathcal{O}_{\alpha} \in D^b(\mathcal{M}_{\text{DR}})$, an element of $\text{Fuk}(\mathcal{M}_{\text{DR}}^{\vee})$ which they interpret as a D -module on the moduli space of \vee -bundles on our curve C . The Hecke eigensheaf property then in turn is deduced from the extra symmetry stemming from the 4-dimensional origin.

First, due to the hyperkähler targets, Kapustin–Witten talk about more structured branes (aka boundary conditions) by proposing that a brane should be either type A or type B with respect to all the three complex structures I, J and K . This way they consider type (B, A, A) , (A, B, A) , (A, A, B) and type (B, B, B) branes on hyperkähler manifolds. For example a type (B, A, A) brane could be an I -holomorphic $\omega_C = \omega_J + i\omega_K$ Lagrangian subvariety together with a local system. Or a type (B, B, B) brane should be a hyperkähler submanifold together with a hyperholomorphic connection on a bundle over it.

In their framework [KW] argue that the mirror (S-dual) of an (B, A, A) brane on the hyperkähler \mathcal{M} should be a (B, B, B) brane on \mathcal{M}^{\vee} . In particular, if we just concentrate on complex structure J , that of \mathcal{M}_{DR} , we see that the mirror of an A brane should be a B brane. The mirror relationships are slightly more subtle [KW, Table 2 p74] in that the A -model in complex structure J should be mirror to the B -model in complex structure K , which in turn, by (1.1), yields our version. This more refined version of mirror symmetry matches a type B brane in complex structure I to another type B brane in complex structure I on the mirror.

This latter correspondence was also formulated by Donagi–Pantev [DP] as a classical limit - a first approximation of (2.4)- as

$$(2.5) \quad \mathcal{S} : D^b(\mathcal{M}_{\text{Dol}}) \simeq D^b(\mathcal{M}_{\text{Dol}}^{\vee})$$

of the homological mirror symmetry (2.4). First [DP] checks that (2.2) generically gives dual abelian varieties as fibers, for every reductive group. Second, they check that generically the Fourier-Mukai transform [Mu] relative to the Hitchin base gives an equivalence like (2.5) which satisfies the additional intertwining of Hecke–Wilson symmetries discussed below. It is expected that (2.5) will have to be modified when extended over certain singular points of the moduli spaces.

Another direction of research - motivated by [KW]’s consideration of hyperkähler branes - lead to new understandings of Lagrangian subvarieties in \mathcal{M}_{Dol} and \mathcal{M}_{DR} and hyperholomorphic sheaves on \mathcal{M} . For example [BS, BGH, Hi3] studied various constructions of such hyperkähler branes in all four different types, and contemplated what their mirror should be. In particular, Hitchin [Hi3] proposed pairs of a (B, A, A) brane on \mathcal{M} and (B, B, B) brane on \mathcal{M}^{\vee} for which he could show that generically over the Hitchin base they are Fourier-Mukai dual. The $= \text{GL}_2$ case of Hitchin’s suggestion was the starting point of

[HMP], where an additional structure, the \mathbb{T} -action, played an important role. This point of view will be explained below in more detail.

Second, due to the 4-dimensional origin of their derivation of (2.4) Kapustin–Witten considered extra symmetries on these categories, arising from line operators in the 4-dimensional theory. Namely, Wilson operators \mathcal{W}^μ attached to representations μ of \mathfrak{g} act on $D^b(\mathcal{M}_{\text{DR}})$ via tensoring with the vector bundle in the representation μ of the \mathfrak{g} -bundle underlying the universal flat connection. On the other side Hecke operators (or t’Hooft operators for the physicists) \mathcal{H}^μ attached to irreducible representations μ of \mathfrak{g} act on the moduli space of \mathbb{C}^* bundles and in turn on D -modules on them. The extra symmetry observation of [KW] is that these operators should intertwine the mirror symmetry of (2.4). We will spell out these operators in a more detailed way in the more symmetric classical limit (2.5) of [DP] in §3.4 below.

The homological mirror symmetry (2.4) with these two additional structures: matching of hyperkähler branes under mirror symmetry, and the Wilson-Hecke symmetry is what we call *enhanced mirror symmetry*. These go beyond the usual homological mirror symmetry of Kontsevich and stem from the supersymmetric and 4-dimensional origins of S -duality.

3. Enhanced mirror symmetry at the tip of the nilpotent cone

The original motivation for the considerations below is to find a way to test the conjectured mirror pairs of (B, A, A) and (B, B, B) branes put forward in [BS, BGH, Hi3]. The only tests so far - which were often carried out in *loc.cit.* - are to check if the proposed mirror pairs are indeed Fourier-Mukai dual relative to the Hitchin maps. This can only be checked generically over the Hitchin base. We would like to see more global checks, in particular ones which can verify mirror symmetry proposals over the 0-fibers of the Hitchin maps: the global nilpotent cones.

We introduced a technique in [HMP, HH] which can verify mirror symmetry proposals over the nilpotent cone by considering the effect of mirror symmetry on morphisms in the corresponding categories. The difficulty to consider the morphisms in our categories arises from the non-compactness of our moduli spaces. For example the vector spaces of morphisms in the derived category $D(\mathcal{M}_{\text{Dol}})$ are typically infinite dimensional. To measure their size, we will be looking at the \mathbb{T} -equivariant structure on them. Recall that the multiplicative group $\mathbb{T} := \mathbb{C}^\times$ of the complex numbers acts on \mathcal{M} by $\lambda : (E, \Phi) \rightarrow (E, \lambda\Phi)$, scalar multiplication of the Higgs field. Here we will be interested in a \mathbb{T} -equivariant extension of the classical limit (2.5) of the geometric Langlands correspondence, which should be as a first approximation an equivalence $\mathcal{S} : D_{\mathbb{T}}(\mathcal{M}_{\text{Dol}}) \sim D_{\mathbb{T}}(\mathcal{M}_{\text{Dol}}^\vee)$ between the \mathbb{T} -equivariant derived categories of \mathcal{M}_{Dol} and $\mathcal{M}_{\text{Dol}}^\vee$. The morphisms between two objects \mathcal{F}_1 and \mathcal{F}_2 in $D_{\mathbb{T}}(\mathcal{M}_{\text{Dol}})$ can be identified with the graded vector space $\text{Ext}^*(X; \mathcal{F}_1, \mathcal{F}_2)$. To measure this graded vector space, we note that \mathbb{T} acts on it, and assuming that the weight spaces are finite dimensional and vanish for large enough weights (which we expect for semiprojective

varieties) we can define the *equivariant Euler form* as

$$\begin{aligned}\chi_{\mathbb{T}}(X; \mathcal{F}_1, \mathcal{F}_2) &= \sum_{k,l} \dim(H^k(R\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2))^l) (-1)^k t^{-l} = \\ &= \sum_{k,l} \dim\left(\mathrm{Hom}_{\mathbf{D}_{coh}(X)}(\mathcal{F}_1, \mathcal{F}_2[k])^l\right) (-1)^k t^{-l} = \\ &= \sum_{k,l} \dim(\mathrm{Ext}^k(X; \mathcal{F}_1, \mathcal{F}_2)^l) (-1)^k t^{-l} \in \mathbb{C}((t)).\end{aligned}$$

With this we expect that \mathcal{S} is an isometry:

$$\chi_{\mathbb{T}}(\mathcal{S}(\mathcal{F}), \mathcal{S}(\mathcal{G})) = \chi_{\mathbb{T}}(\mathcal{F}, \mathcal{G})$$

In [HMP] we managed to check this isometry for several pairs of conjectured mirror branes from [Hi3], while in [HH] we checked this isometry for the conjectured mirror pairs relevant here; see below. In the second part of this paper we will recall some results of [HH] about the mirror of very stable upward flows, introduce and study the multiplicity algebras of the Lagrangian upward flows following [HH1], and finally we will consider what the multiplicity algebra should correspond to on the mirror.

3.1. Very stable Higgs bundles and mirror symmetry

The starting point is the recent paper [HH]. First we recall its formalism.

3.1.1. Białyński-Birula decomposition of semiprojective varieties. Let the multiplicative group $\mathbb{T} := \mathbb{C}^\times$ of the complex numbers act on a (possibly reducible) variety (a reduced separated scheme of finite type over \mathbb{C}). We say that the action is *semiprojective* [HV, §1] if the following three conditions hold

- (1) the action is *linear*, i.e. there is a locally closed \mathbb{T} -equivariant embedding of X into \mathbb{P}^N with a linear action of \mathbb{T} (for example when X is normal and quasi-projective)
- (2) the fixed point subvariety $X^{\mathbb{T}}$ is proper and thus projective
- (3) $\lim_{\lambda \rightarrow 0} \lambda \cdot x$ exists for all $x \in X$

For $\alpha \in X^{\mathbb{T}}$ we define $W_\alpha^+ := \{x \in X \mid \lim_{\lambda \rightarrow 0} \lambda \cdot x = \alpha\}$ the *upward flow* from α and $W_\alpha^- := \{x \in X \mid \lim_{\lambda \rightarrow \infty} \lambda \cdot x = \alpha\}$ the *downward flow* from α . Then we have

$$X = \bigsqcup_{\alpha \in X^{\mathbb{T}}} W_\alpha^+$$

the Białyński-Birula partition of X and we define the projective variety

$$C := \bigsqcup_{\alpha \in X^{\mathbb{T}}} W_\alpha^-$$

to be the *core* of X . We then have the following

Theorem 3.1. ([B-B],[HH, Proposition 2.1,Proposition 2.10]). When $\alpha \in (X^s)^\mathbb{T}$, a \mathbb{T} -fixed point on the smooth locus, then $W_\alpha^\pm \subset X$ are locally closed subvarieties and $W_\alpha^\pm \cong T_\alpha^\pm X$ as \mathbb{T} -varieties. Moreover when $\omega \in \Omega^2(X^s)$ is a homogeneity 1 symplectic form then $W_\alpha^+ \subset X$ and $C \subset X$ are Lagrangian subvarieties.

We call $\alpha \in (X^s)^\mathbb{T}$ and W_α^+ very stable when $W_\alpha^+ \cap C = \{\alpha\}$. Equivalently, $\alpha \in (X^s)^\mathbb{T}$ is very stable if and only if $W_\alpha^+ \cap W_\beta^- \neq \emptyset$ implies $\beta = \alpha$. More generally one can show that the relation $\alpha \leq \beta$ when $W_\alpha^+ \cap W_\beta^- \neq \emptyset$ induces a partial ordering. Then $\alpha \in (X^s)^\mathbb{T}$ is very stable if it is maximal with respect to this ordering. We then have

Theorem 3.2. ([HH, Proposition 2.14].) $\alpha \in (X^s)^\mathbb{T}$ is very stable if and only $W_\alpha^+ \subset X$ is closed.

3.1.2. Białynicki-Birula partition for Higgs bundles. We will work with $= \text{GL}_n$. We will denote by \mathcal{M} the moduli space of semistable rank n degree d Higgs bundles (E, Φ) on a smooth projective curve of genus g . Here E is a rank n vector bundle of degree d on C and the Higgs field $\Phi \in H^0(C; \text{End}(E) \otimes K)$.

We have the Hitchin map [Hi2] given by the characteristic polynomial of the Higgs field:

$$\begin{aligned} h : \quad \mathcal{M} &\rightarrow \mathcal{A} := \times_{i=1}^n H^0(C; K^i) \\ (E, \Phi) &\mapsto \det(x - \Phi). \end{aligned}$$

Then h is a proper map [Hi1, Ni, Si] and a completely integrable Hamiltonian system [Hi2, Fa] with respect to a natural holomorphic symplectic form on \mathcal{M} . In particular, the generic fibers are Lagrangian Abelian varieties, Jacobians of certain spectral curves.

The \mathbb{T} -action on \mathcal{M} is given by $(E, \Phi) \mapsto (E, \lambda\Phi)$. That makes the Hitchin map \mathbb{T} -equivariant if we let \mathbb{T} act on $H^0(C; K^i)$ with weight i . As these weights are all positive on $\mathcal{A} \cong \times_{i=1}^n H^0(C; K^i)$ it is semiprojective, and as the Hitchin map is proper and \mathbb{T} -equivariant we get that our \mathbb{T} -action on \mathcal{M} is also semiprojective. Additionally, we get that the core of \mathcal{A} , i.e. the origin $0 \in \mathcal{A}$ pulls back to the core of \mathcal{M} . For \mathcal{M} the core agrees with the nilpotent cone $h^{-1}(0)_{red} = C$.

Generalising the notion of very stable bundle of Drinfeld and Laumon [Dr2, La1] we can thus define a very stable Higgs bundle as a stable \mathbb{T} -fixed Higgs bundle $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ for which $W_\mathcal{E}^+ \cap h^{-1}(0) = \{\mathcal{E}\}$ the only nilpotent Higgs bundle in its upward flow is itself. Thus by Theorem 3.2 we know $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is very stable exactly when its upward flow is closed. To reformulate in terms of the Hitchin map we have an alternate version of Theorem 3.2.

Theorem 3.3 ([HH1] c.f. [Ze]). $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is very stable if and only if $h^{-1}(0) \cap W_\mathcal{E}^+ = \{\mathcal{E}\}$ if and only $h_\mathcal{E} := W_\mathcal{E}^+ \rightarrow \mathcal{A}$ is proper if and only if it is finite.

Definition 3.4. For $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ define the rational function

$$m_\mathcal{E}(t) := \frac{\chi_\mathbb{T}(\text{Sym}(T_\mathcal{E}^{+*}))}{\chi_\mathbb{T}(\text{Sym}(\mathcal{A}^*))} \in \mathbb{Z}(t).$$

We call it the equivariant multiplicity of $W_\mathcal{E}^+$.

We have the following

Theorem 3.5. ([HH, Corollary 5.4]). When $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is very stable $m_{\mathcal{E}}(t)$ is a polynomial

- with non-negative coefficients, which is
- palindromic and
- monic, such that
- $m_{\mathcal{E}}(1) = m_{F_{\mathcal{E}}}$ is the multiplicity of the component $N_{F_{\mathcal{E}}} \subset N$ in the nilpotent cone.

Let $\ell \in \mathbb{Z}$, $m_i \in \mathbb{Z}_{\geq 0}$ and

$$\delta := (\delta_0, \delta_1, \dots, \delta_{n-1}) \in \text{Jac}_{\ell}(C) \times C^{[m_1]} \times \dots \times C^{[m_{n-1}]}$$

be a vector of representative divisors on C . To this we can construct a type $(1, \dots, 1)$ \mathbb{T} -fixed Higgs bundle $\mathcal{E}_{\delta} = (E_{\delta}, \Phi_{\delta})$ where

$$\mathcal{E}_{\delta} = M_0 \oplus \dots \oplus M_{n-1}$$

is a rank n vector bundle

$$M_i := \mathcal{O}(\delta_0 + \dots + \delta_i)K^{-i}$$

and

$$\Phi_{\delta}|_{M_i} : M_i \rightarrow M_{i+1}K \subset E_{\delta}K$$

is given by the defining section of

$$H^0(C; M_i^{-1}M_{i+1}K) \cong H^0(C; \mathcal{O}(\delta_i)).$$

The following classifies all very stable \mathbb{T} -fixed type $(1, \dots, 1)$ Higgs bundles and gives their equivariant multiplicity.

Theorem 3.6. ([HH, Theorem 4.16,(5.18)]). Let δ be as above and suppose that \mathcal{E}_{δ} is a stable Higgs bundle. Then \mathcal{E}_{δ} is very stable if and only if the effective divisor $\delta_1 + \dots + \delta_{n-1}$ is reduced. Its equivariant multiplicity is given by

$$m_{\mathcal{E}_{\delta}}(t) = \prod_{i=1}^{n-1} \left[\begin{matrix} n \\ i \end{matrix} \right]_t^{m_i},$$

product of t -binomial coefficients.

3.2. Multiplicity algebra and explicit Hitchin system on Lagrangians

The main idea is to study, for $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$, the restricted Hitchin map

$$h_{\mathcal{E}} := h|_{W_{\mathcal{E}}^+} : W_{\mathcal{E}}^+ \rightarrow \mathcal{A}$$

in the framework of the Arnold school [AGV]. It is a \mathbb{T} -equivariant *Lagrangian map* between semiprojective vector spaces (i.e. only positive \mathbb{T} -weights) of the same dimension. Such maps are called *quasi-homogeneous* in [AGV, §12.3].

We recall from Theorem 3.3 that \mathcal{E} is very stable if and only if $h_{\mathcal{E}}^{-1}(0) = \{\mathcal{E}\}$ if and only if $h_{\mathcal{E}}$ is proper. For such maps - called *non-degenerate* in *loc.cit.*- [AGV, §4,5,12] introduces and studies its local multiplicity algebra.

Definition 3.7. When $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$ is very stable define

$$Q_{\mathcal{E}} := Q_{h_{\mathcal{E}}} := \mathbb{C}[W_{\mathcal{E}}^+]/(h_{\mathcal{E}}^{-1}(\mathfrak{m}_0)) = \mathbb{C}[W_{\mathcal{E}}^+]/(h_1, \dots, h_N)$$

the *local multiplicity algebra* of $h_{\mathcal{E}}$ at \mathcal{E} . Here $\mathfrak{m}_0 \subset \mathbb{C}[\mathcal{A}]$ is the maximal ideal at $0 \in \mathcal{A}$ and

$$h_{\mathcal{E}} = (h_1, \dots, h_N) : \mathbb{C}^N \cong W_{\mathcal{E}}^+ \rightarrow \mathbb{C}^N \cong \mathcal{A}$$

in some homogeneous coordinates.

Scheme-theoretically $Q_{\mathcal{E}}$ is just the coordinate ring of the scheme theoretical fiber of $h_{\mathcal{E}}$ over 0 or the scheme-theoretical intersection of $W_{\mathcal{E}}^+ \cap h^{-1}(0)$ of the upward flow with the nilpotent cone. Because $h_{\mathcal{E}}$ is \mathbb{T} -equivariant we will get a \mathbb{T} -action, and thus a grading on $Q_{\mathcal{E}}$. Because of this sometimes we call $Q_{\mathcal{E}}$ the *equivariant multiplicity algebra* of $h_{\mathcal{E}}$ at \mathcal{E} .

Remark 3.8. Note that determining the algebra $Q_{\mathcal{E}}$ explicitly by N generators and N relations gives us coordinates on $W_{\mathcal{E}}^+$ such that the Hitchin map is given *explicitly* by the relations.

Using results of [AGV, §4,5,12] we have the following

Theorem 3.9 ([HH1]). *Let $\mathcal{E} \in \mathcal{M}^s$ be very stable. Then its local multiplicity algebra is*

- (1) *finite dimensional,*
- (2) *graded $Q_{\mathcal{E}} := \bigoplus_{k=0}^m Q_{\mathcal{E}}^k$ such that $Q_{\mathcal{E}}^0 \cong \mathbb{C}$,*
- (3) *Gorenstein, with socle $Q_{\mathcal{E}}^m \cong \mathbb{C}J_{h_{\mathcal{E}}}$, which is one-dimensional and spanned by the Jacobian $J_{h_{\mathcal{E}}}$ of $h_{\mathcal{E}} = (h_1, \dots, h_N)$ and*
- (4) *a Poincaré duality ring. That is it has a natural bilinear pairing $(,) : Q_{\mathcal{E}} \times Q_{\mathcal{E}} \rightarrow \mathbb{C}$ inducing a perfect pairing $Q_{\mathcal{E}}^k \times Q_{\mathcal{E}}^{m-k} \rightarrow \mathbb{C}$ for all k .*
- (5) *Finally, its Poincaré polynomial $\sum_{k=0}^m \dim(Q_{\mathcal{E}}^k)t^k = m_{\mathcal{E}}(t)$ agrees with the equivariant multiplicity.*

This result gives a satisfactory explanation of all the properties of the equivariant multiplicity polynomial we observed in Theorem 3.5. The following then gives an explanation for the appearance of quantum binomial coefficients - which is well-known to be the Poincaré polynomial of a Grassmannian- for the equivariant multiplicity.

Theorem 3.10 ([HH1]). *Let $\delta = (\delta_0, \delta_1, \dots, \delta_{n-1}) \in J_{\ell}(C) \times C^{[m_1]} \times \dots \times C^{[m_{n-1}]}$ be a vector of divisors. Assume that $\delta_1 + \dots + \delta_{n-1}$ is reduced, then \mathcal{E}_{δ} is very stable from Theorem 3.6 and its equivariant multiplicity algebra is*

$$Q_{\delta} \cong \prod_{i=1}^{n-1} H^*(\text{Gr}(i, n); \mathbb{C})^{m_i}$$

the product of the complex cohomology rings of Grassmannians $\text{Gr}(i, n)$ of i -dimensional subspaces in \mathbb{C}^n .

Our attack on Theorem 3.10 starts with understanding the map $h_\delta : W_\delta^+ \rightarrow \mathcal{A}$ using Hecke transformations. We explain it here in the first non-trivial case when

$$\delta_c^k := (\delta_0, \dots, \delta_{n-1})$$

where $\delta_i = 0$ unless $i = k$ and the divisor $\delta_k = c$ is one point. For this we consider the k -th fundamental Hecke correspondence over the Hitchin section $W_0^+ := W_{\mathcal{E}_0}^+$ which coincides with the upward flow of the *canonical uniformising Higgs bundle* $\mathcal{E}_0 = (E_0, \Phi_0)$. Recall [HH, (3.16)] that it is given in chain notation as

$$(E_0, \Phi_0) = (O \oplus K^{-1} \oplus \dots \oplus K^{1-n}, \Phi_0) = O \xrightarrow{1} K^{-1} \xrightarrow{1} \dots \xrightarrow{1} K^{1-n},$$

where morphisms on the arrows are twisted by K . The canonical uniformising Higgs bundle is at the top of the nilpotent cone. We will use Hecke transformations to generate all other type $(1, \dots, 1)$ upward flows from the Hitchin section W_0^+ .

We let

$$(3.1) \quad \mathcal{H}_k := \{(\mathcal{E}_a, V) \in W_0^+ \times \text{Gr}(k, E_0|_c) | \Phi_a|_c(V) \subset V\} \subset W_0^+ \times \text{Gr}(k, E_0|_c).$$

It can be constructed étale locally over \mathcal{A} as the fixed point scheme of the self-map of $\text{Gr}(k, E_0|_c)$ induced by the invertible $\Phi_a|_c - \lambda I : E_0 \rightarrow E_0$ where $\lambda \in \mathbb{C}$ is not an eigenvalue of Φ_a . By performing a Hecke transformation of \mathcal{E}_a at V for a point $(\mathcal{E}_a, V) \in \mathcal{H}_k$ we get that $\mathcal{H}_k \cong W_{\delta_c^k}^+$ and moreover have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_k & \xrightarrow{\cong} & W_{\delta_c^k}^+ \\ \pi_k \downarrow & & \downarrow h_{\delta_c^k} \\ W_0^+ & \xrightarrow{h_0} & \mathcal{A} \end{array}$$

Hence $Q_{\delta_c^k} \cong Q_{\pi_k}$. In turn, the computation of Q_{π_k} can be done in the Grassmannian $\text{Gr}(k, E_0|_c)$ and will yield in two different ways the two isomorphisms

$$(3.2) \quad H^*(\text{Gr}(k, E_0|_c); \mathbb{C}) \cong Q_{\pi_k} \cong \frac{\mathbb{C}[p_1, \dots, p_k, q_1, \dots, q_{n-k}]}{((p_k + \dots + p_1 x^{k-1} + x^k)(q_{n-k} + \dots + q_1 x^{n-k-1} + x^{n-k}) = x^n)}.$$

Of course the isomorphism of the first and third rings gives the well-known presentation of the cohomology ring of the Grassmannian.

In effect, we can think of the determination of the multiplicity algebra in (3.2) to give coordinates on the upward flow $W_{\delta_c^k}^+$ so that the Hitchin map $h_{\delta_c^k}$ becomes explicit - basically given by the relations in the cohomology ring of the Grassmannian.

3.3. Explicit Hitchin system for wobbly Lagrangians

Here we show how one can generalise the technique above to understand multiplicity algebras of wobbly - i.e. not very stable- upward flows using the affine Grassmannian [Gö, Zh]. We start with a generalised notion of the k -th fundamental Hecke correspondence over the Hitchin section. Recall the *affine Grassmannian*

$$\text{Gr} := \text{GL}_n((z))/\text{GL}_n[[z]],$$

where $\mathrm{GL}_n((z)) := \mathrm{GL}_n(\mathbb{C}((z)))$ and $\mathrm{GL}_n[[z]] := \mathrm{GL}_n(\mathbb{C}[[z]])$. It is a projective ind scheme, in particular its reduced is a nested union of projective varieties. It parametrises higher Hecke transformations of a vector bundle at a point on a curve.

Let

$$\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_n) \in P^+ \subset P \cong \mathbb{Z}^n$$

be a dominant weight and

$$z^\mu := \begin{pmatrix} z^{\mu_1} & 0 & \dots & 0 \\ 0 & z^{\mu_2} & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & z^{\mu_n} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}((z))).$$

We note that $\mathrm{GL}_n[[z]]$ acts from the left on Gr with orbit decomposition

$$\mathrm{Gr} = \bigsqcup_{\mu \in P^+} \mathrm{Gr}^\mu = \bigsqcup_{\mu \in P^+} \mathrm{GL}_n[[z]][z^\mu]$$

where Gr^μ are labelled by dominant weights $\mu \in P^+$ as they are the orbits of $[z^\mu]$. We have a natural map

$$(3.3) \quad \mathrm{Gr}^\mu \rightarrow \mathrm{GL}_n[z^\mu] \cong \mathrm{GL}_n/P^\mu$$

given by setting $z = 0$. We note that $\mathrm{GL}_n[z^\mu] \cong \mathrm{GL}_n/P_\mu$ is a partial flag variety and the map (3.3) is a finite rank vector bundle on GL_n/P_μ . We denote by $\overline{\mathrm{Gr}}^\mu$ the reduced of the closure of Gr^μ in Gr . We then have

$$(3.4) \quad \overline{\mathrm{Gr}}^\mu = \bigsqcup_{\mu \geq \lambda \in P^+} \mathrm{Gr}^\lambda$$

where $\mu \geq \lambda$ is meant in the *dominance order* on P^+ i.e. when $\mu - \lambda$ is some sum of positive roots (possibly with multiplicity).

One important example is when $\mu = \omega_k = \underbrace{(1, \dots, 1, 0, \dots, 0)}_k$ is the k -th fundamental weight. Then ω_k is minuscule (minimal in dominance order) and

$$\overline{\mathrm{Gr}}^{\omega_k} = \mathrm{Gr}^{\omega_k} \cong \mathrm{Gr}(k, n)$$

the classical Grassmannian.

For a dominant μ we can now define the Hecke correspondence of type μ over the Hitchin section as

$$(3.5) \quad \mathcal{H}^\mu := \{(\mathcal{E}_a, V) \in W_0^+ \times \mathrm{Gr}^\mu \text{ s.t. } \Phi_a|_{\Delta_c}(V) \subset V\} \subset W_0^+ \times \mathrm{Gr}^\mu,$$

where $\Delta_c \rightarrow C$ is the formal neighbourhood of c in C . In particular $\Delta_c \cong \mathrm{Spec}(\mathbb{C}[[z]])$. We fix a trivialisation $E_0|_{\Delta_c} \cong \mathcal{O}_{\Delta_c}^n$ and a trivialisation $K|_{\Delta_c} \cong \mathcal{O}_{\Delta_c}$ then we can think of $\gamma_a := \Phi_a|_{\Delta_c} \in \mathfrak{gl}_n[[z]]$. Then if $V = [g] \in \mathrm{Gr}^\mu$ is represented by $g \in \mathrm{GL}_n((z))$ the condition $\Phi_a|_{\Delta_c} \in \mathfrak{gl}_n[[z]]$ translates as

$$(3.6) \quad \gamma_a g^{-1} \in \mathfrak{gl}_n[[z]].$$

For a fixed $a \in \mathcal{A}$ this defines the *affine Springer fiber* of γ_a .

By performing a type μ Hecke transformation of \mathcal{E}_0 at $V \in \text{Gr}^\mu$, this will yield a new vector bundle together with a Higgs field, thanks to the invariance condition (3.6). Provided some stability conditions are satisfied the new Higgs bundle will be stable and on the upward flow of $\mathcal{E}_{\delta_c^\mu}$, where

$$(3.7) \quad \delta_c^\mu = (\alpha_n c, \alpha_{n-1} c, \dots, \alpha_1 c)$$

by writing $\mu = \sum_i \alpha_i \omega_i$ in terms of the fundamental weights and $\alpha_i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq n-1$ and $\alpha_n \in \mathbb{Z}$. We claim that such a Hecke transformation will induce an isomorphism and so that we get the following diagram:

$$(3.8) \quad \begin{array}{ccc} \mathcal{H}^\mu & \xrightarrow{\cong} & W_{\delta_c^\mu}^+ \\ \pi_\mu \downarrow & & \downarrow h_{\delta_c^\mu} \\ W_0^+ & \xrightarrow{h_0} & \mathcal{A} \end{array}$$

This way we get $Q_{\pi_\mu} \cong Q_{h_{\delta_c^\mu}}$. Thus we reduced the computation of the equivariant multiplicity algebra for a computation inside Gr^μ by studying the equations describing $\mathcal{H}^\mu \subset W_0^+ \times \text{Gr}^\mu$. It turns out that this reduces to a relatively simple linear algebra computation. We have the following results and conjectures. In order to formulate them we will need to introduce the notion of dominant upward flows and their multiplicity.

Definition 3.11. Let $\mathcal{E} \in \mathcal{M}^{\text{s}\mathbb{T}}$. We call the upward flow $W_{\mathcal{E}}^+$ *dominant* if the Hitchin map $h_{\mathcal{E}} : W_{\mathcal{E}}^+ \rightarrow \mathcal{A}$ is dominant. In this case the induced map on algebra of functions $h_{\mathcal{E}}^* : \mathbb{C}[\mathcal{A}] \rightarrow \mathbb{C}[W_{\mathcal{E}}^+]$ is injective and thus we get an extension $\mathbb{C}[\mathcal{A}] \subset \mathbb{C}(W_{\mathcal{E}}^+)$ of function fields. We define its degree to be the *multiplicity* of $W_{\mathcal{E}}^+$

$$m_{\mathcal{E}} := [\mathbb{C}(W_{\mathcal{E}}^+) : \mathbb{C}[\mathcal{A}]].$$

We note that $m_{\mathcal{E}}$ also agrees with the rank

$$m_{\mathcal{E}} = \dim_{\mathbb{C}[\mathcal{A}]}(M_{\mathcal{E}} \otimes_{\mathbb{C}[\mathcal{A}]} \mathbb{C}[\mathcal{A}])$$

of the $\mathbb{C}[\mathcal{A}]$ -module $M_{\mathcal{E}}$ given by $h_{\mathcal{E}}^*$ and also with the cardinality of the generic fiber

$$m_{\mathcal{E}} = \#(h_{\mathcal{E}}^{-1}(a) \cap W_{\mathcal{E}}^+)$$

for generic $a \in \mathcal{A}$. Notice that these last two definitions make sense for all upward flows. Dominance in turn then will be equivalent with nonzero multiplicity.

We have the following results:

Theorem 3.12 ([HH1]). *Let $\mu = (d+1)\omega_k \in P^+$ for $d \in \mathbb{Z}_{>0}$ and $1 \leq k \leq n-1$ and $c \in \mathbb{C}$. When $\mathcal{E}_{\delta_c^\mu} \in \mathcal{M}^{\text{s}\mathbb{T}}$ the following hold*

- (1) *Hecke modification of type μ of W_0^+ induces $\mathcal{H}^\mu \rightarrow W_{\delta_c^\mu}^+$ an isomorphism*
- (2) *$W_{\mathcal{E}_{\delta_c^\mu}}^+$ is dominant*
- (3) *$m_{\mathcal{E}} = |W \cdot \mu|$ is the order of the Weyl orbit of μ*

(4) we have

$$Q_\mu \cong \mathbb{C}[J_d(Q_{\omega_k})] \cong \mathbb{C}[J_d(\text{Spec}(H^*(\text{Gr}(k, n), \mathbb{C})))] ,$$

where for a scheme X we denote by $J_d(X)$ the $d - 1$ th jet scheme of $\text{Spec}(R)$. In particular $J_d(X)(\mathbb{C}) = \text{Hom}(\text{Spec}(\mathbb{C}[z]/(z^d)), X)$ is the set of $d - 1$ jets in X .

Remark 3.13. We conjecture that (1),(2) and (3) hold for any $\mu \in P^+$.

Remark 3.14. As examples in the $n = 2$ case let us give the multiplicity algebra for $d = 1, 2$ and 3. First we have

$$(3.9) \quad Q_{\delta_c^{2\omega_1}} \cong \mathbb{C}[a_0]/(a_0^2) \cong H^*(\mathbb{P}^1, \mathbb{C}).$$

Then we have

$$(3.10) \quad Q_{\delta_c^{2\omega_1}} \cong \mathbb{C}[a_0, a_1]/(a_0^2, a_0a_1).$$

Note $(a_0^2, a_0a_1) = (a_0, a_1)^2 \cap (a_0)$ thus the scheme theoretical intersection $\text{Spec}(Q_{\delta_c^{2\omega_1}})$ of the upward flow $W_{\delta_c^{2\omega_1}}^+$ and the nilpotent cone $h^{-1}(0)$ is the line (a_0) with a double embedded point at the origin. Note that this upward flow was studied in [HH, §8.2].

For $d = 3$ we have the multiplicity 2 algebra

$$(3.11) \quad Q_{\delta_c^{3\omega_1}} \cong \mathbb{C}[a_0, a_1, a_2]/(a_0^2, a_0a_1, a_0a_2 + a_1^2).$$

Both (3.10) and (3.11) follows from Conjecture 3.12.4 and both can be proved by direct computation in Gr^μ as explained above.

Remark 3.15. It is surprising how complex $J_d(\text{Spec}(H^*(\text{Gr}(k, n), \mathbb{C})))$ can be. In particular, in the $k = 1$ case (i.e. jet schemes of the cohomology ring of projective space) there is only a conjecture about its multiplicity in [Yu, Conjecture III.21].

Remark 3.16. Finally we remark that already for type (2) we have new phenomena. As discussed in [HI4, §5.4] there are multiplicity algebras depending on continuous parameters, in particular they cannot be isomorphic with cohomology rings, because cohomology rings are integral.

3.3.1. Lagrangian closure of W_δ^+ .

Definition 3.17. Let $\mathcal{E} \in \mathcal{M}^{s\mathbb{T}}$. The *Lagrangian closure* $\overline{\overline{W_\mathcal{E}^+}}$ of $W_\mathcal{E}^+$ is the smallest closed union of upward flows containing $W_\mathcal{E}^+$. In other words the Lagrangian closure is the closure in the quotient space by the BB partition.

Using (3.8) and (3.4) we can deduce the following

Theorem 3.18 ([HH1]). *Let $\mu \in P^+$ and $c \in C$. Recall δ_c^μ from (3.7). Assume $\mathcal{E}_{\delta_c^\mu} \in \mathcal{M}^{s\mathbb{T}}$. Then*

$$\overline{\overline{W_\mathcal{E}^+}} = \coprod_{\mu \geq \lambda \in P^+} W_{\delta_c^\lambda}^+,$$

i.e. the upward flows corresponding to dominant weights λ less than or equal to μ in dominance order.

3.4. Towards a classical limit of the geometric Satake correspondence

Finally, we will formulate some conjectures which were the original motivation of much of the previous ideas. In particular, they hint at a new construction of the irreducible representations of $GL_n(\mathbb{C})$, and more generally of any complex reductive group .

The general setup comes from the classical limit (2.5) of the geometric Langlands program, as formulated in [DP]. Here we sketch some of the expectations of this classical limit in a schematic (not completely well-defined) manner. It should be an equivalence of some sort of derived categories of coherent sheaves

$$\mathcal{S} : D^b(\mathcal{M}_{\text{Dol}}) \rightarrow D^b(\mathcal{M}^\vee).$$

Several properties of this equivalence were proposed and some established in [DP]. In particular, \mathcal{S} should be a relative Fourier-Mukai transform along the generic locus in $\mathcal{A}_{\cong, \mathcal{A}^\vee}$. Another crucial property [KW] - which we called enhanced mirror symmetry in §2.5 above - is that \mathcal{S} should intertwine the actions of certain Hecke operators on $D^b(\mathcal{M}_{\text{Dol}})$ and the Wilson operators on $D^b(\mathcal{M}_{\text{Dol}})$. Let $\mu \in X_+(\vee) = X^+(\vee)$ be a dominant character of \vee . We denote by

$$\mathcal{H}^\mu := \{(E, \Phi) \in \mathcal{M}_{\text{Dol}}, [g] \in \text{Gr}^\mu | g^{-1}\Phi_c g \in [[z]]\} \subset \mathcal{M}_{\text{Dol}} \times \text{Gr}^\mu$$

some space of Hecke correspondences at a point $c \in C$. Indeed this gives us

$$\begin{array}{ccc} & \mathcal{H}^\mu & \\ \pi_\mu \swarrow & & \searrow f^\mu \\ \mathcal{M}_{\text{Dol}} & & \mathcal{M}_{\text{Dol}} \end{array}$$

two maps to \mathcal{M}_{Dol} , first the projection π_μ to the first factor, and second f^μ , the Hecke transformation* of (E, Φ) by the compatible Hecke transform $[g] \in \text{Gr}^\mu$, which is expected to induce

$$\mathcal{H}^\mu := f_*^\mu \circ \pi_\mu^* : D^b(\mathcal{M}_{\text{Dol}}) \rightarrow D^b(\mathcal{M}_{\text{Dol}})$$

a Hecke or the physicists' *t'Hooft operator*.

On the other hand we can consider the so-called *Wilson operators*

$$\mathcal{W}^\mu : \begin{array}{ccc} D^b(\mathcal{M}_{\text{Dol}}^\vee) & \rightarrow & D^b(\mathcal{M}_{\text{Dol}}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes \rho_\mu(\mathbb{E}|_{\mathcal{M}_{\text{Dol}}^\vee \times \{c\}}) \end{array}$$

given by tensoring with the universal \vee bundle \mathbb{E} in the representation $\rho_\mu : \vee \rightarrow GL(V_{\rho_\mu})$.

We then expect [DP, KW] that

$$(3.12) \quad \mathcal{W}^\mu \circ \mathcal{S} = \mathcal{S} \circ \mathcal{H}^\mu.$$

There are two more expectations for the classical limit \mathcal{S} , both are motivated from Fourier-Mukai transform where the analogous statements hold. First we expect that for any $\mathcal{F} \in D^b(\mathcal{M}_{\text{Dol}})$ we should have

$$(3.13) \quad (h)_{*(\mathcal{F}) \cong \mathcal{S}(\mathcal{F})|_{\mathbb{W}_0^+}}.$$

* Here we ignore issues about stability.

Second, the structure sheaf of the Hitchin section should be mirror to the structure sheaf of the mirror Higgs moduli space:

$$(3.14) \quad \mathcal{S}(\mathcal{O}_{W_0^+}) \cong \mathcal{O}_{\mathcal{M}_{\text{Dol}}^\vee}.$$

Combining (3.12) with (3.14) we can deduce that

$$\mathcal{S}(\mathcal{H}^\mu(\mathcal{O}_{W_0^+})) = \mathcal{W}^\mu(\mathcal{S}(\mathcal{O}_{W_0^+})) = \mathcal{W}^\mu(\mathcal{O}_{\mathcal{M}_{\text{Dol}}^\vee}).$$

On the one hand we should have

$$\text{supp}(\mathcal{H}^\mu(\mathcal{O}_{W_0^+})) = \overline{W_\mu^+},$$

where W_μ^+ is the upward flow from a certain \mathcal{E}_μ maximally split G -Higgs bundle of type μ at $c \in C$. On the other hand

$$\mathcal{W}^\mu(\mathcal{O}_{\mathcal{M}_{\text{Dol}}^\vee}) = \rho_\mu(\mathbb{E})_c =: \Lambda_\mu,$$

the vector bundle associated to the principal bundle \mathbb{E}_c in the representation ρ_μ .

Thus Kapustin-Witten's (3.12) implies

$$\mathcal{S}(\mathcal{H}^\mu(\mathcal{O}_{W_0^+})) = \Lambda_\mu.$$

We can test this by (3.13)

$$\Lambda_\mu|_{L_{W_0^+}} = \mathcal{S}(\mathcal{H}^\mu(\mathcal{O}_{W_0^+}))|_{L_{W_0^+}} = (h_*)_* (\mathcal{H}^\mu(\mathcal{O}_{W_0^+})).$$

In [HH] we have made arguments that the mirror of the structure sheaf of a very stable type $1, \dots, 1$ upward flow W_δ is

$$\Lambda_\delta := \bigotimes_{i=1}^{n-1} \bigotimes_{j=1}^{m_i} \Lambda^i \mathbb{E}_{c_{ij}},$$

where (\mathbb{E}, Φ) is a universal Higgs bundle on $\mathcal{M} \times C$ and

$$\delta_i = c_{i1} + c_{i2} + \dots \in C^{[m_i]}.$$

In particular, one expectation of mirror symmetry is that

$$h_*(\mathcal{O}_{W_\delta^+}) \cong \Lambda_\delta|_{W_0^+}.$$

This follows from Theorem 3.6 and a direct computation for $\mathcal{X}_{\mathbb{T}}(\mathbb{E}_c)$.

In [HH, §8.2] we proposed that for $n = 2$ the mirror of $\text{Sym}^2(\mathbb{E}_c)$ should be the structure sheaf of the Lagrangian closure $\overline{W_{\delta_c^2}^+}$ where $\delta_c^2 = (0, 2c)$. We can generalise this as follows.

Conjecture 3.19. *Let $c \in C$ and a reductive group. Then we have the following conjectures.*

- (1) *For any $\mu \in X_+(\vee)$ the support of the mirror of $\rho_\mu(\mathbb{E}_c)$ is $\overline{W_{\delta_c^\mu}^+}$*
- (2) *Let $\mu \in X_+(\vee)$ such that the corresponding irreducible \vee representation ρ_μ is multiplicity free. Then the mirror of $\rho_\mu(\mathbb{E}_c)$ is $\mathcal{O}_{\overline{W_{\delta_c^\mu}^+}}$.*

- (3) In the latter case the multiplicity algebra of the restricted Hitchin map $h: \overline{W}_{\delta_c^\mu}^+ \rightarrow \mathcal{A}$ is isomorphic with the cohomology ring of $\overline{\text{Gr}}^\mu$.

Remark 3.20. In [FFR], studying opers in the geometric Langland program, the authors construct a canonical Poincaré duality ring structure on the underlying vector space V_μ of all irreducible representation ρ_μ of \vee . In the case when ρ_μ is multiplicity-free this ring is isomorphic with the cohomology ring $H^*(\overline{\text{Gr}}^\mu)$. Note that according to [Gi, Theorem 1.5] these are precisely the cases when

$$H^*(\overline{\text{Gr}}^\mu) \cong IH^*(\overline{\text{Gr}}^\mu),$$

when the cohomology ring satisfies Poincaré duality. In this case this ring was more carefully studied in [Pa].

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